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Groups Acting On Trees and Contributions to Willis Theory

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STEPHAN TORNIER

MSc ETH in Mathematics
born April 5, 1988
citizen of Germany

Accepted on the recommendation of:

PROF. MARC BURGER, examiner
PROF. GEORGE WILLIS, co-examiner
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A witty quote.

Abstract

This work is concerned with the structure theory of totally disconnected locally compact groups. In a first part, we develop a generalization of Burger–Mozes universal groups acting on regular trees locally like a given permutation group of finite degree. This generalization arises through prescribing the local action on vertex neighbourhoods of a given radius and results in an equally rich and manageable class of groups acting on trees. As an application, we characterize Banks–Elder–Willis k-closures of groups that act locally transitively on the regular tree T_d with an involutive inversion. Our construction also offers a new perspective on the long standing Weiss conjecture in the context of which we recover several known results. Finally, the framework of generalized universal group yields a local-to-global type characterization of the elements which the quasi-center of a non-discrete subgroup of $\mathrm{Aut}(T_d)$ may contain in terms of the group's local action. Most importantly, we show that this characterization is sharp through explicit construction, thus answering a question of Burger for more examples of closed non-discrete subgroups of $\mathrm{Aut}(T_d)$ with non-trivial quasi-center.

The first part ends with a computation of prime localizations of a large class of Burger–Mozes-type groups, including Burger–Mozes universal groups, Le Boudec groups with almost prescribed local action and Lederle's coloured Neretin groups.

The second part contains two works, joint with H. Glöckner and T. Bywaters, and T. Bywaters respectively. Both contribute to Willis theory which studies totally disconnected locally compact groups from the point of view of their endomorphisms. First, we extend results about how the scale and tidy subgroups behave when passing to subgroups or quotients from automorphisms to endomorphisms. Secondly, we offer a geometric characterization of the scale and tidy subgroups associated to endomorphisms, as well as a new tidying procedure in terms of graphs. This is based on prior work of Möller in the case of automorphisms.

Zusammenfassung

Diese Arbeit befasst sich mit der Strukturtheorie total unzusammenhängender lokalkompakter Gruppen. Der erste Teil entwickelt eine Verallgemeinerung der universellen Burger–Mozes-Gruppen, die lokal wie eine gegebene Permutationsgruppe endlichen Grades auf regulären Bäumen wirken. Besagte Verallgemeinerung basiert auf der Festlegung der lokalen Wirkung auf Knotenumgebungen eines vorgegeben Radius, und resultiert in einer gleichermaßen reichhaltigen und handlichen Klasse von Gruppen, die auf Bäumen wirken. Eine erste Anwendung besteht in der Charakterisierung der Banks–Elder–Willis k-Abschlüsse von Gruppen, die lokal transitiv auf dem regulären Baum T_d wirken und eine involutorische Kanteninversion enthalten. Unsere Konstruktion bietet außerdem eine neue Perspektive auf die lang bestehende Weiss'sche Vermutung, in dessen Kontext wir einige bekannte Resultate wiedergewinnen. Schließlich erlangen wir im Rahmen der verallgemeinerten universellen Gruppen eine Charakterisierung der Elemente, die das Quasi-Zentrum einer nicht-diskreten Untergruppe von $\mathrm{Aut}(T_d)$ enthalten kann, in Abhängigkeit von der lokalen Wirkung. Es sei betont, dass sich besagte Charakterisierung durch explizite Konstruktion als strikt erweist. Damit beantworten wir eine Frage von Burger nach neuen Beispielen von abgeschlossenen, nicht-diskreten Untergruppen von $\mathrm{Aut}(T_d)$ mit nicht-trivialem Quasi-Zentrum.

Der erste Teil endet mit der Berechnung der Primlokalisierungen einer großen Klasse von Gruppen des Burger–Mozes Typ. Dies umfasst die universellen Burger–Mozes-Gruppen, Le Boudec-Gruppen mit fast überall vorgeschriebener lokaler Wirkung, und Lederle's gefärbte Versionen von Neretin's Gruppe.

Der zweite Teil enthält zwei Zusammenarbeiten mit H. Glöckner und T. Bywaters beziehungsweise T. Bywaters. Beide leisten einen Beitrag zur Willis-Theorie, die total unzusammenhängende lokalkompakte Gruppen vom Standpunkt ihrer Endomorphismen aus studiert. Zuerst erweitern wir Resultate, die das Verhalten zentraler Konzepte beim Übergang zu Untergruppen oder Quotienten betreffen, von Automorphismen zu Endomorphismen. Anschließend entwickeln wir eine geometrische Beschreibung derselben Konzepte. Dies basiert auf einer bestehenden Arbeit von Möller für den Fall von Automorphismen.

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Introduction and Main Results

In a broad sense, this work is concerned with the structure theory of locally compact groups. A locally compact group G is an extension of its connected component G_0 by the totally disconnected quotient G/G_0:

$$1 \longrightarrow G_0 \longrightarrow G \longrightarrow G/G_0 \longrightarrow 1.$$

Consequently, the study of general locally compact groups splits into connected and totally disconnected such groups via topological group extensions.

Connected locally compact groups are inverse limits of Lie groups by the seminal solution of Hilbert's fifth problem due to Gleason [**Gle52**], Yamabe [**Yam53**], Montgomery–Zippin [**MZ52**] and others. As such, the methods of Lie theory have successfully contributed to their understanding.

Totally disconnected locally compact (t.d.l.c.) groups are nowhere near as well understood as their connected counterparts and exhibit a wealth of phenomena. Nevertheless, recent developments such as [**Wil94**], [**BM00a**], [**CM11**], [**Wes15**], [**RW15**], [**Wil15**] and [**CRW17**] hint at the potential for a general structure theory.

This thesis advances said emerging theory in two largely independent parts. The first one is concerned with the structure theory of groups acting on trees after Burger–Mozes, see [**BM00a**] and [**BM00b**]. These groups form a particularly important class of t.d.l.c. groups for both theoretical and practical reasons.

Part 2 contributes to Willis theory, initiated in [**Wil94**]. This theory studies t.d.l.c. groups from the point of view of their endomorphisms and has lead to numerous unexpected applications. Whereas Chapter V contains joint work with T. Bywaters and H. Glöckner, Chapter VI constitutes joint work with T. Bywaters.

Burger–Mozes Theory and Universal Groups

Every (totally disconnected) locally compact group can be viewed as a directed union of compactly generated open subgroups. Among compactly generated t.d.l.c. groups, automorphism groups of trees stand out for the following reason: Every compactly generated t.d.l.c. group G acts vertex-transitively on a regular graph Γ of finite degree d with compact normal kernel K, known as the Schreier graph or Cayley-Abels graph, see e.g. [**Mon01**, Section 11.3]. In particular, the universal cover of Γ is the d-regular tree T_d and one obtains G/K as a quotient of a cocompact subgroup $\widetilde{G}$ of $\mathrm{Aut}(T_d)$ due to the short exact sequence

$$1 \longrightarrow \pi_1(\Gamma) \longrightarrow \widetilde{G} \longrightarrow G/K \longrightarrow 1.$$

Let Ω be a set of cardinality $d \geq 3$ and let $T_d = (V, E)$ denote the d-regular tree, following Serre's notation [**Ser03**]. Then $\mathrm{Aut}(T_d)$ is a (compactly generated) t.d.l.c. group when equipped with the permutation topology for its action on V. For a subgroup $H \leq \mathrm{Aut}(T_d)$ and a vertex $x \in V$, we let H_x denote the stabilizer of x in H. It induces a permutation group on the set $E(x) := \{e \in E \mid o(e) = x\}$ of edges issuing from x. We say that H is locally "P" if for every $x \in V$ said permutation group satisfies property "P", e.g. being transitive, quasiprimitive or 2-transitive. Refer to Section I.1 for details about permutation groups.

In [**BM00a**], Burger–Mozes develop a remarkable structure theory of closed, non-discrete, locally quasiprimitive subgroups of $\mathrm{Aut}(T_d)$, which resembles the theory of semisimple Lie groups, see Section I.3.

This structure theory is complemented with a particularly accessible class of examples of subgroups of $\mathrm{Aut}(T_d)$ with prescribed local properties: Let $l : E \to \Omega$ be a labelling of T_d, i.e. $l_x := l|_{E(x)} : E(x) \to \Omega$ is a bijection for every $x \in V$ and $l(e) = l(\overline{e})$ for all $e \in E$. Then the map

$$\sigma : \mathrm{Aut}(T_d) \times V \to \mathrm{Sym}(\Omega),\ (g,x) \mapsto l_{gx} \circ g \circ l_x^{-1}$$

captures the *local action* of g at $x \in V$. Now, given $F \leq \mathrm{Sym}(\Omega)$, a subgroup of $\mathrm{Aut}(T_d)$ all of whose local actions are in F can be defined as follows.

Definition. Let $F \leq \mathrm{Sym}(\Omega)$. Set $\mathrm{U}(F) := \{g \in \mathrm{Aut}(T_d) \mid \forall x \in V : \sigma(g,x) \in F\}$.

The following list of properties of $\mathrm{U}(F)$ underlines its utility.

Proposition I.12 ([**BM00a**, Section 3.2]). Let $F \leq \mathrm{Sym}(\Omega)$. Then $\mathrm{U}(F)$ is

(i) closed in $\mathrm{Aut}(T_d)$,
(ii) vertex-transitive,
(iii) compactly generated,
(iv) locally permutation isomorphic to F,
(v) edge-transitive if and only if F is transitive, and
(vi) discrete in $\mathrm{Aut}(T_d)$ if and only if F is semiregular.

For transitive F, the group $\mathrm{U}(F)$ is maximal up to conjugation among vertex-transitive subgroups of $\mathrm{Aut}(T_d)$ that locally act like F, hence the term *universal*.

Proposition I.14 ([**BM00a**, Proposition 3.2.2]). Let $H \leq \mathrm{Aut}(T_d)$ be locally transitive and vertex-transitive. Then there is a labelling of T_d such that $H \leq \mathrm{U}(F)$ where $F \leq \mathrm{Sym}(\Omega)$ is permutation isomorphic to the action of H on balls of radius 1.

The universal groups defined above are a central tool in the study of more general subgroups $\mathrm{Aut}(T_d)$, such as projections of lattices $\Gamma \leq \mathrm{Aut}(T_{d_1}) \times \mathrm{Aut}(T_{d_2})$ which are investigated in [**BM00b**] and [**Rat04**].

We generalize the universal groups by prescribing the local action on balls of a given radius $k \in \mathbb{N}$, the Burger–Mozes construction corresponding to the case $k = 1$. Namely, fix a tree $B_{d,k}$ which is isomorphic to a ball of radius k in the labelled tree T_d and let $l_x^k : B(x,k) \to B_{d,k}$ be the unique label-respecting isomorphism. Then

$$\sigma_k : \mathrm{Aut}(T_d) \times V \to \mathrm{Aut}(B_{d,k}),\ (g,x) \mapsto l_{gx}^k \circ g \circ (l_x^k)^{-1}$$

is the natural generalization of the map σ defined above to the *k-local action*.

Definition II.1. Let $F \leq \mathrm{Aut}(B_{d,k})$. Define

$$\mathrm{U}_k(F) := \{g \in \mathrm{Aut}(T_d) \mid \forall x \in V : \sigma_k(g,x) \in F\}.$$

Properties (i), (ii) and (iii) of $\mathrm{U}(F)$ carry over to $\mathrm{U}_k(F)$ in a straightforward fashion, whereas (v) admits a natural generalization. Concerning (vi), there is a natural *discreteness condition* (D) on $F \leq \mathrm{Aut}(B_{d,k})$ in terms of certain stabilizers in F which holds if and only if $\mathrm{U}_k(F)$ is discrete, generalizing the case $k = 1$. See Section II.3. Property (iv), however, need not hold for $k \geq 2$: The group $\mathrm{U}_k(F^{(k)})$ need not be locally action isomorphic to $F^{(k)}$. We define the following *compatibility condition*, which can be viewed as an interchangeability condition on neighbouring local actions with the appropriate point of view on $F^{(k)}$, see Section II.3.

Definition II.8. Let $F \leq \mathrm{Aut}(B_{d,k})$. Then F satisfies (C) if $\mathrm{U}_k(F)$ locally acts like F.

Numerous examples of subgroups of $\mathrm{Aut}(B_{d,k})$ satisfying the compatibility condition (C) and/or the discretenss condition (D) are given in Section II.3.

Next recall that the quasi-center of a topological group G, denoted by $\mathrm{QZ}(G)$, consists of those elements whose centralizer in G is open. It plays a major role in the Burger–Mozes Structure Theorem I.9.

Proposition II.16. Let $F \leq \mathrm{Aut}(B_{d,k})$. If F satisfies (D) then $\mathrm{QZ}(\mathrm{U}_k(F)) = \mathrm{U}_k(F)$. Otherwise $\mathrm{QZ}(\mathrm{U}_k(F)) = \{\mathrm{id}\}$.

We prove an analogue of the universality statement (Proposition I.14), which not only provides maximality but also a description of the *k-closures*

$$H^{(k)} := \{g \in \mathrm{Aut}(T_d) \mid \forall x \in V\ \exists h_x \in H : g|_{B(x,k)} = h_x|_{B(x,k)}\}$$

of locally transitive groups $H \leq \mathrm{Aut}(T_d)$ containing an involutive inversion, i.e. an inversion of order 2; the notion of k-closures was introduced by Banks–Elder–Willis in [**BEW15**] as a tool to construct simple t.d.l.c. groups, see Section I.2.3.

Theorem II.23. Let $H \leq \mathrm{Aut}(T_d)$ be locally transitive and contain an involutive inversion. Then there is a labelling of T_d such that

$$\mathrm{U}_1(F^{(1)}) \geq \mathrm{U}_2(F^{(2)}) \geq \cdots \mathrm{U}_k(F^{(k)}) \geq \cdots \geq H \geq \mathrm{U}_1(\{\mathrm{id}\})$$

where $F^{(k)} \leq \mathrm{Aut}(B_{d,k})$ is action isomorphic to the action of H on balls of radius k. Furthermore, $H^{(k)} = \mathrm{U}_k(F^{(k)})$.

We show that the assumption that H contains an involutive inversion, which combined with the local transitivity assumption is stronger than vertex-transitivity assumption for the case $k = 1$, is necessary.

Combined with the independence properties P_k ($k \in \mathbb{N}$) (see Section I.2.3), introduced by Banks–Elder–Willis in [**BEW15**] as generalizations of Tits' Independence Property and satisfied by the $\mathrm{U}_k(F^{(k)})$, the universality theorem entails the following characterization of universal groups.

Corollary II.25. Let $H \leq \mathrm{Aut}(T_d)$ be closed, locally transitive and contain an involutive inversion. Then $H = \mathrm{U}_k(F^{(k)})$ if and only if H satisfies Property P_k.

Given $\widetilde{F} \leq \mathrm{Aut}(B_{d,k})$, let $F := \pi\widetilde{F} \leq \mathrm{Sym}(\Omega)$ denote the projection of $\widetilde{F}$ to $\mathrm{Aut}(B_{d,1})$. Whereas we provide an abundance of possible actions $\widetilde{F}$ "above" a given $F \leq \mathrm{Sym}(\Omega)$ in general, we also have the following rigidity.

Theorem II.22. Let $F \leq \mathrm{Sym}(\Omega)$ be 2-transitive with F_ω simple non-abelian for all $\omega \in \Omega$, and let $\widetilde{F} \leq \mathrm{Aut}(B_{d,k})$ with $\pi\widetilde{F} = F$ satisfy (C). Then $\mathrm{U}_k(\widetilde{F})$ equals either

$$\mathrm{U}_2(\Gamma(F)), \quad \mathrm{U}_2(\Delta(F)), \quad \text{or} \quad \mathrm{U}_1(F).$$

Here, $\Gamma(F), \Delta(F) \leq \mathrm{Aut}(B_{d,2})$ satisfy (C) and (D) and therefore yield discrete universal groups. More examples of both discrete and non-discrete universal groups are constructed in the case where either point stabilizers in F are not simple or F is not primitive, see e.g. $\Delta(F,N), \Phi(F,N), \Phi(F,\mathcal{P}) \leq \mathrm{Aut}(B_{d,2})$ in Section II.3.1.

We now present two more applications of universal groups.

On the Weiss Conjecture. The classical Weiss conjecture [**Wei78**] states that for a given locally finite tree T there are only finitely many conjugacy classes of discrete, locally primitive and vertex-transitive subgroups of $\mathrm{Aut}(T)$. This conjecture has been extended by Potočnik–Spiga–Verret in [**PSV12**] and impressive partial results have been obtained by the same authors as well as Guidici–Morgan [**GM14**]. The Weiss conjecture relates to universal groups through the following combination of previous results.

Corollary II.27.Let $H \leq \mathrm{Aut}(T_d)$ be discrete, locally transitive and contain an involutive inversion. Then there is $F^{(k)} \leq \mathrm{Aut}(B_{d,k})$ with (C) and (D), and $H = \mathrm{U}_k(F^{(k)})$.

This suggests to tackle the following weak version of the Weiss conjecture by studying the subgroups of $\mathrm{Aut}(B_{d,k})$ satisfying (C) and (D).

Conjecture II.29. Let $F \leq \mathrm{Sym}(\Omega)$ be primitive. Then there are only finitely many conjugacy classes of discrete subgroups of $\mathrm{Aut}(T_d)$ which locally act like F and contain an involutive inversion.

Given a transitive group $F \leq \mathrm{Sym}(\Omega)$, let $\mathcal{H}_F$ denote the collection of subgroups of $\mathrm{Aut}(T_d)$ which are discrete, locally act like F and contain an involutive inversion. Then the following definition is meaningful by the above Corollary.

Definition II.30. Let $F \leq \mathrm{Sym}(\Omega)$ be transitive. Define

$$\dim_{\mathrm{CD}}(F) := \max_{H \in \mathcal{H}_F} \min \left\{ k \in \mathbb{N} \,\middle|\, \exists F^{(k)} \in \mathrm{Aut}(B_{d,k}) \text{ with (C),(D)}: \ H = \mathrm{U}_k(F^{(k)}) \right\}$$

if the maximum exists and $\dim_{\mathrm{CD}}(F) = \infty$ otherwise.

Conjecture II.29 is now equivalent to the assertion that $\dim_{\mathrm{CD}}(F)$ is finite for every primitive permutation group $F \leq \mathrm{Sym}(\Omega)$. Using the framework of universal groups we recover the following known results in Section II.5.1.

Proposition. Let $F \leq \mathrm{Sym}(\Omega)$ and $P \leq \mathrm{Sym}(\Lambda)$ be transitive for $|\Omega|, |\Lambda| \geq 2$. Then

(i) $\dim_{\mathrm{CD}}(F) = 1$ if and only if F is regular.
(ii) $\dim_{\mathrm{CD}}(F) = 2$ if F_ω has trivial nilpotent radical for all $\omega \in \Omega$.
(iii) $\dim_{\mathrm{CD}}(F \wr P) \geq 3$.

Non-Trivial Quasi-Centers. The discreteness assertion of part (ii) in the Burger–Mozes Structure Theorem I.9 follows from the fact that a non-discrete locally quasiprimitive subgroup of $\mathrm{Aut}(T_d)$ cannot contain any non-trivial quasi-central elliptic elements, see [**BM00a**, Proposition 1.2.1]. The framework of universal groups lends itself to complete this fact to the following theorem.

Theorem II.40. Let $H \leq \mathrm{Aut}(T_d)$ be non-discrete. If H is locally

(i) transitive then $\mathrm{QZ}(H)$ contains no inversion.
(ii) semiprimitive then $\mathrm{QZ}(H)$ contains no non-trivial edge-fixating element.
(iii) quasiprimitive then $\mathrm{QZ}(H)$ contains no non-trivial elliptic element.
(iv) k-transitive ($k \in \mathbb{N}$) then $\mathrm{QZ}(H)$ contains no hyperbolic element of length k.

More importantly, the proof of the above theorem suggests to use groups of the form $\bigcap_{k \in \mathbb{N}} \mathrm{U}_k(F^{(k)})$ for appropriate local actions $F^{(k)}$ in order to *explicitly* construct non-discrete subgroups of $\mathrm{Aut}(T_d)$ whose quasi-centers contain certain types of elements. This leads to the following sharpness result.

Theorem II.41. There is a closed, non-discrete, compactly generated subgroup of $\mathrm{Aut}(T_d)$ which is locally

(i) intransitive and contains a quasi-central inversion.
(ii) transitive and contains a non-trivial quasi-central edge-fixating element.
(iii) semiprimitive and contains a non-trivial quasi-central elliptic element.
(iv) (a) intransitive and contains a quasi-central hyperbolic element of length 1.
(b) quasiprimitive and contains a quasi-central hyperbolic element of length 2.

Part (ii) of this theorem can be strengthened to the following result which shows that Burger–Mozes theory does not carry over to locally transitive groups.

Proposition II.53. There is a closed non-discrete subgroup $H \leq \mathrm{Aut}(T_d)$ which is locally transitive and has non-discrete quasi-center.

In a different direction, Banks–Elder–Willis list $\mathrm{PGL}(2,\mathbb{Q}_p) \leq \mathrm{Aut}(T_{p+1})$ as an example of a group with infinitely many distinct k-closures, see [**BEW15**]. Whereas $\mathrm{PGL}(2,\mathbb{Q}_p)$ has trivial quasi-center because it is simple, the groups constructed in the proof of the theorem above provide a wealth of examples with non-trivial quasi-center. In fact, the following proposition shows that in certain cases such examples have to be of the type constructed in the proof of the above theorem.

Proposition II.73. Let $H \leq \mathrm{Aut}(T_d)$ be closed, non-discrete, locally transitive and contain an involutive inversion. Then $H^{(k)} = \mathrm{U}_k(F^{(k)})$ and $H = \bigcap_{k\in\mathbb{N}} \mathrm{U}_k(F^{(k)})$, where $F^{(k)} \leq \mathrm{Aut}(B_{d,k})$ is action-isomorphic to the action of H on balls of radius k. If, in addition, $\mathrm{QZ}(H) \neq \{\mathrm{id}\}$ then H has infinitely many distinct k-closures.

