# Some properties of group actions on zero-dimensional spaces

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Let $X$ be a locally compact Hausdorff topological space and write $\mathcal{CO}(X)$ for the set of compact open subsets of $X$. Suppose that $X$ is **zero-dimensional**, meaning that $\mathcal{CO}(X)$ forms a base for the topology.

Let $S \subseteq \mathrm{Homeo}(X)$, such that $\mathrm{id}_X \in S$, $S = S^{-1}$ and $\{sU \mid s \in S\}$ is finite for every $U \in \mathcal{CO}(X)$. Let $S^n$ be the set of products of at most $n$ elements of $S$, and let $G = S^\infty = \langle S \rangle$.

Fix some $U \in \mathcal{CO}(X)$. Write $U_0 = U$;

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We think of $U$ as partitioned into a 'core' $U_{+\infty}$ (compact, but not necessarily open) and a sequence of 'shells' $W_n := U_n \setminus U_{n+1}$ indexed by the integers (each of which is compact and open).

### Lemma

(i) There exists $a \geq 0$ such that $U_a = U_\infty$ and $W_m$ is nonempty exactly when $m \in [0, a)$.

(ii) Every $G$-orbit intersecting $U_n \setminus U_{+\infty}$ also intersects $W_m$ for all $m \in [0, n]$.

(iii) There is a $G$-orbit $Gx$ that intersects all of the nonempty shells.

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## Proof

(i) Suppose for some $a \geq 0$ that $W_a = \emptyset$, i.e. $U_a = U_{a+1}$, and let $m \geq 0$. Then

$$U_{a+m} = \bigcap_{g \in S^m} gU_a = \bigcap_{g \in S^m} gU_{a+1} = U_{a+m+1}.$$

(ii) Let $x \in U_n \setminus U_{+\infty}$. Then $x \in W_{n'}$ for some $n' \geq n$, and hence there exists $g \in S$ such that $gx \notin U_{n'}$ (otherwise we would have $x \in U_{n'+1}$), but $gx \in U_{n'-1}$ (since $x \in U_{n'}$). Thus $gx \in W_{n'-1}$. Repeat to get images of $x$ in $W_m$ for all $0 \leq m \leq n'$.

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(iii) Define $P_n = (\bigcup_{g \in S^n} g^{-1} U_n) \setminus U_1$. Then $P_n$ is a compact subset of $U$. Let $I$ be the set of $n > 0$ such that $W_n \neq \emptyset$. Given part (ii) it is enough to show $\bigcap_{n \in I} P_n \neq \emptyset$.

Suppose $x \in P_n$. Then $\exists g \in S, h \in S^{n-1} : ghx \in U_n$, so $hx \in U_{n-1}$ and hence $x \in P_{n-1}$. Thus $(P_n)_{n \in I}$ is a descending sequence.

Suppose $\bigcap_{n \in I} P_n = \emptyset$. Then by compactness $P_n = \emptyset$ for some $n \in I$, that is, $g^{-1} U_n \subseteq U_1$ for all $g \in S^n$. But then $U_n \subseteq \bigcap_{g \in S^n} gU_1 = U_{n+1}$, so $W_n = \emptyset$, contradicting the choice of $n$.

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Alternative incarnation of (iii) (think of $G = X$ acting by conjugation on itself, and $U$ a vertex stabilizer):

## Lemma/Corollary

Let $\Gamma$ be a connected locally finite graph and let $G$ be a closed vertex-transitive group of automorphisms of $\Gamma$. Then exactly one of the following holds:

(i) There is a finite set $v_1, \dots, v_n$ of vertices, such that $\bigcap_{i=1}^{n} G_{v_i} = \{1\}$.

(ii) There is a horoball $H$ in $\Gamma$, such that the pointwise fixator of $H$ in $G$ is nontrivial.

Here we define a **horoball** to be a set of the form $\{v \in V\Gamma : \exists n : d(v, v_n) \leq n\}$, where $(v_n)_{n \geq 0}$ is a set of vertices forming a geodesic ray.

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Hypotheses: Let $X$ be a locally compact zero-dimensional space, $S \subseteq \mathrm{Homeo}(X)$ such that $S = S^{-1}$ and $\{sU \mid s \in S\}$ is finite for every $U \in \mathcal{CO}(X)$, and $G = \langle S \rangle$.