Prime Localizations of Burger–Mozes-type Groups

The concept of prime localization of a totally disconnected locally compact group G was introduced by Reid in [**Rei13**]: Let p be prime. A *local p-Sylow subgroup* of G is a maximal pro-p subgroup of a compact open subgroup of G. The *p-localization* $G_{(p)}$ of G is defined as the commensurator $\mathrm{Comm}_G(S)$ of a local p-Sylow subgroup S of G, equipped with the unique group topology which makes the inclusion of S into $G_{(p)} = \mathrm{Comm}_G(S)$ continuous and open. Reid shows that this yields a dense, locally virtually pro-p subgroup of G whose isomorphism type and G-conjugacy class do not depend on the choice of S. We refer the reader to [**Rei13**] for general properties of prime localization and its applications.

Let $F \leq F' \leq \mathrm{Sym}(\Omega)$. We consider the Burger–Mozes group $\mathrm{U}(F)$ and two locally isomorphic versions of it: The Le Boudec group $\mathrm{G}(F,F')$ acting on T_d almost everywhere like F and elsewhere like F', and Lederle's coloured Neretin groups $\mathrm{N}(F)$ consisting of almost automorphisms of T_d associated to $\mathrm{U}(F)$. See Section I.4 for an introduction to these groups.

For a large family of the above groups, we determine local p-Sylow subgroups in terms of a p-Sylow subgroup of F. By definition of the topologies, any local p-Sylow subgroup of $\mathrm{U}(F)$ is also a local p-Sylow subgroup of $\mathrm{G}(F,F')$ and $\mathrm{N}(F)$. Let $T \subseteq T_d$ denote a finite subtree. The following proposition provides local p-Sylow subgroups of $\mathrm{U}(F)$ in the case where the operations of taking a p-Sylow subgroup and taking point stabilizers commute for F.

Proposition III.1. Let $F \leq \mathrm{Sym}(\Omega)$ and $F(p) \leq F$ a p-Sylow subgroup. Then $\mathrm{U}(F(p))_T$ is a p-Sylow subgroup of $\mathrm{U}(F)_T$ if and only if so is $F(p)_\omega \leq F_\omega$ for all $\omega \in \Omega$.

After collecting criteria and examples for the above situation we determine general subgroups of the p-localization of Burger–Mozes-type groups which we use to identify said p-localization as a group of the same type in certain cases. Recalling that $\mathrm{U}(F) = \mathrm{G}(F,F)$, the following theorem addresses both the Burger–Mozes universal group $\mathrm{U}(F)$ and the Le Boudec groups $\mathrm{G}(F,F')$. It amends [**Rei13**, Lemma 4.2]. We let $\widehat{F}$ denotes the maximal subgroup of $\mathrm{Sym}(\Omega)$ preserving the partition $F\backslash\Omega$ setwise.

Theorem III.8. Let $F \leq F' \leq \widehat{F} \leq \mathrm{Sym}(\Omega)$ and $F(p) \leq F$ a p-Sylow subgroup of F. Assume that we have $F\backslash\Omega = F(p)\backslash\Omega$ and $N_{F'_\omega}(F(p)_\omega) = F(p)_\omega$ for all $\omega \in \Omega$. Then $\mathrm{G}(F,F')_{(p)} = \mathrm{G}(F(p),F')$.

Theorem III.9. Let $F \leq \mathrm{Sym}(\Omega)$ and $F(p) \leq F$ a p-Sylow subgroup. If $F\backslash\Omega = F(p)\backslash\Omega$ and $N_{\widehat{F}_\omega}(F(p)_\omega) = F(p)_\omega$ for all $\omega \in \Omega$ then $\mathrm{N}(F)_{(p)} = \mathrm{N}(F(p))$.

Extending Willis Theory

In [**Wil94**], Willis advances the structure theory of totally disconnected locally compact groups by introducing the notions of *scale* of an automorphism of a t.d.l.c. group and *tidiness* of compact open subgroups for the same automorphism. Being the first major advance in the theory of t.d.l.c. groups for decades, it reignited the hope for a general structure theory of the latter and unexpectedly answered questions in fields as diverse as random walks and ergodic theory [**DSW06**], [**JRW96**], [**PW03**], arithmetic groups [**SW13**] and Galois theory [**CH09**].

This theory was further developed in [**Wil01**], [**Wil04**], [**BW06**], [**Wil07**] and [**BMW12**], among others. We highlight that, searching for the most general natural setting of tidiness and the scale, the definitions were generalized to endomorphisms in [**Wil15**]. For the precise definition, recall that any t.d.l.c. group admits a neighbourhood basis of compact open subgroups by work of van Dantzig [**vD31**]. For a modern treatment, see [**HR12**, (7.7)]. Given a topological group G, we let $\mathrm{End}(G)$ denote the semigroup of continuous homomorphisms from G to itself.

Definition. Let G be a t.d.l.c. group and $\alpha \in \mathrm{End}(G)$. The *scale* of α is

$$s_G(\alpha) = \min\left\{[\alpha(U) : \alpha(U) \cap U] \mid U \leq G \text{ compact open}\right\}.$$

A compact open subgroup $U \leq G$ is *minimizing for* α if $[\alpha(U) : \alpha(U) \cap U] = s(\alpha)$.

It is a cornerstone of Willis theory that U is mimimizing for α if and only if it has a certain structure, which is phrased in terms of the following subgroups of G. Put $U_0 := U$. For $n \in \mathbb{N}_0$, we define $U_{-n} = \bigcap_{k=0}^{n} \alpha^{-k}(U)$ and, inductively, $U_{n+1} := U \cap \alpha(U_n)$. Now set

$$U_+ := \bigcap_{n\in\mathbb{N}_0} U_n, \quad U_- := \bigcap_{n\in\mathbb{N}_0} U_{-n} = \bigcap_{k=0}^{\infty} \alpha^{-k}(U),$$
$$U_{++} := \bigcup_{n\in\mathbb{N}_0} \alpha^n(U_+) \quad \text{and} \quad U_{--} := \bigcup_{n\in\mathbb{N}_0} \alpha^{-n}(U_-).$$

The subgroup U is *tidy above* for α if $U = U_+U_-$, and *tidy below* for α if U_{--} is closed. It is *tidy* for α if it is both tidy above and tidy below for α. Note that this definition of being tidy below deviates from [**Wil15**, Definition 9] but turns out to be equivalent for tidy above subgroups, see [**Wil15**, Proposition 9].

Theorem ([**Wil15**, Theorem 2]). Let G be a t.d.l.c. group, $\alpha \in \mathrm{End}(G)$ and $U \leq G$ compact open. Then U is minimizing for α if and only if it is tidy for α.

Willis complements this theorem with an algorithm, a *tidying procedure*, which turns an arbitrary compact open subgroup of G into one tidy for α.

Whereas statements about automorphisms in this theory frequently utilize continuous invertibility and produce important dual statements by passing to the inverse, statements about endomorphisms often need to be formulated differently and require different techniques of proof. The present work goes through this process for two aspects of the theory.

Scale and Tidiness for Subgroups and Quotients. This section presents joint work with T. Bywaters and H. Glöckner, see [**BGT16**, Section 8].

It is natural to ask how the notions of scale and tidiness introduced above behave with respect to taking subgroups and quotients of the given group. For automorphisms, this was studied in [**Wil01**]. Our first result states that, in the case of endomorphisms, restricting to a closed invariant subgroup can only decrease the scale and thereby generalizes [**Wil01**, Proposition 4.3].

Theorem V.3. Let G be a t.d.l.c. group, $\alpha \in \mathrm{End}(G)$ and $H \leq G$ closed with $\alpha(H) \leq H$. Then $s_H(\alpha|_H) \leq s_G(\alpha)$.

Concerning quotients, we generalize [**Wil01**, Proposition 4.7]. Given $\alpha \in \mathrm{End}(G)$ and $H \trianglelefteq G$ with $\alpha(H) \leq H$, we let $\overline{\alpha} \in \mathrm{End}(G/H)$ be the endomorphism induced by α.

Theorem V.8. Let G be a t.d.l.c. group, $\alpha \in \mathrm{End}(G)$ and $H \trianglelefteq G$ closed with $\alpha(H) \leq H$. Then $s_H(\alpha|_H)s_{G/H}(\overline{\alpha})$ divides $s_G(\alpha)$.

Equality holds for example in the following case, where

$$\mathrm{par}^{-}(\alpha) = \left\{ x \in G \;\middle|\; \begin{array}{c} \exists (x_n)_{n\in\mathbb{N}_0} : \; x_0 = x, \; \forall n \in \mathbb{N} : \; \alpha(x_n) = x_{n-1} \\ \text{and } \{x_n \mid n \in \mathbb{N}_0\} \text{ is precompact} \end{array} \right\}.$$

Proposition V.10. Let G be a t.d.l.c. group, $\alpha \in \mathrm{End}(G)$ and $H \leq \mathrm{par}^{-}(\alpha)$ closed such that $\alpha(H) = H$. Further, let $N \trianglelefteq H$ be closed with $\alpha(N) = N$. Denote by $\overline{\alpha}$ the endomorphism induced by $\alpha|_H$ on H/N. Then $s_H(\alpha|_H) = s_{H/N}(\overline{\alpha})s_N(\alpha|_N)$.

Scale and Tidiness via Graphs. The results presented in this section constitute joint work with T. Bywaters, namely [**BT17**].

An important contribution to Willis theory was made by Möller in [**Möl02**], who, in the case of automorphisms, characterized the notions of scale and tidiness in terms of certain graphs associated to the data (G, α, U). This lead to geometric proofs of known results and provided a new, geometric tidying procedure, as well as a spectral radius type formula for the scale.

We adapt Möller's definitions to the case of endomorphisms. Let G be a t.d.l.c. group. Further, let α be a continuous endomorphism of G and U a compact open subgroup of G. Using a certain graph associated to the data (G, α, U) we give a geometric proof of existence of a subgroup of U which is tidy above for α ([**Wil15**, Proposition 3]), as well as the tidiness below condition ([**Wil15**, Proposition 8]). Combining both yields the following characterization of the scale and tidiness, resembling [**Möl02**, Lemma 3.1] and [**Möl02**, Theorem 3.4], see Lemma VI.1 and Theorem VI.11.

For $i \in \mathbb{N}_0$, define $v_{-i} := \alpha^{-i}(U) \in \mathcal{P}(G)$ and a rooted directed graph Γ_+ by

$$V(\Gamma_+) = \{uv_{-i} \mid u \in U_{++}, \; i \in \mathbb{N}_0\}, \quad E(\Gamma_+) = \{(uv_{-i}, uv_{-i-1}) \mid u \in U, \; i \in \mathbb{N}_0\}.$$

Theorem. Let G be a t.d.l.c. group, $\alpha \in \mathrm{End}(G)$ and $U \leq G$ compact open.

(i) If $\{v_{-i} \mid i \in \mathbb{N}_0\}$ is finite then there is a compact open subgroup U of G with $\alpha(U) \leq U$ and which is tidy for α and $s(\alpha) = 1$.

(ii) If $\{v_{-i} \mid i \in \mathbb{N}_0\}$ is infinite then U is tidy for α if and only if the graph Γ_+ is a directed tree, rooted at v_0 with contant in-valency (excluding the root) equal to 1 and constant out-valency. In this case, $s(\alpha)$ equals said out-valency.

We use this theorem to establish a new, geometric tidying procedure for the case of endomorphisms, see Theorem VI.26. It features yet another graph defined in terms of the data (G, α, U) which admits an action of U_{++}, a fundamental subgroup of G associated to α and U, see Section IV.1. Most of the work goes into showing that this graph admits a quotient with a connected component isomorphic to a regular rooted tree. The stabilizer of its root turns out to be tidy for α.

Theorem VI.26 and associated constructions result in a geometric proof of the fact [**Wil15**, Theorem 2] that tidiness is equivalent to being minimizing, see Theorem VI.34. Using the aforementioned ideas, we obtain a tree representation theorem for a certain natural subsemigroup of $\mathrm{End}(G)$ associated to α, analogous to [**BW04**, Theorem 4.1] for the case of automorphisms.

Finally, we give a simple way to construct endomorphisms of non-compact t.d.l.c groups from certain endomorphisms of compact groups.

Part 1

Groups Acting On Trees With Prescribed Local Action

CHAPTER I

Preliminaries

This chapter collects the necessary preliminaries about permutation groups, groups acting on trees, Burger–Mozes theory and Burger–Mozes type groups. We provide references at the beginning of each section.

1. Permutation Groups

Let Ω be a set. In this section, we collect definitions and results around the group of bijections of Ω, denoted $\mathrm{Sym}(\Omega)$. Refer to [**DM96**], [**Pra96**] and [**GM16**] for more details about the various classes of permutation groups to be introduced.

1.1. Definitions and Examples. Let $F \leq \mathrm{Sym}(\Omega)$. The *degree* of F is $|\Omega|$. For $\omega \in \Omega$, the stabilizer of ω in F is $F_\omega := \{\sigma \in F \mid \sigma\omega = \omega\}$. The subgroup of F generated by its point stabilizers is denoted by $F^+ := \langle\{F_\omega \mid \omega \in \Omega\}\rangle$. The permutation group F is *semiregular*, or *free*, if $F_\omega = \{\mathrm{id}\}$ for all $\omega \in \Omega$; equivalently, if F^+ is trivial. It is *transitive* if its action on Ω is transitive, and *regular* if it is both semiregular and transitive.

Let $F \leq \mathrm{Sym}(\Omega)$ be transitive. The *rank* of F is the number $\mathrm{rank}(F) := |F\backslash\Omega^2|$ of orbits of the diagonal action $\sigma \cdot (\omega, \omega') := (\sigma\omega, \sigma\omega')$ of F on Ω^2. Equivalently, $\mathrm{rank}(F) = |F_\omega\backslash\Omega|$ for all $\omega \in \Omega$. Note that the diagonal $\Delta(\Omega) = \{(\omega,\omega) \mid \omega \in \Omega\}$ is always an orbit of the diagonal action $F \curvearrowright \Omega^2$. The permutation group F is 2-transitive if $\mathrm{rank}(F) = 2$. In other words, it acts transitively on $\Omega^2\backslash\Delta(\Omega)$.

We now define several relevant classes of permutation groups in between the classes of transitive and 2-transitive permutation groups. Let $F \leq \mathrm{Sym}(\Omega)$. A partition $\mathcal{P} : \Omega = \bigsqcup_{i\in I} \Omega_i$ of Ω is *preserved* by F, or *F-invariant*, if for all $\sigma \in F$ we have $\{\sigma\Omega_i \mid i \in I\} = \{\Omega_i \mid i \in I\}$. The partition of Ω as Ω itself, as well as the partition into singletons are *trivial*. A map $a : \Omega \to F$ is *constant with respect to* $\mathcal{P}$ if $a(\omega) = a(\omega')$ whenever $\omega, \omega' \in \Omega_i$ for some $i \in I$.

The permutation group F is *primitive* if it is transitive and preserves no non-trivial partition of Ω, and *imprimitive* otherwise. Given a normal subgroup N of F, the partition of Ω into N-orbits is F-invariant. Consequently, every normal subgroup of a primitive group is transitive. A permutation group is *quasiprimitive* if it is transitive and all its non-trivial normal subgroups are transitive. Finally, a permutation group is *semiprimitive* if it is transitive and all its normal subgroups are either transitive or semiregular. The following chain of implications among properties of permutation groups is immediate from the definitions. We list examples illustrating that each implication is strict. In doing so we refer to the GAP library of small transitive groups [**GAP17**].

2-transitive	$\Rightarrow$	primitive	$\Rightarrow$	quasiprimitive	$\Rightarrow$	semiprimitive	$\Rightarrow$	transitive
		A_3, D_5		$\mathrm{Tr}(12,33) \cong A_5$		$C_4 \trianglerighteq C_2$		$D_4 \trianglerighteq C_2 \times C_2$

Note that every transitive permutation group of prime degree is necessarily primitive as all elements of an F-invariant partition have the same order, and that every simple transitive group is necessarily quasiprimitive.

1.2. Permutation Topology. Given a faithful action of a group H on a discrete set X, or, equivalently, a subgroup $H \leq \mathrm{Sym}(X)$, there is a natural topology on H, termed *permutation topology*, which makes the action map continuous. For example, we equip the automorphism group of a tree with the permutation topology for its action on the vertex set of the tree, see Section 2.2.

As a reference for the following, see e.g. [**Möl10**]. Let X be a set and consider $G := \mathrm{Sym}(X)$. The basic open sets for the permutation topology on G are

$$U_{x,y} := \{g \in G \mid \forall i \in \{1,\dots,n\}: \ g(x_i) = y_i\}$$

with $n \in \mathbb{N}$ and $x = (x_1,\dots,x_n), y = (y_1,\dots,y_n) \in X^n$.

The permutation topology turns G into a topological group. It is Hausdorff and totally disconnected as the following two lemmas show. Recall that a topological space is *zero-dimensional* if it admits a basis consisting of closed open sets.

Lemma I.1. A Hausdorff and zero-dimensional space X is totally disconnected.

Proof. Let $x \in X$. To see that no element $y \in Y$ is contained in the connected component of x it suffices to find disjoint closed open sets containing x and y respectively. Given that X is Hausdorff there are open sets separating x and y. Each contains a closed open set by definition of zero-dimensionality. □

We remark that a locally compact Hausdorff space is zero-dimensional if and only if it is totally disconnected, see [**AT08**].

Lemma I.2. Let X be a set. Then $\mathrm{Sym}(X)$ is Hausdorff and zero-dimensional.

Proof. To see that $\mathrm{Sym}(X)$ is Hausdorff, let $g, h \in \mathrm{Sym}(X)$ be distinct. Then there is $x \in X$ such that $g(x) \neq h(x)$, to the effect that $U_{x,g(x)}$ and $U_{x,h(x)}$ are disjoint open sets containing g and h respectively.

For zero-dimensionality, note that the sets $U_{x,y}$ for $x, y \in X^n$ and $n \in \mathbb{N}$ are open by definition. Now consider $g \in \mathrm{Sym}(X)\backslash U_{x,y}$. Then there is $i \in \{1,\dots,n\}$ such that $g(x_i) \neq y_i$ and $U_{x,g(x)} \subseteq \mathrm{Sym}(X)\backslash U_{x,y}$ contains g. That is, the complement of $U_{x,y}$ is open. Hence the assertion. □

We now show that the permutation topology makes the action map continuous.

Lemma I.3. Let X be a set equipped with the discrete topology. Then the action map $\Phi : \mathrm{Sym}(X) \times X \to X$ given by $(g, x) \mapsto g(x)$ is continuous.

Proof. Let $Y \subseteq X$ (be open). Then $\Phi^{-1}(Y) = \{(g, x) \in \mathrm{Sym}(X) \times X \mid g(x) \in Y\}$. Hence, if $(g, x) \in \Phi^{-1}(Y)$ then so is the open set $U_{x,g(x)} \times \{x\}$ containing (g, x). □

Finally, we characterize compact subsets of $\mathrm{Sym}(X)$.

Proposition I.4. Let X be a set and $H \leq \mathrm{Sym}(X)$. Then H is compact if and only if $H \leq \mathrm{Sym}(X)$ is closed and all its orbits are finite.

Proof. If H is compact, then H is closed in $\mathrm{Sym}(X)$ as $\mathrm{Sym}(X)$ is Hausdorff. Furthermore, $Hx = \Phi|_{H\times\{x\}}$ is compact because Φ is continuous and hence finite.

Conversely, assume that $H \leq \mathrm{Sym}(X)$ is closed and has finite orbits $(X_i)_{i\in I}$. Then $H \leq \prod_{i\in I} \mathrm{Sym}(X_i)$. Since every X_i is finite, $\mathrm{Sym}(X_i)$ is compact and hence so is $\prod_{i\in I} \mathrm{Sym}(X_i)$ by Tychonoff's theorem. Therefore, the conclusion follows if we show that the inclusion map $\prod_{i\in I} \mathrm{Sym}(X_i) \to \mathrm{Sym}(X)$ is continuous. Indeed, an intersection $U_{x,y} \cap \prod_{i\in I} \mathrm{Sym}(X_i)$ restricts only finitely many factors and hence gives rise to an open subset of the product topology. □

2. Generalities of Groups Acting On Trees

In this section, we first recall Serre's [**Ser03**] notation and definitions in the context of graphs and trees, and then collect generalities about automorphisms of trees. We conclude with an important simplicity criterion.

2.1. Definitions and Notation. A *graph* Γ is a tuple (V, E) consisting of a *vertex set* V and an *edge set* E, together with a fixed-point-free involution of E, denoted by $e \mapsto \overline{e}$, and maps $o, t : E \to V$, providing the *origin* and *terminus* of an edge, such that $o(\overline{e}) = t(e)$ and $t(\overline{e}) = o(e)$ for all $e \in E$. Given $e \in E$, the pair $\{e, \overline{e}\}$ is a *geometric edge*. For $x \in V$, we let $E(x) := o^{-1}(x) = \{e \in E \mid o(e) = x\}$ be the set of edges issuing from x. The *valency* of $x \in V$ is $|E(x)|$. A vertex of valency 1 is a *leaf*. A *morphism* between graphs $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$ is a pair (α_V, α_E) of maps $\alpha_V : V_1 \to V_2$ and $\alpha_E : E_1 \to E_2$ preserving the graph structure, i.e. $\alpha_V(o(e)) = o(\alpha_E(e))$ and $\alpha_V(t(e)) = t(\alpha_E(e))$ for all $e \in E$.

For $n \in \mathbb{N}$, let Path_n denote the graph with vertex set $\{0, \ldots, n\}$ and edge set $\{(k, k+1), \overline{(k, k+1)} \mid k \in \{0, \ldots, n-1\}\}$. A *path* of length n in a graph Γ is a morphism γ from Path_n to Γ. It can be identified with $(e_1, \ldots, e_n) \in E(\Gamma)^n$, where e_k is the image of $(k-1, k) \in E(\mathrm{Path}_n)$ for all $k \in \{1, \ldots, n\}$. In this case, γ is a path *from* $o(e_1)$ *to* $t(e_n)$.

Similarly, let $\mathrm{Path}_{\mathbb{N}_0}$ and $\mathrm{Path}_{\mathbb{Z}}$ denote the graphs with vertex sets $\mathbb{N}_0$ and $\mathbb{Z}$, and edge sets $\{(k, k+1), \overline{(k, k+1)} \mid k \in \mathbb{N}_0\}$ and $\{(k, k+1), \overline{(k, k+1)} \mid k \in \mathbb{Z}\}$ respectively. A *half-infinite* path, or *ray*, in a graph Γ is a morphism γ from $\mathrm{Path}_{\mathbb{N}_0}$ to Γ. It can be identified with $(e_k)_{k \in \mathbb{N}} \in E(\Gamma)^{\mathbb{N}}$ where $e_k = \gamma(k-1, k)$ for all $k \in \mathbb{N}$. In this case, γ *originates at*, or *issues from*, $o(e_1)$. An *infinite path*, or *line*, in a graph Γ is a morphism γ from $\mathrm{Path}_{\mathbb{Z}}$ to Γ.

A pair $(e_k, e_{k+1}) = (e_k, \overline{e_k})$ in a path is a *backtracking*. A graph is *connected* if any two of its vertices can be joined by a path. The maximal connected subgraphs of a graph are its *components*.

A *forest* is a graph in which there are no non-backtracking paths $(e_1, \ldots, e_n)$ with $o(e_1) = t(e_n)$ $(n \in \mathbb{N})$. Consequently, a morphism of forests is determined by the underlying vertex map. In particular, a path of length $n \in \mathbb{N}$ in a forest is determined by the images of the vertices of Path_n.

A *tree* is a connected forest. As a consequence of the above, the vertex set V of a tree T admits a natural metric: Given $x, y \in V$, define $d(x, y)$ as the minimal length of a path from x to y. A tree in which every vertex has valency $d \in \mathbb{N}$ is *d-regular tree*. It is unique up to isomorphism and denoted by T_d.

Let $T = (V, E)$ be a tree. For $S \subseteq V \cup E$, the *subtree spanned by S* is the unique minimal subtree of T containing S. For $x \in V$ and $n \in \mathbb{N}_0$, the subtree spanned by $\{y \in V \mid d(y, x) \leq n\}$ is the *ball* of radius n around x, denoted by $B(x, n)$. Similarly, $S(x, n) = \{y \in V \mid d(y, x) = n\}$ is the *sphere* of radius n around x. For a subtree $T' \subseteq T$, let $\pi : V \to V(T')$ denote the closest point projection, i.e. $\pi(x) = y$ whenever $d(x, y) = \min_{z \in T'}(d(x, z))$. In the case of a single edge $e = (v, w) \in E$, the *half-trees* T_v and T_w are the subtrees spanned by $\pi^{-1}(v)$ and $\pi^{-1}(w)$ respectively.

Two rays $\gamma_1, \gamma_2 : \mathrm{Path}_{\mathbb{N}} \to T$ in T are *equivalent*, $\gamma_1 \sim \gamma_2$, if there exist $N, d \in \mathbb{N}$ such that $\gamma_1(n) = \gamma_2(n + d)$ for all $n \geq N$. The *boundary*, or *set of ends*, of T is the set ∂T of equivalence classes of rays in T.

2.2. Automorphism Groups. Let $d \geq 3$ and $T_d = (V, E)$ the d-regular tree. The group of automorphism $\mathrm{Aut}(T_d)$ of T_d, i.e. the group of bijective morphisms from T_d to itself, is our foremost concern. Throughout this work, we equip $\mathrm{Aut}(T_d)$ with the permutation topology for its (faithful) action on $V(T_d)$.

2.2.1. *Notation.* Let $H \leq \mathrm{Aut}(T_d)$. Given a subtree $T \subseteq T_d$, the *pointwise stabilizer* of T in H is denoted by H_T. Similary, the *setwise stabilizer* of T in H is denoted by $H_{\{T\}}$. In the case where T is a single vertex x, the permutation group that H_x induces on $S(x,1)$ is denoted by $H_x^{(1)} \leq \mathrm{Sym}(E(x))$. We say that H is *locally* "P" if for every $x \in V$ the permutation group $H_x^{(1)}$ satisfies property "P", e.g. being tansitive, semiprimitive, quasiprimitive, primitive or 2-transitive. Furthermore, H is *locally k-transitive* ($k \in \mathbb{N}_{\geq 3}$) if H_x acts transitively on the set of non-backtracking paths of length k issuing from x. It is *locally ∞-transitive* if it is locally k-transitive for all $k \in \mathbb{N}$.

The group $\mathrm{Aut}(T_d)$ acts on ∂T_d by $g \cdot [\gamma] := [g \circ \gamma]$. Given an end $[\gamma] \in \partial T_d$, the *stabilizer* of $[\gamma]$ in H is $H_{[\gamma]} = \{h \in H \mid h \circ \gamma \sim \gamma\}$.

We let ${}^{+}H = \langle \{H_x \mid x \in V(T_d)\} \rangle$ denote the subgroup of H generated by vertex-stabilizers and $H^{+} = \langle \{H_e \mid e \in E(T_d)\} \rangle$ the subgroup generated by edge-stabilizers.

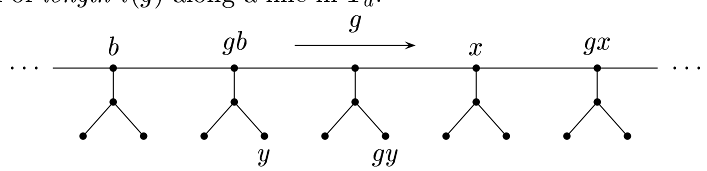
For a subtree $T \subseteq T_d$ and $k \in \mathbb{N}$, let T^k denote the subtree of T_d spanned by $\{x \in V(T_d) \mid d(x,T) \leq k\}$. We set $H^{+_k} = \langle \{H_{e^k} \mid e \in E(T_d)\} \rangle$. Then $H^{+_1} = H^{+}$ and

$$H^{+_k} \trianglelefteq H^{+} \trianglelefteq {}^{+}H \trianglelefteq H.$$

2.2.2. *Classification of Automorphisms.* On a high level, elements of $\mathrm{Aut}(T_d)$ can be distinguished into three disjoint classes which we outline below. We refer the reader to [**GGT16**, Section 2] for details. Let $g \in \mathrm{Aut}(T_d)$. Define

$$l(g) := \min_{x \in V} d(x, gx) \quad \text{and} \quad V(g) := \{x \in V \mid d(x, gx) = l(g)\}.$$

If $l(g) = 0$ then g fixes a vertex. An automorphism of this kind is *elliptic*. Suppose now that $l(g) > 0$. If $V(g)$ is infinite then g is *hyperbolic*. Geometrically, it is a translation of *length* $l(g)$ along a line in T_d.



If $V(g)$ is finite then $l(g) = 1$ and g maps an edge e to $\overline{e}$ and is termed an *inversion*.

2.3. Independence and Simplicity. This section contains an important criterion to obtain simple subgroups of $\mathrm{Aut}(T_d)$. In its base case due to Tits [**Tit70**], it applies to sufficiently large subgroups of $\mathrm{Aut}(T_d)$ satisfying a certain independence property. The generalized version we describe here is due to Banks–Elder–Willis [**BEW15**]. As an alternative reference, see [**GGT16**].

Let c denote a path in T_d (finite, half-infinite or infinite). For every $x \in C$ and $k \in \mathbb{N}_0$, the pointwise stabilizer H_{c^k} of c^k induces an action $H_{c^k}^{(x)} \leq \mathrm{Aut}(\pi^{-1}(x))$ on $\pi^{-1}(x)$. We therefore obtain an injective homomorphism

$$\varphi_c^{(k)} : H_{c^k} \to \prod_{x \in C} H_{c^k}^{(x)}.$$

The subgroup $H \leq \mathrm{Aut}(T_d)$ satisfies *Property P_k* ($k \in \mathbb{N}$) if $\varphi_c^{(k-1)}$ is an isomorphism for every path c in T_d. We remark that in case $H \leq \mathrm{Aut}(T_d)$ is closed, it suffices to check the above properties in the case where c is a single edge. Given a closed subgroup $H \leq \mathrm{Aut}(T_d)$, Property $P^{(k)}$ is satisfied by its *k-closure*

$$H^{(k)} = \{g \in \mathrm{Aut}(T_d) \mid \forall x \in V(T_d)\ \exists h \in H:\ g|_{B(x,k)} = h|_{B(x,k)}\}.$$

Theorem I.5 ([**BEW15**, Theorem 7.3]). Let $H \leq \mathrm{Aut}(T_d)$. If H neither fixes an end of T_d nor stabilizes a proper subtree of T_d setwise, then H satisfy Property P_k and G^{+_k} is either trivial or simple.

3. Burger–Mozes Theory

In [**BM00a**], Burger–Mozes develop a remarkable structure theory of a certain class of groups acting on graphs, resembling the theory of semisimple Lie groups. In order to give the precise structure theorem we introduce further notation.

The fundamental definitions are meaningful in the setting of totally disconnected locally compact groups: Let H be a t.d.l.c. group. We define $H^{(\infty)}$ to be the intersection of all closed normal cocompact subgroups of H, and $\mathrm{QZ}(H)$ to be the subgroup of elements whose centralizer in H is open in H. As a consequence, both $H^{(\infty)}$ and $\mathrm{QZ}(H)$ are topologically characteristic subgroups of H, i.e. they are preserved by continuous automorphisms of H. Alternatively, $H^{(\infty)}$ can be described as the intersection of all open subgroups of finite index.

The next example shows that $H^{(\infty)}$ and $\mathrm{QZ}(H)$ play roles analogous to that of the connected component of the identity and the kernel of the adjoint representation in Lie theory, cf. [**BM00a**, Example 1.1.1.].

Example I.6. Let H be a semisimple p-adic matrix group. Then $H^{(\infty)}$ coincides with the subgroup generated by unipotent elements and $\mathrm{QZ}(H)$ is given by the kernel of the adjoint representation.

The definitions also readily imply that $H^{(\infty)}$ is closed. The next example shows that $\mathrm{QZ}(H)$ need not be so.

Example I.7. Let $H:=\prod_{\mathbb{N}} F$ where F is a finite centerless group. Then $H^{(\infty)}=\{\mathrm{id}\}$ as $\{\mathrm{id}\}$ is cocompact in the compact group H. Furthermore, $\mathrm{QZ}(H)$ is the direct sum $\bigoplus_{\mathbb{N}} F$. In particular, $\mathrm{QZ}(H)$ is dense in H.

Our third example relies on Section II.4.1.

Example I.8. Let $F\leq \mathrm{Sym}(\Omega)$ and $H:=\mathrm{U}(F)\leq \mathrm{Aut}(T_d)$. If F is transitive and generated by point stabilizers then $\mathrm{U}(F)^+$ has index 2 in $\mathrm{U}(F)$ and is simple. Thus $H^{(\infty)}=\mathrm{U}(F)^+$. Furthermore, $\mathrm{QZ}(\mathrm{U}(F)^+)=\{\mathrm{id}\}$.

Recall that any discrete normal subgroup of a topological group is central. From the definitions we can therefore deduce that every cocompact normal subgroup of H contains $H^{(\infty)}$ and that $\mathrm{QZ}(H)$ contains all discrete normal subgroups of H. The subquotient $H^{(\infty)}/\mathrm{QZ}(H^{(\infty)})$ of H therefore has a chance to be topologically simple. Whereas Examples I.6 and I.7 show that nothing much can be said about the size of $H^{(\infty)}$ and $\mathrm{QZ}(H)$ in general, Burger–Mozes show that good control can be obtained in the case of closed non-discrete subgroups of $\mathrm{Aut}(\Gamma)$, where Γ is a connected graph, satisfying certain local transitivity properties. The following result summarizes their structure theory in the case of regular trees to which the present work contributes. It is a combination of Proposition 1.2.1, Corollary 1.5.1, Theorem 1.7.1 and Corollary 1.7.2 in [**BM00a**].

Theorem I.9. Let $H\leq \mathrm{Aut}(T_d)$ be closed, non-discrete and locally quasiprimitive.

(i) $H^{(\infty)}$ is minimal closed normal cocompact in H.
(ii) $\mathrm{QZ}(H)$ is maximal discrete normal, and non-cocompact in H.
(iii) $H^{(\infty)}/\mathrm{QZ}(H^{(\infty)})=H^{(\infty)}/(\mathrm{QZ}(H)\cap H^{(\infty)})$ admits minimal, non-trivial closed normal subgroups; finite in number, H-conjugate and topologically simple.

If, in addition, H is locally primitive then

(iv) $H^{(\infty)}/\mathrm{QZ}(H^{(\infty)})$ is a direct product of topologically simple groups.

4. Burger–Mozes-type Groups

In this section we introduce several classes of groups acting on (regular) trees. First, we concern ourselves with Burger–Mozes universal groups, introduced by Burger–Mozes in [**BM00a**, Section 3.2] as a complement to their structure theory. Chapter II develops a versatile generalization of these groups.

Secondly, we recall a locally isomorphic generalization of these groups due to Le Boudec [**Bou16**]. Among his examples are t.d.l.c. groups which are virtually simple and contain no lattices, i.e. discrete cofinite subgroups.

Finally, we introduce a recently developed generalization of Neretin's group [**Ner03**] due to Lederle [**Led17**]. She shows that most of these groups do not contain lattices, generalizing the same result for Neretin's group [**BCGM12**].

In Chapter III, we compute the p-localizations of a large subclass of the three types of Burger–Mozes groups and primes p.

Let Ω be a set of cardinality $d \geq 3$ and let $T_d = (V, E)$ denote the d-regular tree. A labelling l of T_d is a map $l : E \to \Omega$ such that for every $x \in V$ the map $l_x := l|_{E(x)} : E(x) \to \Omega,\ y \mapsto l(y)$ is a bijection and for all $e \in E$ we have $l(e) = l(\overline{e})$.

4.1. Burger–Mozes Groups. The original introduction of Burger–Mozes universal groups in [**BM00a**, Section 3.2] has been expanded in the introductory article [**GGT16**] which we follow closely. Most results are generalized in Chapter II.

Consider the labelled tree T_d introduced above. The *local actions* of automorphisms are captured by the map

$$\sigma : \mathrm{Aut}(T_d) \times X \to \mathrm{Sym}(\Omega),\ (g, x) \mapsto \sigma(g, x) := l_{gx} \circ g \circ l_x^{-1}.$$

Given any permutation group $F \leq \mathrm{Sym}(\Omega)$, we can define a subgroup of $\mathrm{Aut}(T_d)$ all of whose local actions are in F as follows.

Definition I.10. Let $F \leq \mathrm{Sym}(\Omega)$ and l a labelling of T_d. Define

$$\mathrm{U}^{(l)}(F) := \{g \in \mathrm{Aut}(T_d) \mid \forall x \in V :\ \sigma(g, x) \in F\}.$$

The map σ satisfies a *cocycle identity*: For all $g, h \in \mathrm{Aut}(T_d)$ and $x \in V$ we have $\sigma(gh, x) = \sigma(g, hx)\sigma(h, x)$. As a consequence, $\mathrm{U}^{(l)}(F)$ is a subgroup of $\mathrm{Aut}(T_d)$.

Passing to a different labelling amounts to passing to a conjugate of $\mathrm{U}^{(l)}(F)$ inside $\mathrm{Aut}(T_d)$. We therefore omit explicit reference to the labelling from here on.

Remark I.11. Let $F \leq \mathrm{Sym}(\Omega)$. Elements of $\mathrm{U}(F)$ are readily constructed: Given $v, w \in V(T_d)$ and $\tau \in F$, define $g : B(v, 1) \to B(w, 1)$ by setting $g(v) = w$ and $\sigma(g, v) = \tau$. Given a collection of permutations $(\tau_\omega)_{\omega \in \Omega}$ such that $\tau(\omega) = \tau_\omega(\omega)$ for all $\omega \in \Omega$ there is a unique extension of g to $B(v, 2)$ such that $\sigma(g, v_\omega) = \tau_\omega$ where $v_\omega \in S(v, 1)$ is the unique vertex with $l(v, v_\omega) = \omega$. Then proceed iteratively.

The following proposition collects several elementary properties of Burger–Mozes groups. We refer the reader to [**GGT16**, Section 4] for proofs. Alternatively, a generalized version of this result is contained in Section II.1.

Proposition I.12. Let $F \leq \mathrm{Sym}(\Omega)$. Then $\mathrm{U}(F)$ is

(i) closed in $\mathrm{Aut}(T_d)$,
(ii) vertex-transitive,
(iii) compactly generated,
(iv) locally permutation isomorphic to F,
(v) edge-transitive if and only if F is transitive, and
(vi) discrete in $\mathrm{Aut}(T_d)$ if and only if F is semiregular.

Part (iii) of Proposition I.12 relies on the following result which we include for future reference.

Lemma I.13. The group $\mathrm{U}_1(\{\mathrm{id}\})$ is finitely generated.

Proof. Fix $x \in V$. For every $\omega \in \Omega$, let $\iota_\omega \in \mathrm{U}_1(\{\mathrm{id}\})$ denote the unique label-respecting inversion of the edge $e_\omega \in E$ with origin x and label ω. Then $\mathrm{U}_1(\{\mathrm{id}\})$ is generated by $\{\iota_\omega \mid \omega \in \Omega\}$: Every element of $\mathrm{U}_1(\{\mathrm{id}\})$ is determined by its image on v, so the assertion follows from vertex-transitivity of $\langle\{\iota_\omega \mid \omega \in \Omega\}\rangle$: Let $y \in V\backslash\{x\}$ and let $(\omega_1, \ldots, \omega_n)$ be the labels appearing in the geodesic from x to y. Then $\iota_{\omega_1} \circ \cdots \circ \iota_{\omega_n} \in \mathrm{U}_1(\{\mathrm{id}\})$ maps x to y. □

The name *universal group* is due to the following maximality statement whose proof should be compared with the proof of Theorem II.23.

Proposition I.14. Let $H \leq \mathrm{Aut}(T_d)$ be locally transitive and vertex-transitive. Then there is a labelling l of T_d such that $H \leq \mathrm{U}^{(l)}(F)$ where $F \leq \mathrm{Sym}(\Omega)$ is action isomorphic to the action of H on balls of radius 1.

Proof. Fix $b \in V$ and a bijection $l_b : E(b) \to \Omega$. Then the local action of H at b is given by $F := l_b \circ H_b|_{E(b)} \circ l_b^{-1}$. We now inductively define a legal labelling $l : E \to \Omega$ such that $H \leq \mathrm{U}^{(l)}(F)$. Set $l|_{E(b)} := l_b$ and suppose inductively that l is defined on $E(b,n) := \bigcup_{x \in B(b,n-1)} E(x)$. To extend l to $E(b, n+1)$, let $x \in S(b,n)$ and let $e_x \in E$ be the unique edge with $o(e_x) = x$ and $d(b, t(e_x)) + 1 = d(b,x)$. Since H is vertex-transitive and locally transitive, there is an element $\iota_{e_x} \in H$ which inverts the edge e_x. Using ι_{e_x} we may extend l to $E(x)$ by setting $l|_{E(x)} := l \circ \iota_{e_x}$.

To check the inclusion $H \leq \mathrm{U}^{(l)}(F)$, let $x \in V$ and $h \in H$. If $(b, b_1, \ldots, b_n, x)$ and $(b, b_1', \ldots, b_m', h(x))$ denote the unique reduced paths from b to x and $h(x)$, then

$$s := \iota_{e_{b_1'}} \cdots \iota_{e_{b_m'}} \iota_{e_{h(x)}} \circ h \circ \iota_x \iota_{e_{b_n}} \cdots \iota_{e_{b_2}} \iota_{e_{b_1}} \in H_b$$

and we have $\sigma(h,x) = \sigma(s,b) \in F$ by the cocycle identity satisfied by the map σ. □

4.2. Le Boudec Groups.

In [**Bou16**], Le Boudec introduces groups acting on T_d locally like a given permutation group $F \leq \mathrm{Sym}(\Omega)$ *almost* everywhere. The precise definition reads as follows.

Definition I.15. Let $F \leq \mathrm{Sym}(\Omega)$. Define

$$\mathrm{G}(F) := \{g \in \mathrm{Aut}(T_d) \mid \sigma(g,x) \in F \text{ for almost all } x \in V\}.$$

Notice that $\mathrm{U}(F)$ is a subgroup of $\mathrm{G}(F)$. We equip $\mathrm{G}(F)$ with the unique group topology making the inclusion $\mathrm{U}(F) \rightarrowtail \mathrm{G}(F)$ continous and open. It exists essentially due to the fact that $\mathrm{G}(F)$ commensurates a compact open subgroup of $\mathrm{U}(F)$, see [**Bou16**, Lemma 3.2]. We state explicitly that this topology differs from the subspace topology of $\mathrm{Aut}(T_d)$, see e.g. Proposition I.18 below. However, it entails that $\mathrm{G}(F)$ is locally isomorphic to $\mathrm{U}(F)$.

Given $g \in \mathrm{G}(F)$, a vertex $v \in V$ with $\sigma(g,v) \notin F$ is a *singularity*. The local action at singularities is restricted as follows.

Lemma I.16 ([**Bou16**, Lemma 3.3]). Let $F \leq \mathrm{Sym}(\Omega)$ and $g \in \mathrm{G}(F)$ with a singularity $v \in V$. Then $\sigma(g,v)$ preserves the partition $F\backslash\Omega$ of Ω into F-orbits setwise.

For $F \leq \mathrm{Sym}(\Omega)$, the maximal subgroup of $\mathrm{Sym}(\Omega)$ which preserves the partition $F\backslash\Omega = \bigsqcup_{i \in I} \Omega_i$ setwise is the direct product $\widehat{F} := \prod_{i \in I} \mathrm{Sym}(\Omega_i)$. Combined with Lemma I.16, this suggests the following extension of Definition I.15.