**Theorem (Auslander–Glasner–Weiss; R.)**

Let $U \in \mathcal{CO}(X)$ and write $U_{+\infty} = \bigcap_{g \in G} gU$. Then the following are equivalent:

(i) Given $x \in U$ and $y \in U_{+\infty}$ such that $y \in \overline{Gx}$, then $x \in \overline{Gy}$.

(ii) For all $V \in \mathcal{CO}(U)$, there is a finite subset $F$ of $G$ such that $V_{+\infty} = \bigcap_{g \in F} gV$.

(iii) $U_{+\infty}$ is open and there is a $G$-invariant quotient map $\phi : U_{+\infty} \to Y$, such that $G$ acts trivially on $Y$ and minimally on each fibre of $\phi$.

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**Distal** action: if $(g_i x, g_i y) \to (z, z)$ as $i \to \infty$, then $x = y$.
In particular, if $\overline{Gy}$ is compact and $y \in \overline{Gx}$, then $\overline{Gx} = \overline{Gy}$.

## Corollary

Suppose that $G$ acts distally on $X$ and that every orbit has compact closure. Then $\{gV \mid g \in G\}$ is finite for every $V \in \mathcal{CO}(X)$. In particular, the action of $G$ is equicontinuous.

(If $X$ is the Cantor set, then $G \leq \mathrm{Homeo}(X)$ acts equicontinuously if and only if there is a compatible $G$-invariant metric on $X$, or equivalently $X$ is the boundary of some locally finite rooted tree on which $G$ acts by automorphisms.)

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A locally compact group $G$ is **distal** (as a topological group) if it acts distally on itself by conjugation; equivalently, no conjugacy class of $G$ accumulates at the identity. For example: nilpotent groups; discrete groups; compact groups; any residually distal group is distal.

t.d.l.c. group = “totally disconnected locally compact group”. T.d.l.c. groups are zero-dimensional; in fact the cosets of compact open *subgroups* form a base for the topology (Van Dantzig).

Corollary (Willis; Caprace–Monod; R.)

Let $G$ be a compactly generated t.d.l.c. group. Then $G$ is distal if and only if the cosets of open *normal* subgroups of $G$ form a base for the topology.

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## Proposition (Caprace–Monod; R.–Wesolek)

Let $G$ be a compactly generated t.d.l.c. group and let $U$ be a compact open subgroup of $G$.

(i) Let $(K_i)_{i \in \mathbb{N}}$ be a sequence of closed normal subgroups such that $K_i \to \{1\}$ as $i \to \infty$. Then for $i$ large enough, $K_i \cap U$ is normal in $G$.

(ii) Suppose that $\bigcap_{g \in G} gUg^{-1} = \{1\}$ and that $G$ has no nontrivial discrete normal subgroup. Then every nontrivial closed normal subgroup of $G$ contains a minimal one.

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Let $G$ be a t.d.l.c. group and let $H$ be a compactly generated group of automorphisms of $G$. Write $\mathrm{Res}_G(H)$ for the intersection of all open $H$-invariant subgroups of $G$.

**Theorem (R.)**

(i) There is an $H$-invariant open subgroup of the form $V\mathrm{Res}_G(H)$ for some compact open subgroup $V$ of $G$. Moreover, $\mathrm{Res}_G(H)$ is normal in $V\mathrm{Res}_G(H)$.

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Let $G$ be a group acting faithfully on a space $X$, and given $Y \subseteq X$, write $\mathrm{rist}_G(Y)$ for the set of elements that fix $X \setminus Y$ pointwise. The action is **micro-supported** if $\mathrm{rist}_G(Y) \neq \{1\}$ for every nonempty open $Y$.

### Theorem (Caprace–R.–Willis)

Let $G$ be a compactly generated t.d.l.c. group with faithful continuous action by homeomorphisms on the Cantor set $X$. Suppose that $G$ has a compact open subgroup $U$, such that $U$ is micro-supported on $X$ and $\bigcap_{g \in G} gUg^{-1} = \{1\}$. Then there is a partition of $X$ into clopen sets $B_1, \dots, B_n$ such that for every $A \in \mathcal{CO}(X) \setminus \{\emptyset\}$, there is $g \in G$ and $1 \leq i \leq n$ such that $B_i \subseteq gA$.

If $G$ is topologically simple, then the action is also minimal, and consequently $G$ is not amenable.

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