Definition I.17. Let $F \leq F' \leq \widehat{F} \leq \mathrm{Sym}(\Omega)$. Set $\mathrm{G}(F, F') := \mathrm{G}(F) \cap \mathrm{U}(F')$.

We remark that $\mathrm{G}(F,F)=\mathrm{U}(F)$ and $\mathrm{G}(F,\widehat{F})=\mathrm{G}(F)$. In this sense, the groups $\mathrm{G}(F,F')$ interpolate between $\mathrm{U}(F)$ and $\mathrm{G}(F)$. Le Boudec shows that for certain choices of F and F', the groups $\mathrm{G}(F,F')$ are virtually simple and contain no lattices, see [**Bou16**, Introduction]. For future reference we include the following fact.

Proposition I.18. Let $F \lneq F' \leq \widehat{F} \leq \mathrm{Sym}(\Omega)$ and $b \in V(T_d)$. Then $\mathrm{G}(F,F')_b$ is non-compact and residually discrete.

Proof. The vertex stabilizer $\mathrm{G}(F,F')_b$ can be written as the (strictly) increasing union $\mathrm{G}(F,F')_b = \bigcup_{n\in\mathbb{N}} K_n$ of the open sets K_n, consisting of the elements of $\mathrm{G}(F,F')_b$ whose singularities are contained in $B(b,n)$. Hence it is non-compact.

As to residual discreteness, an identity neighbourhood basis of $\mathrm{G}(F,F')_b$ consisting of open normal subgroups is given by the collection $\left(\mathrm{G}(F,F')_{B(b,n)}\right)_{n\in\mathbb{N}}$. □

4.3. Lederle Groups. As before, we consider the d-regular tree $T_d=(V,E)$ with a labelling and a base vertex $b \in V$. Further, let $F \leq \mathrm{Sym}(\Omega)$. In [**Led17**], Lederle introduces a locally isomorphic version of $\mathrm{U}(F)$ resembling Neretin's group [**Ner03**] and thereby generalizes Neretin's construction.

Towards a precise definition, we recall the following from [**Led17**, Section 3.2]: A finite subtree $T \subseteq T_d$ is *complete* if it contains b and all its non-leaf vertices have valency d. We denote the set of leaves of T by $L(T) \subseteq V(T_d)$. Given a leaf $v \in L(T)$, let T_v denote the subtree of T_d spanned by v and those vertices outside T whose closest vertex in T is v. Then define $T_d\backslash T := \bigsqcup_{v\in L(T)} T_v$, a forest of $|L(T)|$ trees.

Let $H \leq \mathrm{Aut}(T_d)$. Given finite complete subtrees $T,T' \subseteq T_d$ with $|L(T)| = |L(T')|$, a forest isomorphism $\varphi : T_d\backslash T \to T_d\backslash T'$ such that for every $v \in L(T)$ there is $h_v \in H$ with $\varphi|_{T_v} = h_v|_{T_v}$ is an *H-honest almost automorphism of T_d*. Two H-honest almost automorphisms of T_d given by $\varphi : T_d\backslash T_1 \to T_d\backslash T_1'$ and $\psi : T_d\backslash T_2 \to T_d\backslash T_2'$ are *equivalent* if there exists a finite complete subtree $T \supseteq T_1 \cup T_2$ with $\varphi|_{T_d\backslash T} = \psi|_{T_d\backslash T}$. Notice that for any finite complete subtree $T \supseteq T_1$ there is a unique finite complete subtree $T' \supseteq T_1'$ and representative $\varphi' : T_d\backslash T \to T_d\backslash T'$ of φ; analogously for T_1'. Hence we may pick a finite complete subtree $T \supseteq T_1' \cup T_2$ and representatives of φ and ψ with codomain and domain equal to $T_d\backslash T$ respectively, thus allowing for a composition of equivalence classes of H-honest almost automorphisms. Lederle's coloured Neretin groups (original notation $\mathcal{F}(\mathrm{U}(F))$) can now be defined as follows.

Definition I.19. Let $F \leq \mathrm{Sym}(\Omega)$. Set

$$\mathrm{N}(F) := \{[\varphi] \mid \varphi \text{ is a } \mathrm{U}(F)\text{-honest almost autormorphism of } T_d\}.$$

Observe that $\mathrm{N}(F) \cap \mathrm{Aut}(T_d) = \mathrm{G}(F)$. As before, there exists a unique group topology on $\mathrm{N}(F)$ such that the inclusion $\mathrm{U}(F) \rightarrowtail \mathrm{N}(F)$ is open and continuous. This is essentially due to the fact that $\mathrm{N}(F)$ commensurates a compact open subgroup of $\mathrm{U}(F)$, see [**Led17**, Proposition 2.24].

We mention that most Lederle groups contain no lattices, see [**Led17**, Theorem 1.2]. This generalizes the same assertion for Neretin's group obtained in [**BCGM12**]. In this context, Lederle also produces new examples of locally compact, compactly generated, simple groups without lattices.

Overall, we have the following continuous and open injections, capturing that all involved groups have isomorphic open subgroups:

$$\mathrm{U}(F) \rightarrowtail \mathrm{G}(F) \rightarrowtail \mathrm{N}(F).$$

CHAPTER II

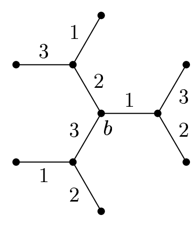
Universal Groups

We present a generalization of Burger–Mozes universal groups that arises via prescribing the local action on balls of a given radius $k \in \mathbb{N}$ around vertices. The Burger–Mozes construction corresponds to the case $k = 1$. Whereas many properties of their construction carry over to this new setting in a straightforward fashion, others require a more careful analysis. We proceed by exhibiting examples and (non)-rigidity phenomena of our construction. The universality statement given in Theorem II.23 provides both a characterization of the generalized universal groups and the k-closures of groups that act locally transitively with an involutive inversion on the d-regular tree. The discrete case discussed in Section 5, utilizes Theorem II.23 to suggest a new approach to the Weiss conjecture stating that for a given locally finite tree T there are only finitely many conjugacy classes of discrete, vertex-transitive and locally primitive subgroups of $\mathrm{Aut}(T)$. It also shows that the additional assumption in Theorem II.23 compared to [**BM00a**, Proposition 3.2.2] is indeed necessary. Finally, Section 7 applies the framework of universal groups to groups acting with non-trivial quasi-center. We characterize the type of elements that the quasi-center of a non-discrete subgroup of $\mathrm{Aut}(T_d)$ can have in terms of its local action and explicitly construct groups with non-trivial quasi-centers to show that said characterization is sharp.

1. Definition and Basic Properties

1.1. Definition. Let Ω be a set of cardinality $d \geq 3$ and let $T_d = (V, E)$ denote the d-regular tree. Recall that a labelling l of T_d is a map $l : E \to \Omega$ such that for every $x \in V$ the map $l_x : E(x) \to \Omega,\ y \mapsto l(y)$ is a bijection and for all $e \in E$ we have $l(e) = l(\overline{e})$.

Given $k \in \mathbb{N}$, fix a labelled tree $B_{d,k}$ with center b which is isomorphic to a ball of radius k in T_d and whose labelling arises from a labelling of T_d via such an isomorphism. For example, $B_{3,2}$ may be as on the side. Then for every $x \in V$, there is a unique label-respecting isomorphism



$$l_x^k : B(x,k) \to B_{d,k}.$$

These maps allow us to capture the *k-local actions* of automorphisms via the map

$$\sigma_k : \mathrm{Aut}(T_d) \times X \to \mathrm{Aut}(B_{d,k}),\ (g,x) \mapsto \sigma_k(g,x) := l_{gx}^k \circ g \circ (l_x^k)^{-1}.$$

Definition II.1. Let $F \leq \mathrm{Aut}(B_{d,k})$ and l a labelling of T_d. Define

$$\mathrm{U}_k^{(l)}(F) := \{g \in \mathrm{Aut}(T_d) \mid \ \forall x \in V : \ \sigma_k(g,x) \in F\}.$$

The following lemma states that the maps σ_k satisfy a cocycle identity which immediately implies that $\mathrm{U}_k^{(l)}(F)$ is a subgroup of $\mathrm{Aut}(T_d)$ for every $F \leq \mathrm{Aut}(B_{d,k})$.

Lemma II.2. Let $x \in V$ and $g, h \in \mathrm{Aut}(T_d)$. Then $\sigma_k(gh, x) = \sigma_k(g, hx)\sigma_k(h, x)$.

Proof. We readily compute

$$\begin{aligned}\sigma_k(gh,x) =& l^k_{(gh)x} \circ gh \circ (l^k_x)^{-1} = l^k_{(gh)x} \circ g \circ h \circ (l^k_x)^{-1} = \\ &= l^k_{(gh)x} \circ g \circ (l^k_{hx})^{-1} \circ l^k_{hx} \circ h \circ (l^k_x)^{-1} = \sigma_k(g,hx)\sigma_k(h,x).\end{aligned}$$

for all $x \in V$ and all $g, h \in \mathrm{Aut}(T_d)$. □

1.2. Basic Properties. Note that the group $\mathrm{U}_1^{(l)}(F)$ of Definition II.1 for $F \leq \mathrm{Aut}(B_{d,1}) \cong \mathrm{Sym}(\Omega)$ coincides with the Burger–Mozes universal group $\mathrm{U}_{(l)}(F)$ introduced in [**BM00a**, Sec. 3.2] and Section 4.1. Several basic properties of the latter carry over to our generalized situation. First of all, passing between labellings of T_d amounts to conjugating in $\mathrm{Aut}(T_d)$.

Lemma II.3. For every quadruple (l, l', x, x') of labellings l, l' of T_d and vertices $x, x' \in V$, there is a unique automorphism $g \in \mathrm{Aut}(T_d)$ with $gx = x'$ and $l' = l \circ g$.

Proof. Set $gx := x'$. Now assume inductively that g is uniquely determined on $B(x,n)$ $(n \in \mathbb{N}_0)$ and let $v \in S(x,n)$. Then g is also uniquely determined on $E(v)$ by the requirement $l' = l \circ g$, namely $g|_{E(v)} := l|_{E(gv)}^{-1} \circ l'|_{E(v)}$. □

Corollary II.4. Let $F \leq \mathrm{Aut}(B_{d,k})$. Further, let l and l' be labellings of T_d. Then the groups $\mathrm{U}_k^{(l)}(F)$ and $\mathrm{U}_k^{(l')}(F)$ are conjugate in $\mathrm{Aut}(T_d)$.

Proof. Choose $x \in V$. Let $\tau \in \mathrm{Aut}(T_d)$ denote the automorphism of T_d associated to (l, l', x, x) by Lemma II.3, then $\mathrm{U}_k^{(l)}(F) = \tau \mathrm{U}_k^{(l')}(F)\tau^{-1}$. □

In the following, we shall therefore omit the reference to an explicit labelling.

Proposition II.5. Let $F \leq \mathrm{Aut}(B_{d,k})$. Then $\mathrm{U}_k(F)$ is a

(i) closed subgroup of $\mathrm{Aut}(B_{d,k})$, and
(ii) vertex-transitive.

Proof. As to (i), note that if $g \notin \mathrm{U}_k(F)$ then $\sigma_k(g,x) \notin F$ for some $x \in V$. In this case, the open neighbourhood $\{h \in \mathrm{Aut}(T_d) \mid h|_{B(x,k)} = g|_{B(x,k)}\}$ of g in $\mathrm{Aut}(T_d)$ is also contained in the complement of $\mathrm{U}_k(F)$.

For (ii), let $x, x' \in V$ and let $g \in \mathrm{Aut}(T_d)$ be the automorphism of T_d associated to (l, l, x, x') by Lemma II.3. Then $g \in \mathrm{U}_k(F)$ as $\sigma_k(g,v) = \mathrm{id} \in F$ for all $v \in V$. □

The following result is now a consequence of Proposition II.5 and Lemma I.13.

Corollary II.6. Let $F \leq \mathrm{Aut}(B_{d,k})$. Then $\mathrm{U}_k(F)$ is a compactly generated, totally disconnected, locally compact Hausdorff group.

Proof. The group $\mathrm{U}_k(F)$ is totally disconnected locally compact Hausdorff as a closed subgroup of $\mathrm{Aut}(T_d)$. To show compact generation, fix $x \in V$. Then $\mathrm{U}_k(F)$ is generated by the join of the compact set $\mathrm{U}_k(F)_x$ and the finite generating set of $\mathrm{U}_1(\{\mathrm{id}\}) = \mathrm{U}_k(\{\mathrm{id}\}) \leq \mathrm{U}_k(F)$ given in the proof of Lemma I.13: Indeed, for $\alpha \in \mathrm{U}_k(F)$ pick β in the finitely generated, vertex-transitive subgroup $\mathrm{U}_1(\{\mathrm{id}\})$ of $\mathrm{U}_k(F)$ such that $\beta(\alpha x) = x$. Then $\beta\alpha \in \mathrm{U}_k(F)_x$ and the assertion follows. □

Proposition II.7. Let $F \leq \mathrm{Aut}(B_{d,k})$. Then $\mathrm{U}_k(F)$ satisfies Property P_k.

Proof. Let $e \in E$. Clearly, $\mathrm{U}_k(F)_{e^k} \supseteq \mathrm{U}_k(F)_{e^k,T_y} \cdot \mathrm{U}_k(F)_{e^k,T_x}$. Conversely, consider $g \in \mathrm{U}_k(F)_{e^k}$ and define $g_y \in \mathrm{Aut}(T_d)$ and $g_x \in \mathrm{Aut}(T_d)$ by

$$\sigma_k(g_y, v) = \begin{cases} \sigma_k(g,v) & v \in V(T_x) \\ \mathrm{id} & v \in V(T_y) \end{cases} \quad \text{and} \quad \sigma_k(g_x, v) = \begin{cases} \mathrm{id} & v \in V(T_x) \\ \sigma_k(g,v) & v \in V(T_y) \end{cases}$$

respectively. Then $g_y \in \mathrm{U}_k(F)_{e^k,T_y}$, $g_x \in \mathrm{U}_k(F)_{e^k,T_x}$ and $g = g_y \circ g_x$. □

2. Compatibility and Discreteness

We now generalize parts (iv) and (vi) of Proposition I.12 to the generalized setting. This results in a compatibility condition (C) and a discreteness condition (D) on subgroups $F \leq \mathrm{Aut}(\mathrm{B_{d,k}})$ that hold if and only if the associated universal group locally acts like F and is discrete respectively.

2.1. Compatibility. First, we ask whether $\mathrm{U}_k(F)$ locally acts like F, that is whether the actions $\mathrm{U}_k(F)_x \curvearrowright B(x,k)$ and $F \curvearrowright B_{d,k}$ are isomorphic for every $x \in V$. Whereas this always holds for $k=1$ by Lemma II.3 it need not be true for $k \geq 2$, see Example II.9, the issue being (non)-compatibility among elements of F. The condition developed in this section allows for computations. A more practical version from a theoretical viewpoint follows in Section 3.

We introduce the following notation for vertices in the labelled tree (T_d, l): Given $x \in V$ and $w = (\omega_1, \dots, \omega_n) \in \Omega^n$ $(n \in \mathbb{N}_0)$, set $x_w := \gamma_{x,w}(n)$ where

$$\gamma_{x,w} : \mathrm{Path}_n^{(w)} := \underset{0}{\bullet}\overset{w_1}{\text{———}}\underset{1}{\bullet}\overset{w_2}{\text{———}}\underset{2}{\bullet}\text{——}\ \cdots\ \text{—}\!\!\rightarrow\!\underset{n}{\bullet} \to T_d$$

is the unique label-respecting morphism sending 0 to $x \in V$. If w is the empty word, set $x_w := x$. Whenever admissible, we also adopt this notation in the case of $B_{d,k}$ and its labelling. In particular, $S(x,n)$ is in natural bijection with the set $\Omega^{(n)} := \{(\omega_1, \dots, \omega_n) \in \Omega^n \mid \forall k \in \{1, \dots, n-1\} : \ \omega_k \neq \omega_{k+1}\}$.

Now, let $x \in V$ and suppose that $\alpha \in \mathrm{U}_k(F)_x$ realizes $a \in F$ at x, that is

$$\alpha|_{B(x,k)} = (l_x^k)^{-1} \circ a \circ l_x^k.$$

Then given the condition that $\sigma_k(\alpha, x_\omega)$ be in F for all $\omega \in \Omega$, we obtain the following necessary condition on F for $\mathrm{U}_k(F)$ to act like F at $x \in V$:

$$\forall a \in F \ \forall \omega \in \Omega : \ \exists a_\omega \in F : \ (l_x^k)^{-1} \circ a \circ l_x^k|_{S_\omega} = (l_{\alpha x_\omega}^k)^{-1} \circ a_\omega \circ l_{x_\omega}^k|_{S_\omega}$$

where $S_\omega := B(x,k) \cap B(x_\omega, k) \subseteq T_d$. Set $T_\omega := l_x^k(S_\omega) \subseteq B_{d,k}$. Then the above condition can be rewritten as

$$\forall a \in F \ \forall \omega \in \Omega : \ \exists a_\omega \in F : \ a_\omega|_{T_\omega} = l_{\alpha x_\omega}^k \circ (l_x^k)^{-1} \circ a \circ l_x^k \circ (l_{x_\omega}^k)^{-1}|_{T_\omega}.$$

Now observe the following: First of all, αx_ω depends only on a. Secondly, the subtree T_ω of $B_{d,k}$ does not depend on x, and thirdly, $\iota_\omega := l_x^k|^{T_\omega} \circ (l_x^k)^{-1}|_{T_\omega}$ is the unique non-trivial, involutive and label-respecting automorphism of T_ω, given by

$$\iota_\omega := l_x^k\big|^{T_\omega} \circ (l_{x_\omega}^k)^{-1}\big|_{T_\omega} : T_\omega \to S_\omega \to T_\omega, \ b_w \mapsto x_{\omega w} \mapsto b_{\omega w}$$

for admissible words w. Hence the above condition may be rewritten as

$$\text{(C)} \qquad \forall a \in F \ \forall \omega \in \Omega : \ \exists a_\omega \in F : \ a_\omega|_{T_\omega} = \iota_{a(\omega)} \circ a \circ \iota_\omega.$$

In this situation we shall say that a_ω *is compatible with* a *in direction* ω.

Proposition II.8. Let $F \leq \mathrm{Aut}(B_{d,k})$. Then $\mathrm{U}_k(F)$ locally acts like F if and only if F satisfies the compatibility condition (C).

Proof. By the above, condition (C) is necessary. To show that it is also sufficient, let $v \in V$ and $a \in F$. We aim to define an automorphism $\alpha \in \mathrm{U}_k(F)$ which realizes a at v. This forces us to set

$$\alpha|_{B(v,k)} := (l_v^k)^{-1} \circ a \circ l_v^k.$$

Now, assume inductively that α is defined consistently on $B(v,n)$ in the sense that $\sigma_k(\alpha, x) \in F$ for all $x \in B(v,n)$ with $B(x,k) \subseteq B(v,n)$. In order to extend α to $B(v, n+1)$, let $x \in S(v, n-k+1)$ and let $\omega \in \Omega$ be the unique label such that $x_\omega \in S(v, n-k)$. Applying condition (C) to the pair $(c := \sigma_k(\alpha, x_\omega), \omega)$ provides an element $c_\omega \in F$ such that

$$(l^k_{\alpha x_\omega})^{-1} \circ c \circ l^k_{x_\omega}\big|_{S_\omega} = (l^k_{\alpha x})^{-1} \circ c_\omega \circ l^k_x\big|_{S_\omega}$$

where $S_\omega := B(x,k) \cap B(x_\omega, k)$ and we have realized

$$\iota_\omega \text{ as } l^k_{x_\omega}\big|^{T_\omega} \circ (l^k_x)^{-1}\big|_{T_\omega} \quad \text{and} \quad \iota_{c(\omega)} \text{ as } l^k_{\alpha x}\big|^{T_{c(\omega)}} \circ (l^k_{\alpha x_i})^{-1}\big|_{T_{c(\omega)}}.$$

Now extend α consistently to $B(v, n+1)$ by setting $\alpha|_{B(x,k)} := (l^k_{\alpha x})^{-1} \circ c_\omega \circ l^k_x$. □

Example II.9. Let $\Omega := \{1,2,3\}$ and $a \in \mathrm{Aut}(B_{3,2})$ the element which swaps the leaves x_{12} and x_{13} of $B_{3,2}$. Then $F := \langle a \rangle = \{\mathrm{id}, a\}$ does not contain an element compatible with a in direction $1 \in \Omega$ and hence does not satisfy condition (C).

To make the verification of condition (C) viable, we record the following reduction to generating sets: For $a, b \in F \leq \mathrm{Aut}(B_{d,k})$ and $c := ab \in F$ we have

$$\begin{aligned} c_\omega|_{T_\omega} = \iota_{c(\omega)} \circ a \circ b \circ \iota_\omega &= \big(\iota_{c(\omega)} \circ a \circ \iota_{b(\omega)}\big) \circ \big(\iota_{b(\omega)} \circ b \circ \iota_\omega\big) \\ &= \big(\iota_{a(b(\omega))} \circ a \circ \sigma_{b(\omega)}\big) \circ \big(\iota_{b(\omega)} \circ b \circ \iota_\omega\big) \end{aligned}$$

Thus if $C_F(a,\omega)$ denotes the set of elements in F which are compatible with $a \in F$ in direction $\omega \in \Omega$ then $C_F(ab,\omega) \supseteq C_F(b, a\omega) C_F(a,\omega)$. It therefore suffices to check condition (C) on a generating set of F.

Given $S \subseteq \Omega$, we also define the *compatibility set* $C_F(a,S) := \bigcap_{\omega \in S} C_F(a,\omega)$, the set of elements in F which are compatible with $a \in F$ in all directions from S.

As a consequence, we obtain the following description of the local action of $\mathrm{U}_k(F)$ if F does not satisfy condition (C).

Corollary II.10. Let $F \leq \mathrm{Aut}(B_{d,k})$. Then F has a unique maximal subgroup $C(F)$ which satisfies condition (C). Furthermore, $\mathrm{U}_k(F) = \mathrm{U}_k(C(F))$.

Proof. By the above, $C(F) := \langle H \leq F \mid H \text{ satisfies (C)} \rangle \leq F$ satisfies condition (C). Clearly, it is the unique maximal such subgroup of F.

By definition, $\mathrm{U}_k(C(F)) \leq \mathrm{U}_k(F)$. Conversely, suppose $g \in \mathrm{U}_k(F) \backslash \mathrm{U}_k(C(F))$. Then there is $x \in V$ such that $\sigma_k(g,x) \in F \backslash C(F)$ and the group

$$C(F) \lneq \langle C(F), \{\sigma_k(g,x) \mid x \in V\} \rangle \leq F$$

satisfies condition (C), too, as can be seen by setting $\sigma_k(g,x)_\omega := \sigma_k(g, x_\omega)$. This contradicts the maximality of $C(F)$. □

Remark II.11. Let $F \leq \mathrm{Aut}(B_{d,k})$ satisfy (C). Elements of $\mathrm{U}_k(F)$ are readily constructed: Given $v, w \in V(T_d)$ and $a \in F$, define $g : B(v,k) \to B(w,k)$ by setting $g(v) = w$ and $\sigma(g,v) = a$. Given a collection of actions $(a_\omega)_{\omega \in \Omega}$ such that $a_\omega \in C(\alpha,\omega)$ for all $\omega \in \Omega$ there is a unique extension of g to $B(v,k+1)$ such that $\sigma_k(g, v_\omega) = a_\omega$. Proceed iteratively.

2.2. Discreteness. The group $F \leq \mathrm{Aut}(B_{d,k})$ also determines whether or not $\mathrm{U}_k(F)$ is discrete. In fact, the following proposition generalizes the fact that a Burger-Mozes universal group is discrete if and only if its local action is semiregular.

Proposition II.12. Let $F \leq \mathrm{Aut}(B_{d,k})$ satisfy condition (C). Then $\mathrm{U}_k(F) \leq \mathrm{Aut}(T_d)$ is discrete if and only if F satisfies

$$\text{(D)} \qquad \forall \omega \in \Omega : \ F_{T_\omega} = \{\mathrm{id}\}.$$

Proof. Fix $v \in V$. A subgroup $H \leq \mathrm{Aut}(T_d)$ is non-discrete if and only if for every $n \in \mathbb{N}$ there is $h \in H \backslash \{\mathrm{id}\}$ such that $h|_{B(v,n)} = \mathrm{id}$.

Suppose that $\mathrm{U}_k(F)$ is non-discrete. Then there are $n \in \mathbb{N}_{\geq k}$ and $\alpha \in \mathrm{U}_k(F)$ such that $\alpha|_{B(v,n)} = \mathrm{id}$ and $\alpha|_{B(v,n+1)} \neq \mathrm{id}$. Hence there is $x \in S(v, n-k+1)$ with

$a := \sigma_k(\alpha, x) \neq \mathrm{id}$. In particular, $a \in F_{T_\omega}\backslash\{\mathrm{id}\}$ where ω is the label of the unique edge e with $o(e) = x$ and $d(v,x) = d(v,t(e)) + 1$.

Conversely, suppose that $F_{T_\omega} \neq \{\mathrm{id}\}$ for some $\omega \in \Omega$. For every $n \in \mathbb{N}_{\geq k}$, we define an automorphism $\alpha \in \mathrm{U}_k(F)$ with $\alpha|_{B(v,n)} = \mathrm{id}$ and $\alpha|_{B(v,n+1)} \neq \mathrm{id}$: If $\alpha|_{B(v,n)} = \mathrm{id}$, then $\sigma_k(\alpha, x) \in F$ for all $x \in B(v, n-k)$. Next, choose $e \in E$ with $x := o(e) \in S(v, n-k+1)$ and $t(e) \in S(v, n-k)$ such that $l(e) = \omega$. We extend α to $B(x,k)$ by $\alpha|_{B(x,k)} := l_x^k \circ s \circ (l_x^k)^{-1}$ where $s \in F_{T_\omega}\backslash\{\mathrm{id}\}$. Finally, we extend α to T_d using condition (C). □

As we shall investigate the discrete case later on in Section 5, we define condition (CD) on $F \leq \mathrm{Aut}(B_{d,k})$ to be the conjunction of (C) and (D). The following description is then immediate from the above:

$$\text{(CD)} \qquad \forall a \in F\ \forall \omega \in \Omega:\ \exists!\ a_\omega \in F:\ a_\omega|_{T_\omega} = \iota_{a(\omega)} \circ a \circ \iota_\omega.$$

In this case, an element of $\mathrm{U}_k(F)_x$ is determined by its action on $B(x,k)$. Hence $\mathrm{U}_k(F)_x \cong F$ for all $x \in V$ and $\mathrm{U}_k(F)_{(x,y)} \cong F_{(b,b_\omega)}$ for all adjacent $x, y \in V$ with $l(x,y) = \omega$. Also, F admits a unique map $z : F \times \Omega \to F,\ (a,\omega) \mapsto a_\omega$ which for all $a, b \in F$ and $\omega \in \Omega$ satisfies

(i) $z(a,\omega) \in C_F(a,\omega)$,
(ii) $z(ab,\omega) = z(a,b\omega)z(b,\omega)$, and
(iii) $z(z(a,\omega),\omega) = a$,

We shall refer to a map z as above as an *involutive compatibility cocycle* of F. In particular, z restricts to an automorphism $z_\omega := z(-,\omega)|_{F_{(b,b_\omega)}} \in \mathrm{Aut}(F_{(b,b_\omega)})$ of order at most 2 for every $\omega \in \Omega$.

2.3. Group Structure. For $\widetilde{F} \leq \mathrm{Aut}(B_{d,k})$, let $F := \pi\widetilde{F} \leq \mathrm{Sym}(\Omega)$ denote the projection of $\widetilde{F}$ onto $\mathrm{Aut}(B_{d,1}) \cong \mathrm{Sym}(\Omega)$. As an illustration, we record that the structure of $\mathrm{U}_k(\widetilde{F})$ is particularly simple if F is regular.

Proposition II.13. Let $\widetilde{F} \leq \mathrm{Aut}(B_{d,k})$ satisfy condition (C). Suppose that $F := \pi\widetilde{F}$ is regular. Then $\mathrm{U}_k(\widetilde{F}) = \mathrm{U}_1(F) \cong F * \mathbb{Z}/2\mathbb{Z}$.

Proof. Fix $b \in V$. Since F is transitive, $\mathrm{U}_k(\widetilde{F})$ is generated by $\mathrm{U}_k(\widetilde{F})_b$ and an involution ι inverting an edge with origin b. Given $\alpha \in \mathrm{U}_k(\widetilde{F})_b$, regularity of F implies that $\sigma_1(\alpha, x) = c_1(\alpha, b) \in F$ for all $x \in V$. The subgroups $H_1 := \mathrm{U}_k(\widetilde{F})_b \cong F$ and $H_2 := \langle\iota\rangle$ of $\mathrm{U}_k(\widetilde{F})$ generate a free product within $\mathrm{U}_k(F)$ by the ping-pong lemma: Put $X_1 := V(T_b)$ and $X_2 := V(T_{b_\omega})$. Any non-trivial element of H_1 maps X_2 into X_1 be regularity of F. Also, $\iota \in H_2$ maps X_1 into X_2 by definition. □

More generally, Bass-Serre theory [**Ser03**] identifies the universal groups $\mathrm{U}_k(F)$ as amalgamated free products.

Proposition II.14. Let $F \leq \mathrm{Aut}(B_{d,k})$ with πF transitive satisfy (C) (and D). Then

$$\mathrm{U}_k(F) \cong \mathrm{U}_k(F)_x \underset{\mathrm{U}_k(F)_{(x,y)}}{*} \mathrm{U}_k(F)_{\{x,y\}} \left(\cong F \underset{F_{(b,b_\omega)}}{*} (F_{(b,b_\omega)}) \rtimes \mathbb{Z}/2\mathbb{Z})\right)$$

for any edge $(x,y) \in E$, where $\omega = l(x,y)$ and the action of $\mathbb{Z}/2\mathbb{Z}$ on $F_{(b,b_\omega)}$ is given by $z_\omega \in \mathrm{Aut}(F_{(b,b_\omega)})$.

Corollary II.15. Let $F, F' \leq \mathrm{Aut}(B_{d,k})$ satisfy (CD). If $\varphi : F \to F'$ is an isomorphism such that $\varphi(F_{(b,b_\omega)}) = F'_{(b,b_{\omega'})}$ for some $\omega, \omega' \in \Omega$, then $\mathrm{U}_k(F) \cong \mathrm{U}_k(F')$. □

Note that Corollary II.15 applies to conjugate subgroups of $\mathrm{Aut}(B_{d,k})$ with (CD).

2.4. The Burger–Mozes Subquotient. Here, we determine the Burger–Mozes subquotient $H^{(\infty)}/\mathrm{QZ}(H^{(\infty)})$ of Theorem I.9 for certain universal groups.

Proposition II.16. Let $F \leq \mathrm{Aut}(B_{d,k})$. If F satisfies (D) then $\mathrm{QZ}(\mathrm{U}_k(F)) = \mathrm{U}_k(F)$. Otherwise, $\mathrm{QZ}(\mathrm{U}_k(F)) = \{\mathrm{id}\}$.

Proof. If F satisfies (D) then $\mathrm{U}_k(F)$ is discrete and hence $\mathrm{QZ}(\mathrm{U}_k(F)) = \mathrm{U}_k(F)$. Conversely, if F does not satisfy (D) then Proposition II.7 implies that any half-tree stabilizer in $\mathrm{U}_k(F)$ is non-trivial: Let $T \subseteq T_d$ be a half-tree. Then $T \in \{T_x, T_y\}$ for an edge $e := (x, y) \in E$. Since $\mathrm{U}_k(F)$ is non-discrete and has satisfies Property P_k by Proposition II.7, the stabilizer $\mathrm{U}_k(F)_{e^k} = \mathrm{U}_k(F)_{e^k,T_y} \cdot \mathrm{U}_k(F)_{e^k,T_x}$ is non-trivial. In particular, either $\mathrm{U}_k(F)_{T_x}$ or $\mathrm{U}_k(F)_{T_y}$ is non-trivial. Then both are non-trivial in view of the existence of label-respecting inversions. Hence so is $\mathrm{U}_k(F)_T$.

Therefore, $\mathrm{U}_k(F)$ has Property H of Möller–Vonk [**MV12**, Definition 2.3] and [**MV12**, Proposition 2.6] implies that $\mathrm{U}_k(F)$ has trivial quasi-center. □

Proposition II.17. Let $F \leq \mathrm{Aut}(B_{d,k})$ with $\pi F \leq \mathrm{Sym}(\Omega)$ semiprimitive satisfy (C) but not (D). Then $\mathrm{U}_k(F)^{(\infty)} = \mathrm{U}_k(F)^{+_k}$.

Proof. The subgroup $\mathrm{U}_k(F)^{+_k} \leq \mathrm{U}_k(F)$ is open, hence closed, and normal by definition. Since $\mathrm{U}_k(F)$ does not satisfy (D) it is also non-discrete. By Corollary II.43, we conclude that $\mathrm{U}_k(F)^{+_k} \geq \mathrm{U}_k(F)^{(\infty)}$. However, since $\mathrm{U}_k(F)$ satisfies Property P_k by Proposition II.7, the group $\mathrm{U}_k(F)^{+_k}$ is simple by Theorem I.5. Hence $\mathrm{U}_k(F)^{+_k} = \mathrm{U}_k(F)^{(\infty)}$. □

In particular, $\mathrm{U}_k(F)^{+_k}$ is a non-discrete, totally disconnected locally compact simple group in the case of Proposition II.17. If πF is quasiprimitive, then $\mathrm{U}_k(F)^{+_k}$ is cocompact in $\mathrm{U}_k(F)$ by [**BM00a**, Proposition 1.2.1] and therefore compactly generated by [**MŚ59**].

Overall, we may record $\mathrm{U}_k(F)^{(\infty)}/\mathrm{QZ}(\mathrm{U}_k(F)^{(\infty)}) = \mathrm{U}_k(F)^{+_k}$ in the quasiprimitive case, using [**BM00a**, Proposition 1.2.1 (4)].

3. Examples

In this section, we construct various classes of examples of subgroups of $\mathrm{Aut}(B_{d,k})$ satisfying (C) or (CD), and prove a rigidity result for certain local actions.

First, we introduce a workable realization of $\mathrm{Aut}(B_{d,k})$ as well as the conditions (C) and (CD). Essentially, we view an automorphism α of $B_{d,k}$ as the collection $\{\sigma_{k-1}(\alpha, v) \mid v \in B(b, 1)\}$: Let $\mathrm{Aut}(B_{d,1}) \cong \mathrm{Sym}(\Omega)$ be the natural isomorphism and for $k \geq 2$ identify $\mathrm{Aut}(B_{d,k})$ with its image under the map

$$\mathrm{Aut}(B_{d,k}) \to \mathrm{Aut}(B_{d,k-1}) \ltimes \prod_{\omega \in \Omega} \mathrm{Aut}(B_{d,k-1}),\ \alpha \mapsto (\sigma_{k-1}(\alpha, b), (\sigma_{k-1}(\alpha, b_\omega))_\omega)$$

where $\mathrm{Aut}(B_{d,k-1})$ acts on $\prod_{\omega\in\Omega} \mathrm{Aut}(B_{d,k-1})$ by permuting the factors according to its action on $S(b, 1) \cong \Omega$. In addition, for every $\omega \in \Omega$ consider the map

$$p_\omega : \mathrm{Aut}(B_{d,k}) \to \mathrm{Aut}(B_{d,k-1}) \times \mathrm{Aut}(B_{d,k-1}),\ \alpha \mapsto (\sigma_{k-1}(\alpha, b), \sigma_{k-1}(\alpha, b_\omega))$$

whose image we interpret as a relation on $\mathrm{Aut}(B_{d,k-1})$. The conditions (C) and (D) for a subgroup $F \leq \mathrm{Aut}(B_{d,k})$ now read as follows.

(C) $\quad \forall \omega \in \Omega :\ p_\omega(F)$ is symmetric

(D) $\quad \forall \omega \in \Omega :\ p_\omega|_F^{-1}(\mathrm{id}, \mathrm{id}) = \{\mathrm{id}\}$

3.1. The case $k = 2$. We first consider the case $k = 2$ which suffices in certain situations, see Theorem II.22. Consider the map $\gamma : \mathrm{Sym}(\Omega) \to \mathrm{Aut}(B_{d,2})$ which maps $a \in \mathrm{Sym}(\Omega)$ to $(a, (a, \ldots, a)) \in \mathrm{Aut}(B_{d,2})$ using the realization of $\mathrm{Aut}(B_{d,2})$ defined above. Given $F \leq \mathrm{Sym}(\Omega)$, the image

$$\Gamma(F) := \mathrm{im}(\gamma|_F) = \{(a, (a, \ldots, a)) \mid a \in F\} \cong F$$

is a subgroup of $\mathrm{Aut}(B_{d,2})$ isomorphic to F which satisfies (CD). Indeed, its compatibility cocycle is given by $z : \Gamma(F) \times \Omega \to \Gamma(F),\ (\gamma(a),\omega) \mapsto \gamma(a)$. Notice that $\Gamma(F)$ implements the restriction of the diagonal action $F \curvearrowright \Omega^2$ to $\Omega^{(2)} \cong S(b,2)$.

Clearly, $\mathrm{U}_2(\Gamma(F)) = \{\alpha \in \mathrm{Aut}(T_d) \mid \exists a \in F :\ \forall x \in V :\ c_\omega(\alpha, x) = a\} =: \mathrm{D}(F)$, following the notation of [**BEW15**]. Moreover, we have the following description of all subgroups $F^{(2)} \leq \mathrm{Aut}(B_{d,2})$ which satisfy (C), project onto F and contain $\Gamma(F)$.

Proposition II.18. Let $F \leq \mathrm{Sym}(\Omega)$. Given $K \leq \prod_{\omega\in\Omega} F_\omega \cong \ker\pi \leq \mathrm{Aut}(B_{d,2})$, there is $F^{(2)} \leq \mathrm{Aut}(B_{d,2})$ with (C) and fitting into the split exact sequence

$$1 \longrightarrow K \overset{\iota}{\rightarrowtail} F^{(2)} \underset{\gamma}{\overset{\pi}{\rightleftarrows}} F \longrightarrow 1$$

if and only if K is invariant under the action $F \curvearrowright \prod_{\omega\in\Omega} F_\omega$ given by

$$a \cdot (a_\omega)_{\omega\in\Omega} := (a a_{a^{-1}(\omega)})_{\omega\in\Omega}$$

In the split situation of Proposition II.18 we also denote $F^{(2)}$ by $\Sigma(K)$.

Proof. If there is an exact sequence as above then $K \trianglelefteq F^{(2)}$ is invariant under conjugation by $\Gamma(F) \leq F^{(2)}$. Conversely, if K is invariant under the given action, then $F^{(2)} := \{(a, (a a_\omega)_\omega) \mid a \in F,\ \forall \omega \in \Omega : a_\omega \in F_\omega\}$ fits into the sequence. Note that $F^{(2)}$ contains K and $\Gamma(F)$, and is a subgroup: For $(a, (a a_\omega)_\omega),\ (b, (b b_\omega)_\omega) \in F^{(2)}$,

$$(a, (a a_\omega)_\omega)(b, (b b_\omega)_\omega) = (ab, (a a_{b(\omega)} b b_\omega)) = (ab, (ab \circ b^{-1} a_{b(\omega)} b b_\omega)_\omega) \in F^{(2)}$$

by assumption. In particular, $F^{(2)} = \langle \Gamma(F), K\rangle$. We now check condition (C) on generators of $F^{(2)}$. As before, $\gamma(a) \in C(\gamma(a),\omega)$ for all $a \in F$ and $\omega \in \Omega$. Further, given $k \in K$, we have $\gamma(\mathrm{pr}_\omega k)k^{-1} \in C(k,\omega)$ for all $\omega \in \Omega$. □

Both the construction Γ and Proposition II.18 generalize to non-trivial involutive compatibility cocycles of F. The following subgroups of $\mathrm{Aut}(B_{d,2})$ are of this type: Let $F \leq \mathrm{Sym}(\Omega)$ be transitive. Fix $\omega_0 \in \Omega$ and let $N \trianglelefteq F_{\omega_0}$ be normal. Furthermore, fix elements $f_\omega \in F$ $(\omega \in \Omega)$ satisfying $f_\omega(\omega_0) = \omega$ and define

$$\Delta(F,N) := \{(a, (f_{a(\omega)} f_\omega^{-1} \circ f_\omega a_{\omega_0} f_\omega^{-1})_\omega) \mid a \in F,\ a_{\omega_0} \in N\} \cong F \times N,$$
$$\Phi(F,N) := \{(a, (a \circ f_\omega a_{\omega_0}^{(\omega)} f_\omega^{-1})_\omega) \mid a \in F,\ \forall \omega \in \Omega :\ a_{\omega_0}^{(\omega)} \in N\} \cong F \ltimes N^d.$$

Note that in the case of $\Delta(F,N)$ we have chosen $z(a,\omega) := f_{a(\omega)} f_\omega^{-1}$ for all $a \in F$ and $\omega \in \Omega$ but in general any involutive compatibility cocycle z of F for which $\Gamma(F)$ and $\{(\mathrm{id}, (f_\omega a_{\omega_0} f_\omega^{-1})_\omega) \mid \omega \in \Omega\}$ commute works. The groups $\Phi(F,N)$ satisfy (C) and the groups $\Delta(F,N)$ satisfy (CD). We abbreviate $\Delta(F) := \Delta(F, F_{\omega_0})$ and $\Phi(F) := \Phi(F, F_{\omega_0})$. Notice that $\Phi(F)$ can also be defined without assuming transitivity of F, namely

$$\Phi(F) := \{(a, (a_\omega)_\omega) \mid a \in F,\ \forall \omega \in \Omega :\ a_\omega \in C_F(a,\omega)\} \cong F \ltimes \prod_{\omega\in\Omega} F_\omega$$

It is then plain that $\mathrm{U}_2(\Phi(F)) = \mathrm{U}_1(F)$ for every $F \leq \mathrm{Sym}(\Omega)$. More generally, assume that $F \leq \mathrm{Sym}(\Omega)$ preserves a partition $\mathcal{P} : \Omega = \bigsqcup_{i\in I} \Omega_i$. Set

$$\Phi(F,\mathcal{P}) := \{(a, (a_\omega)_\omega) \mid a \in F,\ a_\omega \in C_F(a,\omega) \text{ constant w.r.t. } \mathcal{P}\} \cong F \ltimes \prod_{i\in I} F_{\Omega_i}.$$

The group $\Phi(F,\mathcal{P})$ satisfies (C) and plays a major role in Section 7.

Example II.19. In this example we investigate Proposition II.18 for primitive dihedral groups: Set $F := D_p \leq S_p$ for some prime $p \geq 3$. Then $F_i \cong (\mathbb{F}_2, +)$. Hence $U := \prod_{i=1}^p F_i$ is a p-dimensional vector space over $\mathbb{F}_2$ and the F-action on it reduces to permuting coordinates. In case $2 \in (\mathbb{Z}/p\mathbb{Z})^*$ is primitive we show that there are only the following four F-invariant subspaces of U: The trivial subspace, the diagonal subspace $\langle(1,\ldots,1)\rangle$, the whole space and $K := \ker\sigma \cong \mathbb{F}_2^{(p-1)}$ where

$\sigma : U \to \mathbb{F}_2,\ (v_1, \dots, v_p)^T \mapsto \sum_{i=1}^{p} v_i$. Notice that K is an F-invariant subspace because σ is an F-invariant homomorphism. It is a conjecture of Artin that there are infinitely many such primes, the list starting with 3, 5, 11, 13 . . ., see [**Slo**, A001122].

Suppose that $W \leq U$ is F-invariant. It suffices to show that $K \leq W$ as soon as $W \cap \ker \sigma$ contains a non-trivial element w. To see this, we show that the orbit of w under the cyclic group $\langle \varrho \rangle = C_p \leq D_p$ generates a $(p-1)$-dimensional subspace of K which hence equals K: Indeed, the rank of the circulant matrix $C := (w, \varrho w, \varrho^2 w, \dots, \varrho^{(p-1)} w)$ equals $p - \deg(\gcd(x^p - 1, f(x)))$ where $f(x) \in \mathbb{F}_2[x]$ is the polynomial $f(x) = w_p x^{p-1} + \dots + w_2 x + w_1$, see e.g. [**Day60**, Corollary 1]. The polynomial $x^p - 1 \in \mathbb{F}_2[x]$ factors into the irreducibles $(x^{p-1} + x^{p-2} + \dots + x + 1)(x - 1)$ by the assumption on p. Since f has an even number of non-zero coefficients, we conclude that $\mathrm{rank}(C) = p - 1$.

3.2. General case. We now extend the constructions Γ and Φ to arbitrary k. Given $F \leq \mathrm{Aut}(B_{d,k})$ with (C), define the subgroup

$$\Phi_k(F) := \{(\alpha, (\alpha_\omega)_\omega) \mid \alpha \in F,\ \forall \omega \in \Omega :\ \alpha_\omega \in C_F(\alpha, \omega)\}$$

of $\mathrm{Aut}(B_{d,k+1})$. Clearly, $\Phi_k(F)$ satisfies (C) and $\mathrm{U}_{k+1}(\Phi_k(F)) = \mathrm{U}_k(F)$. Concerning the construction Γ we have the following.

Lemma II.20. Let $F \leq \mathrm{Aut}(B_{d,k})$ satisfy (C). Then there exists $\Gamma_k(F) \leq \mathrm{Aut}(B_{d,k+1})$ satisfying (CD) and such that $\pi_k : \Gamma_k(F) \to F$ is an isomorphism if and only if F admits an involutive compatibility cocycle.

Proof. If F admits an involutive compatibility cocycle z, define

$$\Gamma_k(F) := \{(\alpha, (z(\alpha, \omega))_\omega) \mid \alpha \in F\} \leq \mathrm{Aut}(B_{d,k+1}).$$

Then $\gamma_k : F \to \Gamma_k(F),\ \alpha \mapsto (\alpha, (z(\alpha, \omega))_\omega)$ is an isomorphism and the involutive compatibility cocycle of $\Gamma_k(F)$ is given by $\widetilde{z} : (\gamma_k(\alpha), \omega) \mapsto \gamma_k(z(\alpha, \omega))$. Conversely, if a group $\Gamma_k(F)$ as above exists, set $z : (\alpha, \omega) \mapsto \mathrm{pr}_\omega\, \pi_k^{-1} \alpha$. □

Let $F \leq \mathrm{Aut}(B_{d,k})$ with (C) and $l > k$. Set $\Gamma^l(F) := \Gamma_{l-1} \circ \dots \circ \Gamma_k(F)$ for an implicit sequence of involutive compatibility cocycles and $\Phi^l(F) := \Phi_{l-1} \circ \dots \circ \Phi_k(F)$.

Example II.28 provides a group $E \leq \mathrm{Aut}(B_{3,2})$ that satisfies (C), admits an involutive compatibility cocycle but does not satisfy (CD).

3.3. A rigid case. For certain $F \leq \mathrm{Sym}(\Omega)$ the groups $\Gamma(F)$, $\Delta(F)$ and $\Phi(F)$ already yield all possible $\mathrm{U}_k(\widetilde{F})$. The argument is based on Sections 3.4 and 3.5 of [**BM00a**]. The following lemma is due to M. Guidici by personal communication.

Lemma II.21. Let $F \leq \mathrm{Sym}(\Omega)$ be 2-transitive with F_ω simple non-abelian for all $\omega \in \Omega$. Then every extension of F_ω $(\omega \in \Omega)$ by F is equivalent to the direct product.

Proof. Let $1 \to F_\omega \to F^{(2)} \to F \to 1$ be an extension of F_ω by F. In particular, F_ω can be regarded as a subgroup of $F^{(2)}$ and we may consider the conjugation map $\varphi : F^{(2)} \to \mathrm{Aut}(F_\omega)$. We show that $K := \ker \varphi = C_{F^{(2)}}(F_\omega) \trianglelefteq F^{(2)}$ complements F_ω in $F^{(2)}$. Since F_ω is non-abelian, we have $K \cap F_\omega = \{\mathrm{id}\}$ whence $K \times F_\omega \leq F^{(2)}$. Now consider $F^{(2)}/(K \times F_\omega) \leq \mathrm{Out}(F_\omega)$ which is solvable by Schreier's conjecture. Since $F^{(2)}/F_\omega \cong F$ is not solvable we conclude $K \neq \{\mathrm{id}\}$. Now, by a theorem of Burnside, every 2-transitive permutation group F is either almost simple or affine.

In the first case, F is actually simple: Let $N \trianglelefteq F$. Then $F_\omega \cap N \trianglelefteq F_\omega$. Hence either $F_\omega \cap N = \{\mathrm{id}\}$ or $F_\omega \cap N = F_\omega$. Since F is 2-transitive and hence primitive, every normal subgroup acts transitively. In the first case, N is regular which contradicts F being almost simple. Hence the second case holds and $N = NF_\omega = F$. Now $F^{(2)}/(K \times F_\omega)$ is a proper quotient of F and hence trivial. Therefore $F^{(2)} = K \times F_\omega$ and $K \cong F^{(2)}/F_\omega \cong F$. In the second case, $F = F_\omega \ltimes C_p^d$ $(d \in \mathbb{N})$ and $\{\mathrm{id}\} \neq K \cong$

$K \cdot F_\omega/F_\omega \trianglelefteq F$ contains the unique minimal normal subgroup $C_p^d \trianglelefteq K \trianglelefteq F$. Since $F_\omega \cong F/C_p^d$ is non-abelian simple whereas $F^{(2)}/(K \times F_\omega)$ is solvable, we conclude that $K \neq C_p^d$. But $F/C_p^d \cong F_\omega$ is simple, so $K \times F_\omega = F^{(2)}$. □

Theorem II.22. Let $F \leq \mathrm{Sym}(\Omega)$ be 2-transitive with F_ω simple non-abelian for all $\omega \in \Omega$, and let $\widetilde{F} \leq \mathrm{Aut}(B_{d,k})$ with $\pi\widetilde{F} = F$ satisfy (C). Then $\mathrm{U}_k(\widetilde{F})$ equals either

$$\mathrm{U}_2(\Gamma(F)), \quad \mathrm{U}_2(\Delta(F)) \quad \text{or} \quad \mathrm{U}_2(\Phi(F)) = \mathrm{U}_1(F).$$

Proof. We may assume $k \geq 2$. Since $\widetilde{F} \leq \mathrm{Aut}(B_{d,k})$ satisfies (C) so does the restriction $F^{(2)} := \pi_2\widetilde{F} \leq \Phi(F) \leq \mathrm{Aut}(B_{d,2})$. Consider the projection $\pi : F^{(2)} \twoheadrightarrow F$ and fix $\omega_0 \in \Omega$. We have $\ker\pi \leq \prod_{\omega\in\Omega} F_\omega \cong F_{\omega_0}^d$ and $\mathrm{pr}_\omega \ker\pi \trianglelefteq F_{\omega_0}$ for all $\omega \in \Omega$ because $F^{(2)}$ satisfies (C). Since F_{ω_0} is simple, $\ker\pi \trianglelefteq F^{(2)}$ and F is transitive this implies that either $\mathrm{pr}_\omega \ker\pi = \{\mathrm{id}\}$ for all $\omega \in \Omega$ or $\mathrm{pr}_\omega \ker\pi = F_{\omega_0}$ for all $\omega \in \Omega$. In the first case, $\pi : F^{(2)} \to F$ is an isomorphism and $F^{(2)}$ satisfies (CD) which implies $F^{(2)} = \Gamma(F)$ and hence $\mathrm{U}_k(\widetilde{F}) = \mathrm{U}_2(\Gamma(F))$ for some involutive compatibility cocycle of F.

In the second case, Section 3.4.3 of [**BM00a**] implies that $\ker\pi \leq F_{\omega_0}^d$ is a product of subdiagonals preserved by the primitive action of F on the index set of $F_{\omega_0}^d$. Therefore, either there is just one block and $\ker\pi \cong F_{\omega_0}$, or all blocks are singletons and $\ker\pi \cong F_{\omega_0}^d$. In the first case, we conclude $F^{(2)} = \Delta(F)$ using Lemma II.21 which satisfies (CD) and therefore $\mathrm{U}_k(\widetilde{F}) = \mathrm{U}_2(\Delta(F))$.

Now assume that $\ker\pi \cong F_{\omega_0}^d$. We aim to show that $\widetilde{F} = \Phi^k(F)$ which implies $\mathrm{U}_k(\widetilde{F}) = \mathrm{U}_2(\Phi(F)) = \mathrm{U}_1(F)$. To this end, we introduce the following notation: Given $\omega \in \Omega$ and $B_{d,k}$, set $S_n(b,\omega) = \{x \in S(b,n) \mid d(x,b) = d(x,b_\omega) + 1\}$ for $n \leq k$, $a(n) := |S_n(b,\omega)|$ and $c(n) := |S(b,n)|$. Further, let $F^{(n)} \leq \mathrm{Aut}(B_{d,n})$ $(n \in \mathbb{N})$ denote the local actions of $\mathrm{U}_k(\widetilde{F})$.

First of all, note that $\mathrm{U}_k(\widetilde{F})$ is non-discrete by the Thompson-Wielandt Theorem, see [**BM00a**, Theorem 2.1.1]: The group $\Phi(F)_{T_\omega} \cong F_{\omega_0}^{d-1}$ cannot be a p-group given that F_{ω_0} is simple non-abelian. Thus $K_n := \mathrm{stab}_{F^{(n)}}(B(b,n-1)) \leq F_{\omega_0}^{c(n-1)}$ is non-trivial for all $n \in \mathbb{N}$.

We now inductively prove that $F^{(n)}$ acts transitively on $S(b,n)$ for all $n \in \mathbb{N}$ which holds for $n = 2$. Since $F^{(n+1)}$ satisfies (C), the projection onto each factor of $K_{n+1} \leq F_{\omega_0}^{c(n)}$ is subnormal in F_{ω_0}. Since F_{ω_0} is simple, $F^{(n)}$ acts transitively on $S(b,n)$ by the induction hypothesis, and K_{n+1} is non-trivial this implies that $\mathrm{pr}_x K_{n+1} = F_{\omega_0}$ for all $x \in S(b,n)$. Hence $F^{(n+1)}$ acts transitively on $S(b,n+1)$. Thus $\mathrm{U}_k(\widetilde{F})$ is locally ∞-transitive.

We now inductively prove that $F^{(n)} = \Phi_{n-1}(F^{(n-1)})$ for all $n \in \mathbb{N}$. This holds for $n = 2$. As a consequence of the above argument, K_{n+1} is a product of subdiagonals preserved by the transitive action of $F^{(n+1)}$ on $S(b,n)$. The associated block decomposition $(B_j)_{j\in J}$ of $S(b,n)$ satisfies $|B_j \cap S_n(b,\omega)| \leq 1$ for all $j \in J$ and $\omega \in \Omega$: Since $K_n \cong F_{\omega_0}^{c(n-1)}$ by the induction hypothesis we conclude $K_{n+1}|_{S_{n+1}(b,\omega)} \cong F_{\omega_0}^{a(n)}$ because $K_{n+1} = \mathrm{stab}_{F^{(n+1)}}(B(b,n)) \trianglelefteq \mathrm{stab}_{F^{(n+1)}}(B(b_\omega,n-1)) \cong K_n$. However, any such block decomposition has to be the decomposition into singletons: Assume that $|B_j| \geq 2$ for some $j \in J$ and choose $\omega,\omega' \in \Omega$ such that $B_j \cap S_n(b,\omega) = x$ and $B(j) \cap S_n(b,\omega') = x'$. Further, choose $y \in S_n(b,\omega')\backslash\{x'\}$. Then $y \in B_{j'}$ for some $j' \in J\backslash j$. Since $\mathrm{U}_k(F^{(k)})$ is locally ∞-transitive, there is $a \in F^{(n+1)}$ such that $ax = x$ and $ax' = y$. However, this implies $aB_j = B_j$ and $aB_j = B_{j'}$ which contradicts the assumption $j \neq j'$. □

See [**BM00a**, Example 3.3.1] for examples of permutation groups satisfying the assumptions of Theorem II.22. If F does not have simple point stabilizers or preserves a non-trivial partition, further universal groups are given by $\mathrm{U}_2(\Delta(F,N))$, $\mathrm{U}_2(\Phi(F,N))$ and $\mathrm{U}_2(\Phi(F,\mathcal{P}))$, see Section 3.1.

4. Universality

Let $\widetilde{F} \leq \mathrm{Aut}(B_{d,k})$ satisfy (C). Suppose that $F := \pi\widetilde{F}$ is transitive. Then $\mathrm{U}_k(\widetilde{F}) \leq \mathrm{Aut}(T_d)$ is locally transitive, satisfies Property P_k and contains an involutive inversion. In this section we show that these properties characterize locally transitive universal groups and thereby determine the k-closures of all locally transitive groups containing an involutive inversion. Recall that the k-closure of $H \leq \mathrm{Aut}(T_d)$ is the group

$$H^{(k)} := \{g \in \mathrm{Aut}(T_d) \mid \forall x \in V : \exists h \in H : \ g|_{B(x,k)} = h|_{B(x,k)}\}.$$

Theorem II.23. Let $H \leq \mathrm{Aut}(T_d)$ be locally transitive and contain an involutive inversion. Then there is a labelling l of T_d such that

$$\mathrm{U}_1(F^{(1)}) \geq \mathrm{U}_2(F^{(2)}) \geq \cdots \mathrm{U}_k(F^{(k)}) \geq \cdots \geq H \geq \mathrm{U}_1(\{\mathrm{id}\})$$

where $F^{(k)} \leq \mathrm{Aut}(B_{d,k})$ is action isomorphic to the action of H on balls of radius k. Furthermore, $H^{(k)} = \mathrm{U}_k(F^{(k)})$.

Proof. We first construct a labelling l of T_d such that $H \geq \mathrm{U}_1^{(l)}(\{\mathrm{id}\})$: Fix $b \in V$ and choose a bijection $l_b : E(b) \to \Omega$. The assumptions provide an involutive inversion $\iota_\omega \in H$ of the edge (b, b_ω) for each $\omega \in \Omega$. Using these, we define the announced labelling inductively: Set $l|_{E(b)} := l_b$. Assume that l is defined on $E(b,n)$ and for $e \in E(b, n+1)$ put $l(e) := l(\iota_\omega(e))$ if b_ω is part of the unique reduced path from b to $o(e)$. Since the ι_ω $(\omega \in \Omega)$ have order 2, we have $\sigma_1(\iota_\omega, x) = \mathrm{id}$ for all $\omega \in \Omega$ and $x \in V$. Thus $\langle\{\iota_\omega \mid \omega \in \Omega\}\rangle = \mathrm{U}_1^{(l)}(\{\mathrm{id}\}) \leq H$.

Now let $h \in H$ and $x \in V$. Further, let $(b, b_1, \ldots, b_n, x)$ and $(b, b_1', \ldots, b_m', h(x))$ be the unique reduced paths from b to x and $h(x)$ respectively. Since $\mathrm{U}_1^{(l)}(\{\mathrm{id}\}) \leq H$, the latter in particular contains the unique label-respecting inversion ι_e about every edge e in the above paths. Then

$$s := \iota_{(b_1',b)}^{-1} \cdots \iota_{(b_m',b_{m-1}')}^{-1} \iota_{(h(x),b_m')}^{-1} \circ h \circ \iota_{(x,b_n)} \cdots \iota_{(b_2,b_1)} \iota_{(b_1,b)} \in H$$

stabilizes b and the cocycle identity implies for every $k \in \mathbb{N}$:

$$\sigma_k(h,x) = \sigma_k(\iota_{(h(x),b_m')} \cdots \iota_{(b_1',b)} \circ s \circ \iota_{(b_1,b)}^{-1} \cdots \iota_{(x,b_n)}^{-1}, x) = \sigma_k(s,b) \in F^{(k)}.$$

where $F^{(k)} \leq \mathrm{Aut}(B_{d,k})$ is defined by $l_b^k \circ H_b|_{B(b,k)} \circ (l_b^k)^{-1}$. The remaining assertions are now immediate from [**BEW15**, Theorem 5.4]. □

Remark II.24. Retain the notation of Theorem II.23. By Proposition I.14, there is a labelling l of T_d such that $\mathrm{U}_1^{(l)}(F^{(1)}) \geq H$ regardless of the minimal order of an inversion. This labelling may be distinct from the one of Theorem II.23 which fails without assuming the existence of an involutive inversion: For example, a vertex-stabilizer of the group G_2^1 of Example II.28 is action isomorphic to $\Gamma(S_3)$ but $G_2^1 \not\leq \mathrm{U}_2^{(l)}(\Gamma(S_3))$ for any labelling l because $(G_2^1)_{\{b,b_i\}} \cong \mathbb{Z}/4\mathbb{Z}$ whereas

$$\mathrm{U}_2^{(l)}(\Gamma(S_3))_{\{b,b_i\}} \cong \Gamma(S_3)_{(b,b_i)} \rtimes \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$$

by Proposition II.14.

The following corollary of Theorem II.23 characterizes universal groups as the locally transitive subgroups of $\mathrm{Aut}(T_d)$ which contain an involutive inversion and satisfy an independence property.

Corollary II.25. Let $H \leq \mathrm{Aut}(T_d)$ be closed, locally transitive and contain an involutive inversion. Then there is a labelling l of T_d and a group $F^{(k)} \leq \mathrm{Aut}(B_{d,k})$ such that $H = \mathrm{U}_k(F^{(k)})$ if and only if H satisfies Property P_k.

Proof. If $H = \mathrm{U}_k(F^{(k)})$ then H has Property P_k by Proposition II.7. Conversely, if H satisfies Property P_k then $H = \overline{H} = H^{(k)} = \mathrm{U}_k(F^{(k)})$ by virtue of [**BEW15**, Theorem 5.4] and Theorem II.23. □

To complement Theorem II.23 we record the following criterion for certain discrete subgroups of $\mathrm{Aut}(T_d)$ to contain an involutive inversion.

Proposition II.26. Let $H \leq \mathrm{Aut}(T_d)$ be discrete and locally transitive with odd order point stabilizers. If H contains an inversion then it contains an involutive one.

Proof. Let $k_0 \in \mathbb{N}_0$ be minimal such that stabilizers in H of balls of radius k_0 about edges in T_d are trivial. Let $\iota \in H$ be an inversion of an edge $e \in E$. Then $\iota^2 \in H_e$. Hence we are done if $k_0 = 0$. Otherwise the smallest integer $n_1 \in \mathbb{N}$ such that $(\iota^2)^{n_1} \in H_{B(1,e)}$ is odd by the assumptions on the local action of H. Iteratively, the smallest integer $n_k \in \mathbb{N}$ such that $(\iota^2)^{n_k} \in H_{B(k,e)}$ is odd for every $k \leq k_0$ and we conclude that $\iota^{n_{k_0}}$ is an involutive inversion. □

In Proposition II.26, we may for example assume that H be vertex-transitive. Combined with local transitivity this implies the existence of an inversion.

Primitive permutation groups with odd order point stabilizers were classified in [**LS91**]. For instance, they include $\mathrm{PSL}(2,q)$ for all $q \equiv 3 \bmod 4$.

5. The Discrete Case

In this section we study the universal group construction in the discrete case. This provides Remark II.24 showing that the assumptions of Theorem II.23 are necessary and offers a new approach to the long standing Weiss conjecture, stating in particular that there are only finitely many conjugacy classes of discrete, vertex-transitive, locally primitive subgroups of $\mathrm{Aut}(T_d)$.

The following straightforward consequence of Theorem II.23 identifies certain groups relevant to the Weiss conjecture as universal groups for local actions satisfying condition (CD).

Corollary II.27. Let $H \leq \mathrm{Aut}(T_d)$ be discrete, locally transitive and contain an involutive inversion. Then there is $k \in \mathbb{N}$ and a labelling l of T_d such that $H = \mathrm{U}_k^{(l)}(F_k)$ where $F_k \leq \mathrm{Aut}(B_{d,k})$ is action isomorphic to the action of H on balls of radius k.

Proof. Note that discreteness of H implies Property P_k for every $k \in \mathbb{N}$ such that stabilizers in H of balls of radius k in T_d are trivial and apply Corollary II.25. □

Hence studying the class of groups given in Corollary II.27 reduces to studying subgroups of $\mathrm{Aut}(B_{d,k})$ ($k \in \mathbb{N}$) which satisfy (CD). By Corollary II.15, any two conjugate such groups yield isomorphic universal groups. In this sense, it suffices to examine conjugacy classes of subgroups of $\mathrm{Aut}(B_{d,k})$. This can be done computationally using the description of conditions (C) and (D) developed in Section 2, using e.g. GAP [**GAP17**].

Example II.28. Consider the case $d = 3$. By [**Tut47**], [**Tut59**] and [**DM80**], there are, up to conjugacy, seven discrete, vertex-transitive and locally transitive subgroups of $\mathrm{Aut}(T_3)$. We denote them by G_1, G_2, G_2^1, G_3, G_4, G_4^1 and G_5. They have known amalgamated free product structure and presentation. A subscript n indicates that the respective group acts regularly on non-backtracking paths of length n in T_3, and determines the isomorphism class of the (finite) vertex stabilizer which is of order $3 \cdot 2^{n-1}$. The respective group contains an involutive inversion if and only if it has no superscript. The minimal order of an inversion in G_2^1 and G_4^1 is 4. See also [**CL89**]. By Corollary II.27, the groups G_n ($n \in \{1, \ldots, 5\}$) are of the form $\mathrm{U}_k(F)$. We recover their local actions in the following table of conjugacy class representatives of subgroups $\widetilde{F}$ of $\mathrm{Aut}(B_{3,2})$ and $\mathrm{Aut}(B_{3,3})$ which satisfy condition (C) and project onto a transitive subgroup of S_3. The list is complete for $k=2$, and for $k=3$ in the case of (CD).

Description of $\widetilde{F}$	k	$\pi\widetilde{F}$	$\lvert\widetilde{F}\rvert$	(CD)	i.c.c.
$\Phi(A_3)$	2	A_3	3	Yes	
$\Gamma(S_3)$	2	S_3	6	Yes	
$\Delta(S_3)$	2	S_3	12	Yes	
$\Sigma(K)$	2	S_3	24	No	No
E	2	S_3	24	No	Yes
$\Phi(S_3)$	2	S_3	48	No	No
Description of $\widetilde{F}$	k	$\pi_2\widetilde{F}$	$\lvert\widetilde{F}\rvert$	(CD)	i.c.c.
$\Gamma_2(E)$	3	E	24	Yes	
$\Delta_2(E)$	3	E	48	Yes	

The column labelled "i.c.c." records whether the respective group admits an involutive compatibility cocycle which can be determined computationally in [**GAP17**]. Recall that this is automatic if (CD) is satisfied. The kernel K stems from Example II.19. The split example $\Sigma(K)$, after Proposition II.18, is isomorphic to an exceptional group termed E but the two are not conjugate within $\mathrm{Aut}(B_{3,2})$.

Using the above, we conclude $G_1 = \mathrm{U}_1(A_3)$, $G_2 = \mathrm{U}_2(\Gamma(S_3))$, $G_3 = \mathrm{U}_2(\Delta(S_3))$, $G_4 = \mathrm{U}_3(\Gamma_2(E))$ and $G_5 = \mathrm{U}_3(\Delta_2(E))$. It appears likely that the groups G_2^1 and G_4^1 can be described as universal groups with prescribed local action on balls around edges, in which one prevents involutive inversions to begin with.

5.1. On the Weiss Conjecture. The long standing Weiss conjecture [**Wei78**] states that for a given locally finite tree T there are only finitely many conjugacy classes of discrete, vertex-transitive, locally primitive subgroups of $\mathrm{Aut}(T)$. It is typically studied from the point of view of finite graphs. See Potočnic–Spiga–Verret [**PSV12**] for a description and a generalization of the conjecture to semiprimitive local action. Promising partial results were obtained in the same article as well as by Guidici–Morgan in [**GM14**].

Corollary II.27 suggests to restrict to discrete, locally primitive subgroups of $\mathrm{Aut}(T_d)$ containing an involutive inversion.

Conjecture II.29. Let $F \le \mathrm{Sym}(\Omega)$ be primitive. Then there are only finitely many conjugacy classes of discrete subgroups of $\mathrm{Aut}(T_d)$ which locally act like F and contain an involutive inversion.

Given a transitive group $F \le \mathrm{Sym}(\Omega)$, let $\mathcal{H}_F$ denote the collection of subgroups of $\mathrm{Aut}(T_d)$ which are discrete, locally act like F and contain an involutive inversion. Then the following definition is meaningful by Corollary II.27.

Definition II.30. Let $F \le \mathrm{Sym}(\Omega)$ be transitive. Define

$$\dim_{\mathrm{CD}}(F) := \max_{H\in\mathcal{H}_F} \min \left\{ k\in\mathbb{N} \mid \exists F^{(k)} \in \mathrm{Aut}(B_{d,k}) \text{ with (CD)}:\ H = \mathrm{U}_k(F^{(k)}) \right\}$$

if the maximum exists and $\dim_{\mathrm{CD}}(F) = \infty$ otherwise.

Conjecture II.29 is equivalent to the statement that $\dim_{\mathrm{CD}}(F)$ is finite whenever $F \le \mathrm{Sym}(\Omega)$ is primitive.

The remainder of this section is devoted to determining the dimension of certain classes of permutation groups. As a start, transitive permutation groups of dimension 1 are readily characterized.

Lemma II.31. Let $F \leq \mathrm{Sym}(\Omega)$ be transitive. Then $\dim_{\mathrm{CD}}(F) = 1$ if and only if F is regular.

Proof. If F is regular, then $\dim_{\mathrm{CD}}(F) = 1$ by Proposition II.13. Conversely, if $\dim_{\mathrm{CD}}(F) = 1$ then necessarily $\mathrm{U}_2(\Delta(F)) = \mathrm{U}_1(F)$. Hence $\Gamma(F) \cong \Delta(F)$ which implies that F_ω is trivial for all $\omega \in \Omega$. That is, F is regular. □

The next proposition provides a large class of primitive groups of dimension 2. For its proof, we first record the following relations between various characteristic subgroups of a finite group. Recall that the socle of a group is the subgroup generated by its minimal normal subgroups. These form a direct product.

Lemma II.32. Let G be a finite group. Then the following statements are equivalent.

(i) The socle $\mathrm{soc}(G)$ has no abelian factor.
(ii) The solvable radical $\mathcal{O}_\infty(G)$ is trivial.
(iii) The nilpotent radical $\mathrm{Fit}(G)$ is trivial.

Proof. If $\mathrm{soc}(G)$ has no abelian factor then $\mathcal{O}_\infty(G)$ is trivial: A non-trivial solvable normal subgroup of G would contain a solvable minimal normal subgroup of G which is necessarily abelian. Hence (i) implies (ii). Statement (ii) implies (iii) by definition. Finally, if $\mathrm{soc}(G)$ has an abelian factor then G has a (minimal) normal abelian and hence nilpotent subgroup. Thus (iii) implies (i). □

Proposition II.33. Let $F \leq \mathrm{Sym}(\Omega)$ be primitive non-regular and assume that F_ω has trivial nilpotent radical for all $\omega \in \Omega$. Then $\dim_{\mathrm{CD}}(F) = 2$.

Proof. Suppose that $F^{(2)} \leq \mathrm{Aut}(B_{d,2})$ has (C) and that

$$1 \to \ker\pi \to F^{(2)} \xrightarrow{\pi} F \to 1$$

is exact. Fix $\omega_0 \in \Omega$. Then $\ker\pi \leq \prod_{\omega\in\Omega} F_\omega \cong F_{\omega_0}^d$. Since $F^{(2)}$ has (C) we get $\mathrm{pr}_\omega \ker\pi \trianglelefteq F_{\omega_0}$ for all $\omega \in \Omega$. Since F is transitive these projections furthermore coincide with the same $N \trianglelefteq F_{\omega_0}$. Now consider $F^{(2)}_{T_\omega} = \ker \mathrm{pr}_\omega|_{\ker\pi} \trianglelefteq \ker\pi$ for some $\omega \in \Omega$. Either $F^{(2)}_{T_\omega}$ is trivial in which case $F^{(2)}$ has (CD) or $F^{(2)}_{T_\omega}$ is non-trivial. In the latter case, suppose that $N_{\omega,\omega'} := \mathrm{pr}_{\omega'} F^{(2)}_{T_\omega}$ is non-trivial for some $\omega' \in \Omega$. Then $N_{\omega,\omega'}$ is subnormal in F_{ω_0} as $\{\mathrm{id}\} \neq N_{\omega,\omega'} \trianglelefteq N \trianglelefteq F_{\omega_0}$. As a consequence, $N_{\omega,\omega'}$ has trivial nilpotent radical since F_{ω_0} does. Hence the Thompson-Wielandt Theorem [**Tho70**], [**Wie71**] (cf. [**BM00a**, Theorem 2.1.1]) implies that there is no $F^{(k)} \leq \mathrm{Aut}(B_{d,k})$ ($k \geq 3$) with $\pi_2 F^{(k)} = F^{(2)}$ and (CD). Therefore $\dim_{\mathrm{CD}}(F) \leq 2$. Lemma II.31 implies that equality holds. □

We now list several classes of permutation groups that Proposition II.33 includes; see [**LPS88**] for a statement of the O'Nan-Scott classification theorem of finite primitive groups to which the following types refer.

(i) A_n, S_n ($n \geq 6$) acting on $\{1, \ldots, n\}$ (which are of almost simple type (AS)).
(ii) Primitive groups of twisted wreath type (TW).
(iii) Primitive groups of type (HS).

This follows from combining Lemma II.32 with the following observations: For every $F \in \{A_n, S_n \mid n \geq 6\}$, point stabilizers have socle isomorphic to the simple non-abelian group A_{n-1}. Point stabilizers in primitive groups of type (TW) have trivial solvable radical by [**DM96**, Theorem 4.7B], and point stabilizers in primitive groups of type (HS) have simple non-abelian socle, see [**LPS88**].

Example II.34. By Example II.28, we have $\dim_{\mathrm{CD}}(S_3) \geq 3$ and it was shown in [**DM80**] that in fact $\dim_{\mathrm{CD}}(S_3) = 3$. Computationally constructing involutive compatibility cocycles one can show that $\dim_{\mathrm{CD}}(F) \geq 3$ for the dihedral groups $F \in \{D_4, D_6\}$ and their natural permutation actions.

To contrast the primitive case, we show that non-trivial transitive wreath products have dimension at least 3. The proof illustrates the use of involutive compatibility cocycles. Recall that for $F \leq \mathrm{Sym}(\Omega)$ and $P \leq \mathrm{Sym}(\Lambda)$ the wreath product $F \wr P := F^{|\Lambda|} \rtimes P$ admits a natural imprimitive permutation action on $\Omega \times \Lambda$ given by $((a_\lambda)_\lambda, \sigma) \cdot (\omega, \lambda') := (a_{\sigma(\lambda')}\omega, \sigma\lambda')$ with blocks $\Omega \times \Lambda = \bigsqcup_{\lambda \in \Lambda} \Omega \times \{\lambda\}$.

Proposition II.35. Let Ω and Λ be finite sets such that $|\Omega|, |\Lambda| \geq 2$. Furthermore, let $F \leq \mathrm{Sym}(\Omega)$ and $P \leq \mathrm{Sym}(\Lambda)$ be transitive. Then $\dim_{\mathrm{CD}}(F \wr P) \geq 3$.

Proof. We define a subgroup $W(F,P) \leq \mathrm{Aut}(B_{\Omega\times\Lambda,2})$ which projects onto $F \wr P$, satisfies (C), does not satisfy (CD) but admits an involutive compatibility cocycle. This suffices by Lemma II.20. For $\lambda \in \Lambda$, let ι_λ denote the λ-th embedding of F into $F \wr P = \left(\prod_{\lambda\in\Lambda} F\right) \rtimes P$. Recall the map γ from Section 3.1 and consider

$$\begin{aligned} \gamma_\lambda &: F \to \mathrm{Aut}(B_{\Omega\times\Lambda,2}),\ a \mapsto (\iota_\lambda(a), ((\iota_\lambda(a))_{(\omega,\lambda)}, (\mathrm{id})_{(\omega,\lambda'\neq\lambda)})), \\ \gamma_\lambda^{(2)} &: F \to \mathrm{Aut}(B_{\Omega\times\Lambda,2}),\ a \mapsto (\mathrm{id}, ((\mathrm{id})_{(\omega,\lambda)}, (\iota_\lambda(a))_{(\omega,\lambda'\neq\lambda)})). \end{aligned}$$

Furthermore, let ι denote the embedding of P into $F \wr P$. We define

$$W(F,P) := \langle \gamma_\lambda(a), \gamma_\lambda^{(2)}(a), \gamma(\iota(\varrho)) \mid \lambda \in \Lambda,\ a \in F,\ \varrho \in P\rangle.$$

In order to show that $W(F,P)$ admits an involutive compatibility cocycle, we first determine its group structure. Consider the subgroups

$$V := \langle \gamma_\lambda(a) \mid \lambda \in \Lambda,\ a \in F\rangle \quad \text{and} \quad \overline{V} := \langle \gamma_\lambda^{(2)}(a) \mid \lambda \in \Lambda,\ a \in F\rangle.$$

Then $W(F,P) = \langle V, \overline{V}, \Gamma(\iota(P))\rangle$. Now observe that $V \cong F^{|\Lambda|}$ and $\overline{V} \cong F^{|\Lambda|}$ commute, intersect trivially and are normalized by $\Gamma(\iota(P))$ which permutes the factors of each product. Therefore

$$W(F,P) \cong (V \times \overline{V}) \rtimes P \cong (F^{|\Lambda|} \times F^{|\Lambda|}) \rtimes P.$$

An involutive compatibility cocycle z of $W(F,P)$ may now be defined by setting

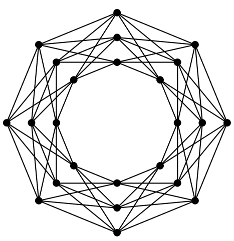
$$z(\gamma_\lambda(a), (\omega,\lambda')) := \begin{cases} \gamma_\lambda(a) & \lambda = \lambda' \\ \gamma_\lambda^{(2)}(a) & \lambda \neq \lambda' \end{cases},\ z(\gamma_\lambda^{(2)}(a), (\omega,\lambda')) := \begin{cases} \gamma_\lambda^{(2)}(a) & \lambda = \lambda' \\ \gamma_\lambda(a) & \lambda \neq \lambda' \end{cases}$$

for all $\lambda \in \Lambda$, $a \in F$ and $\varrho \in P$ and $z(\gamma(\iota(\varrho)), (\omega,\lambda)) := \gamma(\iota(\varrho))$. Note that the map z extends to an involutive compatibility cocycle of $V \times \overline{V} \leq W(F,P)$ which in turn extends to $W(F,P)$. □

Actually, much more than Proposition II.35 holds true for particular wreath products. For instance, there is the following well-known construction, c.f. [**MSV14**].

Proposition II.36. Let $m \geq 2$. Then $\dim_{\mathrm{CD}}(S_m \wr S_2) = \infty$.

Proof. We give a family of $2m$-regular finite graphs $(\Gamma_n)_{n\geq 3}$ whose automorphism groups yield amalgams with the right properties: Let $C(m,n)$ be the graph with vertex set $\{1,\ldots,m\} \times \{1,\ldots,n\}$ where (i,j) is connected to (i',j') via an edge if and only $j' \in \{j \pm 1\}$ (cyclically). For example, $C(3,8)$ is given below.



Then $G^{m,n} := \mathrm{Aut}(C(m,n)) \cong S_m \wr D_n$. If (v,w) is any edge of $C(m,n)$ then the vertex stabilizer $G_v^{m,n} \cong S_m^{n-1} \rtimes S_2$ has the 1-local action $S_m^2 \rtimes S_2 = S_m \wr S_2$. Furthermore, the subgroup $D_n \leq G^{m,n}$ provides an involutive inversion of (v,w). Via the coset construction, the amalgam

$$G_v^{m,n} \underset{G_{(v,w)}^{m,n}}{*} G_{\{v,w\}}^{m,n}$$

yields a discrete group $\widetilde{G}^{m,n}$ acting vertex-transitively on $T_{2m} = (V,E)$ with local action $S_m \wr S_2$ and an involutive inversion. Let $(x,y) \in E(T_{2m})$ lie over (v,w). Then $|\widetilde{G}_x^{m,n}| = |G_v^{m,n}|$ tends to infinity as n does. Thus $\dim_{\mathrm{CD}}(S_m \wr S_2) = \infty$. □

6. A Bipartite Version

We now present a bipartite version of the universal groups introduced in Section 1. It plays a critical role in the proof of Theorem II.41 below. Retain the notation of Section 1, let $V = V_1 \sqcup V_2$ be a regular bipartition of $V(T_d)$, and $b \in V_1$.

6.1. Definition and Basic Properties. The groups to be defined are subgroups of ${}^+\mathrm{Aut}(T_d) \leq \mathrm{Aut}(T_d)$, the maximal subgroup of $\mathrm{Aut}(T_d)$ preserving the bipartition $V = V_1 \sqcup V_2$. Alternatively, it can be described as the subgroup generated by all point stabilizers, or all edge-stabilizers.

Definition II.37. Let $F^{(2k)} \leq \mathrm{Aut}(B_{d,2k})$. Define

$$\mathrm{BU}_{2k}(F^{(2k)}) := \{\alpha \in {}^+\mathrm{Aut}(T_d) \mid \forall v \in V_1(T_d): \ \sigma_{2k}(\alpha, v) \in F^{(2k)}\}.$$

Note that $\mathrm{BU}_{2k}(F^{(2k)})$ is a subgroup of ${}^+\mathrm{Aut}(T_d)$ thanks to Lemma II.2 and the assumption that it is a subset of ${}^+\mathrm{Aut}(T_d)$.

As before, $\mathrm{BU}_{2k}(F^{(2k)})$ is a closed subgroup of $\mathrm{Aut}(T_d)$ and transitive on both V_1 and V_2. We also recover compact generation and thereby the following.

Lemma II.38. Let $F^{(2k)} \leq \mathrm{Aut}(B_{d,2k})$. Then $\mathrm{BU}_{2k}(F^{(2k)})$ is a compactly generated, totally disconnected locally compact group.

Proof. The group $\mathrm{BU}_{2k}(F^{(2k)})$ is totally disconnected and locally compact as a closed subgroup of $\mathrm{Aut}(T_d)$. Compact generation relies on the Lemma II.39 below, showing that $\mathrm{BU}_2(\{\mathrm{id}\}) = \mathrm{U}_1(\{\mathrm{id}\}) \cap {}^+\mathrm{Aut}(T_d)$ is finitely generated. Given that it is also transitive on V_1 (and V_2) we conclude that $\mathrm{BU}_{2k}(F^{(2k)})$ is compactly generated by $\mathrm{BU}_{2k}(F^{(2k)})_b$ and the finite generating set of the V_1-transitive group $\mathrm{BU}_2(\{\mathrm{id}\})$ given in Lemma II.39. □

Given $v \in V(T_d)$ and $w \in \Omega^{(2)}$, let $t_w^{(v)} \in \mathrm{Aut}(T_d)$ denote the unique label-preserving translation with $t_w^{(v)}(v) = v_w$.

Lemma II.39. The group $\mathrm{BU}_2(\{\mathrm{id}\})$ is finitely generated by $\{t_w^{(b)} \mid w \in \Omega^{(2)}\}$.

Proof. Argue by induction on $k \in \mathbb{N}$ that b can be mapped to b_w for any $w \in \Omega^{(2k)}$ by a unique element of $\langle\{t_w \mid w \in \Omega^{(2)}\}\rangle \leq \mathrm{U}_1(\{\mathrm{id}\}) \cap {}^+\mathrm{Aut}(T_d)$, using the fact that each t_w is label-preserving.

Now, let $h \in \mathrm{U}_1(\{\mathrm{id}\}) \cap {}^+\mathrm{Aut}(T_d)$ be non-trivial. Since ${}^+\mathrm{Aut}(T_d) = \mathrm{Aut}(T_d)^+$, the element h is hyperbolic of even length. Pick $v \in V_1$ on the axis of h. Then there is $t \in \langle\{t_w \mid w \in \Omega^{(2)}\}\rangle$ such that $t(b) = v$ and $t^{-1}ht$ is a hyperbolic element whose axis contains b. Thus $t^{-1}ht \in \langle\{t_w \mid w \in \Omega^{(2)}\}\rangle$ by the above and so is h. □

6.2. Compatibility and Discreteness. In order to describe the compatibility and discreteness condition in the bipartite setting, we first introduce a workable realization of $\mathrm{Aut}(B_{d,2k})$ ($k \in \mathbb{N}$), similar to the one given at the beginning of Section 3. Let $\mathrm{Aut}(B_{d,1}) \cong \mathrm{Sym}(\Omega)$ and $\mathrm{Aut}(B_{d,2})$ be as before. For $k \geq 2$, we inductively identify $\mathrm{Aut}(B_{d,2k})$ with its image under

$$\mathrm{Aut}(B_{d,2k}) \to \mathrm{Aut}(B_{d,2(k-1)}) \ltimes \prod_{w \in \Omega^{(2)}} \mathrm{Aut}(B_{d,2(k-1)})$$

$$\alpha \mapsto (\sigma_{2(k-1)}(\alpha, b), (\sigma_{2(k-1)}(\alpha, b_w))_w))$$

where $\mathrm{Aut}(B_{d,2(k-1)})$ acts on $\Omega^{(2)}$ by permuting factors according to its action on $S(b,2) \cong \Omega^{(2)}$. In addition, consider the map $\mathrm{pr}_w : \mathrm{Aut}(B_{d,2k}) \to \mathrm{Aut}(B_{d,2(k-1)})$, $\alpha \mapsto \sigma_{2(k-1)}(\alpha, b_w)$ for every $w \in \Omega^{(2)}$, as well as

$$p_w : \mathrm{Aut}(B_{d,2k}) \to \mathrm{Aut}(B_{d,2(k-1)}) \times \mathrm{Aut}(B_{d,2(k-1)}), \ \alpha \mapsto (\pi_{2(k-1)}(\alpha), \mathrm{pr}_w(\alpha))$$

For $k \geq 2$, conditions (C) and (D) for $F \leq \mathrm{Aut}(B_{d,2k})$ now read as follows.

(C) $\forall \alpha \in F \ \forall w \in \Omega^{(2)} \ \exists \alpha_w \in F: \ \pi_{2(k-1)}(\alpha_w) = \mathrm{pr}_w(\alpha), \ \mathrm{pr}_{\overline{w}}(\alpha_w) = \pi_{2(k-1)}(\alpha)$

(D) $\forall w \in \Omega^{(2)}: \ p_w|_F^{-1}(\mathrm{id}, \mathrm{id}) = \{\mathrm{id}\}$

For $k = 1$ we have, using the maps p_ω $(\omega \in \Omega)$ as in Section 3,

(C) $\forall \alpha \in F \ \forall w = (\omega_1, \omega_2) \in \Omega^{(2)} \ \exists \alpha_w \in F: \ \mathrm{pr}_{\omega_2}(\alpha_w) = \mathrm{pr}_{\omega_1} \alpha.$

(D) $\forall \omega \in \Omega: \ p_\omega|_F^{-1}(\mathrm{id}, \mathrm{id}) = \{\mathrm{id}\}.$

The discreteness conditions are proven as in Proposition II.12. We do not introduce new notation for any of the above as the context always implies which condition is to be considered. The definition of the compatibility sets $C_F(\alpha, S)$ for $F \leq \mathrm{Aut}(B_{d,2k})$ and $S \subseteq \Omega^{(2)}$ carries over from Section 2 in a straightforward fashion.

Similar to the non-bipartite case, given $F \leq \mathrm{Aut}(B_{d,2k})$ with (C), we set

$$\Psi_{2k}(F) := \{(\alpha, (\alpha_w)_{w \in \Omega^{(2)}}) \mid \alpha \in F, \ \forall w \in \Omega^{(2)}: \ \alpha_w \in C_F(\alpha, w)\} \leq \mathrm{Aut}(B_{d,2(k+1)}).$$

Then $\Psi_{2k}(F) \leq \mathrm{Aut}(B_{d,2(k+1)})$ satisfies (C) and $\mathrm{BU}_{2(k+1)}(\Psi_{2k}(F)) = \mathrm{BU}_{2k}(F)$. Given $l > k$, we also set $\Psi^{2l}(F) := \Psi_{2(l-1)} \circ \cdots \circ \Psi_{2k}(F)$, c.f. Section 3.2.

More examples of bipartite universal groups are contained in Section 7.5 below.

7. Non-Trivial Quasi-Centers

We now apply the framework of universal groups to the study of subgroups of $\mathrm{Aut}(T_d)$ with non-trivial quasi-center, motivated by Burger–Mozes theory as outlined in Section 3 of Chapter I and questions about lattices in products of trees as studied in [**BM00b**] and [**Rat04**], specifically [**Rat04**, Conjecture 2.63].

The discreteness assertion of part (ii) in Theorem I.9 follows from the fact that a non-discrete locally quasiprimitive subgroup of $\mathrm{Aut}(T_d)$ cannot contain any non-trivial quasi-central elliptic elements by [**BM00a**, Proposition 1.2.1]. We now complete this fact to the following local-to-global type characterization of the quasi-central elements a subgroup of $\mathrm{Aut}(T_d)$ can cointain in terms of its local action.

Theorem II.40. Let $H \leq \mathrm{Aut}(T_d)$ be non-discrete. If H is locally

(i) transitive then $\mathrm{QZ}(H)$ contains no inversion.
(ii) semiprimitive then $\mathrm{QZ}(H)$ contains no non-trivial edge-fixating element.
(iii) quasiprimitive then $\mathrm{QZ}(H)$ contains no non-trivial elliptic element.
(iv) k-transitive ($k \in \mathbb{N}$) then $\mathrm{QZ}(H)$ contains no hyperbolic element of length k.

The assertions of Theorem II.40 are sharp in the following sense.

Theorem II.41. There is a closed, non-discrete, compactly generated subgroup of $\mathrm{Aut}(T_d)$ which is locally

(i) intransitive and contains a quasi-central inversion.
(ii) transitive and contains a non-trivial quasi-central edge-fixating element.
(iii) semiprimitive and contains a non-trivial quasi-central elliptic element.
(iv) (a) intransitive and contains a quasi-central hyperbolic element of length 1.
(b) quasiprimitive and contains a quasi-central hyperbolic element of length 2.

Proof. (Theorem II.40). Fix a labelling of T_d and let $H \leq \mathrm{Aut}(T_d)$ be non-discrete.

For (i), assume that H is locally transitive and $\iota \in \mathrm{QZ}(H)$ inverts the edge $(b, b_\omega) \in E(T_d)$. By definition, the centralizer of ι in H is open. Hence there is $n \in \mathbb{N}$ such that $H_{B(b,n)}$ commutes with ι. Thus for all $h \in H_{B(b,n)}$ and $k \in \mathbb{N}$:

$$\begin{aligned}\sigma_k(\iota, b)\sigma_k(h, b) &= \sigma_k(\iota, hb)\sigma_k(h, b) = \sigma_k(\iota h, b) \\ &= \sigma_k(h\iota, b) = \sigma_k(h, \iota b)\sigma_k(\iota, b) = \sigma_k(h, b_\omega)\sigma_k(\iota, b).\end{aligned}$$

Therefore, $\sigma_k(h, b_\omega) = \sigma_k(\iota, b)\sigma_k(h, b)\sigma_k(\iota, b)^{-1}$ for all $k \in \mathbb{N}$. Now, since H is non-discrete, we may assume without loss of generality that $H_{B(b,n)}$ acts non-trivially on $B(b, n+1)$. Let $h' \in H_{B(b,n)} \backslash H_{B(b,n+1)}$. Then there is $\omega' \in \Omega$ with $\sigma_n(h', b_{\omega'}) \neq \mathrm{id}$. Furthermore, since H is locally transitive, there is $g \in H_b$ with $g^{-1}b_\omega = b_{\omega'}$. For the element $gh'g^{-1} \in H_{B(b,n)}$ we have $\sigma_n(gh'g^{-1}, b) = \mathrm{id}$ but

$$\begin{aligned}\sigma_n(gh'g^{-1}, b_\omega) &= \sigma_n(g, h'g^{-1}b_\omega)\sigma_n(h', g^{-1}b_\omega)\sigma_n(g^{-1}, b_\omega)\\ &= \sigma_n(g, g^{-1}b_\omega)\sigma_n(h', b_{\omega'})\sigma_n(g^{-1}, b_\omega)\\ &= \sigma_n(g, g^{-1}b_\omega)\sigma_n(h', b_{\omega'})\sigma_n(g, g^{-1}b_\omega)^{-1} \neq \mathrm{id}\end{aligned}$$

because $\sigma_n(h', b_{\omega'}) \neq \mathrm{id}$ by assumption. This contradicts the assumption that ι commutes with $H_{B(b,n)}$ elaborated above. Hence the assertion.

Part (ii) is based on a variation of [**BM00a**, Lemma 1.4.2] given in Proposition II.42 below and the observation [**BM00a**, 1.3.5] according to which a non-discrete group $H \leq \mathrm{Aut}(T_d)$ cannot have cofinite quasi-center. Hence part (i) of Proposition II.42 applies and $\mathrm{QZ}(H)$ acts freely on $E(T_d)$.

Part (iii) follows from [**BM00a**, Lemma 1.4.2] and [**BM00a**, 1.3.5]. The closedness assumption of [**BM00a**, Proposition 1.2.1] is unnecessary for its second part.

For part (iv), assume that H is locally k-transitive and that $\tau \in \mathrm{QZ}(H)$ is a translation of length k. Let $b \in V$ be a vertex on the axis of τ. Then $\tau b = b_w$ for some path $w = (\omega_1, \ldots, \omega_k) \in \Omega^{(k)}$. By definition, the centralizer of τ in H is open. Hence there is $n \in \mathbb{N}_{\geq k}$ such that $H_{B(b,n)}$ commutes with τ. Thus for all $h \in H_{B(b,n)}$ and $l \in \mathbb{N}$:

$$\begin{aligned}\sigma_l(\tau, b)\sigma_l(h, b) &= \sigma_l(\tau, hb)\sigma_l(h, b) = \sigma_l(\tau h, b)\\ &= \sigma_l(h\tau, b) = \sigma_l(h, \tau b)\sigma_l(\tau, b) = \sigma_l(h, b_w)\sigma_l(\tau, b).\end{aligned}$$

Therefore, $\sigma_l(h, b_w) = \sigma_l(\tau, b)\sigma_l(h, b)\sigma_l(\tau, b)^{-1}$ for all $l \in \mathbb{N}$. Now, since H is non-discrete, there is $m \in \mathbb{N}_{\geq n}$ such that $H_{B(b,m)}$ acts non-trivially on $B(b, m+1)$. Let $h' \in H_{B(b,m)} \backslash H_{B(b,m+1)}$ and define l via $k + l = m + 1$. Then there is $w' \in \Omega^{(k)}$ such that $\sigma_l(h', b_{w'}) \neq \mathrm{id}$. Furthermore, since H is locally k-transitive there is $g \in H_b$ with $g^{-1}b_{w'} = b_w$. Then $gh'g^{-1} \in H_{B(b,m)}$ satisfies $\sigma_l(gh'g^{-1}, b) = \mathrm{id}$ but

$$\begin{aligned}\sigma_l(gh'g^{-1}, b_w) &= \sigma_l(g, h'g^{-1}b_w)\sigma_l(h', g^{-1}b_w)\sigma_l(g^{-1}, b_w)\\ &= \sigma_l(g, g^{-1}b_w)\sigma_l(h', b_{w'})\sigma_l(g^{-1}, b_w)\\ &= \sigma_l(g, g^{-1}b_w)\sigma_l(h', b_{w'})\sigma_l(g, g^{-1}b_w)^{-1} \neq \mathrm{id}\end{aligned}$$

because $\sigma_l(h', b_{w'}) \neq \mathrm{id}$ by assumption. This contradicts the assumption that τ commutes with $H_{B(b,m)} \leq H_{B(b,n)}$ elaborated above. Hence the assertion. □

The following result referenced to in the proof of Theorem II.40 generalizes [**BM00a**, Proposition 1.4.2] to semiprimitive actions.

Proposition II.42. Let $H \leq \mathrm{Aut}(T_d)$ be locally semiprimitive and $N \trianglelefteq H$. Define

$$V_1(N) := \{x \in V(T_d) \mid N_x \curvearrowright S(x,1) \text{ is transitive and not semiregular}\}$$
$$V_2(N) := \{x \in V(T_d) \mid N_x \curvearrowright S(x,1) \text{ is semiregular}\}.$$

Then one of the following holds.

(i) $V(T_d) = V_2(N)$ and N acts freely on $E(T_d)$.
(ii) $V(T_d) = V_1(N)$ and N acts transitively on the set of geometric edges of T_d.
(iii) $V(T_d) = V_1(N) \sqcup V_2(N)$ is an H-invariant bipartition of $V(T_d)$ and $B(x,1)$ is a fundamental domain for the action of N on T_d for any $x \in V_2(N)$.

Proof. Since H is locally semiprimitive, we have $V(T_d) = V_1(N) \sqcup V_2(N)$. If N does not act freely on $E(T_d)$ then there is an edge $e \in E(T_d)$ with $N_e \neq \{\mathrm{id}\}$ and consequently an N_e-fixed vertex $x \in V(T_d)$ for which $N_x \curvearrowright S(x,1)$ is not semiregular and hence transitive. Then $V_1(N) \neq \emptyset$. Now, either $V_2(N) = \emptyset$ in which case N is locally transitive and we are in case (ii), or $V_2(N) \neq \emptyset$. Being locally transitive, H acts transitively on the set of geometric edges it thus has at most two orbits in $V(T_d)$. Given that both $V_1(N)$ and $V_2(N)$ are non-empty and H-invariant, they constitute exactly said orbits. Since any pair of adjacent vertices (x, y) is a fundamental domain for the H-action on $V(T_d)$, we conclude that if $y \in V_2(N)$ then $x \in V_1(N)$. Thus every leaf of $B(y, 1)$ is in $V_1(N)$ and we are in case (iii) by [**BM00a**, 1.3.1]. □

We also include the natural generalization of [**BM00a**, Proposition 1.2.1 3)].

Corollary II.43. Let $H \leq \mathrm{Aut}(T_d)$ be locally semiprimitive and $N \trianglelefteq H$ closed. Then either N is discrete and $N \leq \mathrm{QZ}(H)$, or N is cocompact and $H^{(\infty)} \leq N$.

Proof. By Proposition II.42, the closed normal subgroup N of H is either discrete or cocompact. The assertion hence follows from the definitions and the fact that every discrete normal subgroup of a topological group is central. □

Before proceeding to the proof of Theorem II.41, we complement part (iv) of Theorem II.40 with the following result inspired by [**BM00a**, Proposition 3.1.2] and [**Rat04**, Conjecture 2.63].

Proposition II.44. Let $H \leq \mathrm{Aut}(T_d)$ be non-discrete and locally semiprimitive. If all orbits of $H \curvearrowright \partial T_d$ are uncountable then $\mathrm{QZ}(H)$ contains no hyperbolic elements.

Proof. Let $S \subseteq \partial T_d$ be the collection of fixed points of hyperbolic elements in $\mathrm{QZ}(H)$. Since $\mathrm{QZ}(H) \trianglelefteq H$, the set S is H-invariant. Also, $\mathrm{QZ}(H)$ is discrete by Theorem II.40 and therefore countable as a subgroup of the second-countable group H which inherits second-countability from $\mathrm{Aut}(T_d)$. We conclude that S is countable and therefore empty in view of the assumption. □

Theorem II.41 is proven by construction in the consecutive sections. Whereas parts (i) to (iv) (a) all rely on a construction of the form $H := \bigcap_{k \in \mathbb{N}} \mathrm{U}_k(F^{(k)})$ for appropriate local actions $F^{(k)} \leq \mathrm{Aut}(B_{d,k})$, part (iv) (b) utilizes the bipartite version of the universal groups developed in Section 6. All sections appear similar at first glance but vary in detail.

7.1. Theorem II.41 (i). For certain intransitive $F \leq \mathrm{Sym}(\Omega)$ we construct a group $H(F) \leq \mathrm{Aut}(T_d)$ which is closed, non-discrete, compactly generated, vertex-transitive, locally acts like F and contains a quasi-central involutive inversion.

Let $F \leq \mathrm{Sym}(\Omega)$. Assume that the partition $F \backslash \Omega = \bigsqcup_{i \in I} \Omega_i$ of Ω into F-orbits has at least three elements and $F_{\Omega_i} \neq \{\mathrm{id}\}$ for all $i \in I$.

Fix an orbit Ω_0 of size at least 2 and $\omega_0 \in \Omega_0$. Define actions $F^{(k)} \leq \mathrm{Aut}(B_{d,k})$ for $k \in \mathbb{N}$ inductively by $F^{(1)} := F$ and

$$F^{(k+1)} := \{(\alpha, (\alpha_\omega)_\omega) \mid \alpha \in F^{(k)},\ \alpha_\omega \in C_{F^{(k)}}(\alpha, \omega) \text{ is constant w.r.t. } F \backslash \Omega,\ \alpha_{\omega_0} = \alpha\}.$$

Proposition II.45. The actions $F^{(k)} \leq \mathrm{Aut}(B_{d,k})$ $(k \in \mathbb{N})$ defined above satisfy:

(i) Every $\alpha \in F^{(k)}$ is self-compatible in directions from Ω_0.
(ii) The compatibility set $C_{F^{(k)}}(\alpha, \Omega_i)$ is non-empty for all $\alpha \in F^{(k)}$ and $i \in I$. In particular, the group $F^{(k)}$ satisfies (C).
(iii) The compatibility set $C_{F^{(k)}}(\mathrm{id}, \Omega_i)$ is non-trivial for all $\Omega_i \neq \Omega_0$. In particular, the group $F^{(k)}$ does not satisfy (D).

Proof. We prove all three properties simultaneously by induction: For $k = 1$, the assertions (i) and (ii) are trivial. The third translates to F_{Ω_i} being non-trivial for all $\Omega_i \neq \Omega_0$ which is an assumption. Now, assume that all properties hold for $F^{(k)}$. Then the definition of $F^{(k+1)}$ is meaningful because of (i) and it is a subgroup of $\mathrm{Aut}(B_{d,k+1})$ because F preserves $F\backslash\Omega$. Assertion (i) is now evident. Statements (ii) carries over from $F^{(k)}$ to $F^{(k+1)}$. So does (iii) since $|F\backslash\Omega| \geq 3$. □

Definition II.46. Retain the above notation. Define $H(F) := \bigcap_{k\in\mathbb{N}} \mathrm{U}_k(F^{(k)})$.

The group $H(F)$ is vertex-transitive, compactly generated and contains an involutive inversion because $\mathrm{U}_1(\{\mathrm{id}\}) \leq H(F)$. Also, $H(F)$ is closed as an intersection of closed sets. The 1-local action of H is given by $F = F^{(1)}$ because $\mathrm{D}(F) \leq H(F)$.

Lemma II.47. Let F be as above. Then $H(F)$ is non-discrete.

Proof. A non-trivial element $h \in H(F)$ fixing $B(b,n)$ for a given $n \in \mathbb{N}$ is readily constructed using Proposition II.45: Consider $\alpha_n := \mathrm{id} \in F^{(n)}$. By parts (i) and (iii) of Proposition II.45 as well as the definition of $F^{(n+1)}$, there is a non-trivial element $\alpha_{n+1} \in F^{(n+1)}$ with $\pi_n\alpha_{n+1} = \alpha_n$. Applying parts (i) and (ii) of Proposition II.45 repeatedly, we obtain non-trivial elements $\alpha_k \in F^{(k)}$ for all $k \geq n+1$ with $\pi_k\alpha_{k+1} = \alpha_k$ for all $k \geq n+1$. Set $\alpha_k := \mathrm{id} \in F^{(k)}$ for all $k \leq n$ and define $h \in \mathrm{Aut}(T_d)_b$ by fixing b and setting $\sigma_k(h,b) := \alpha_k \in F^{(k)}$. Since $F^{(l)} \leq \Phi^l(F^{(k)})$ for all $k \leq l$ we conclude that $h \in \bigcap_{k\in\mathbb{N}} \mathrm{U}_k(F^{(k)}) = H(F)$. □

Proposition II.48. Let F be as above. Then $\mathrm{QZ}(H(F))$ contains an involutive inversion.

Proof. Fix $b \in V(T_d)$. We show that $\mathrm{QZ}(H(F))$ contains the label-preserving inversion ι_ω of the edge (b, b_ω) for all $\omega \in \Omega_0$: Indeed, let $h \in H(F)_{B(b,1)}$ and $\omega \in \Omega_0$. Then $h\iota_\omega(b) = b_\omega = \iota_\omega h(b)$ and

$$\sigma_k(h\iota_\omega, b) = \sigma_k(h, \iota_\omega b)\sigma_k(\iota_\omega, b) = \sigma_k(h, b_\omega) = \sigma_k(\iota_\omega, hb)\sigma_k(h,b) = \sigma_k(\iota_\omega h, b)$$

for all $k \in \mathbb{N}$ since $h \in \mathrm{U}_{k+1}(F^{(k+1)})$. That is, ι_ω commutes with $H(F)_{B(b,1)}$. □

7.2. Theorem II.41 (ii). For certain transitive $F \leq \mathrm{Sym}(\Omega)$ we construct a group $H(F) \leq \mathrm{Aut}(T_d)$ which is closed, non-discrete, compactly generated, vertex-transitive, locally acts like F and has non-discrete quasi-center.

Let $F \leq \mathrm{Sym}(\Omega)$ be transitive. Assume that F preserves a non-trivial partition $\mathcal{P} = (\Omega_i)_{i\in I}$ of Ω and $F_{\Omega_i} \neq \{\mathrm{id}\}$ for all $i \in I$. Further, suppose that F^+ is abelian and preserves $\mathcal{P}$ setwise.

Example II.49. Let $F' \leq \mathrm{Sym}(\Omega)$ be regular abelian and $P \leq \mathrm{Sym}(\Lambda)$ be regular. Then $F := F' \wr P \leq \mathrm{Sym}(\Omega \times \Lambda)$ satisfies the above properties as $F^+ = \prod_{\lambda\in\Lambda} F'$.

Define actions $F^{(k)} \leq \mathrm{Aut}(B_{d,k})$ for $k \in \mathbb{N}$ inductively by $F^{(1)} := F$ and

$$F^{(k+1)} := \{(\alpha, (\alpha_\omega)_\omega) \mid \alpha \in F^{(k)},\ \alpha_\omega \in C_{F^{(k)}}(\alpha,\omega) \text{ constant w.r.t. } \mathcal{P}\}$$

for all $k \in \mathbb{N}$. Then we have the following.

Proposition II.50. The actions $F^{(k)} \leq \mathrm{Aut}(B_{d,k})$ $(k \in \mathbb{N})$ defined above satisfy:

(i) The compatibility set $C_{F^{(k)}}(\alpha, \Omega_i)$ is non-empty for all $\alpha \in F^{(k)}$ and $i \in I$. In particular, the group $F^{(k)}$ satisfies (C).
(ii) The compatibility set $C_{F^{(k)}}(\mathrm{id}, \Omega_i)$ is non-trivial for all $i \in I$. In particular, the group $F^{(k)}$ does not satisfy (D).
(iii) The group $F^{(k)} \cap \Phi^k(F^+)$ is abelian.

Proof. We prove all three properties simultaneously by induction: For $k = 1$, assertion (i) is trivial whereas (iii) is an assumption. The second translates to F_{Ω_i} being non-trivial for all $i \in I$ which is an assumption. Now, assume that all properties hold for $F^{(k)}$. Then the definition of $F^{(k+1)}$ is meaningful because of (i) and it is a subgroup of $\mathrm{Aut}(B_{d,k})$ because F preserves $\mathcal{P}$. Statement (ii) carries over from $F^{(k)}$ to $F^{(k+1)}$. Finally, (iii) follows inductively because F^+ preserves $\mathcal{P}$ setwise. □

Definition II.51. Retain the above notation. Define $H(F) := \bigcap_{k\in\mathbb{N}} \mathrm{U}_k(F^{(k)})$.

The group $H(F)$ is vertex-transitive, compactly generated and contains an involutive inversion because $\mathrm{U}_1(\{\mathrm{id}\}) \leq H(F)$. Also, $H(F)$ is closed as an intersection of closed sets. The 1-local action of H is given by $F = F^{(1)}$ because $\mathrm{D}(F) \leq H(F)$.

Lemma II.52. Let F be as above. Then $H(F)$ is non-discrete.

Proof. A non-trivial element $h \in H(F)$ fixing $B(b, n)$ for a given $n \in \mathbb{N}$ is readily constructed using Proposition II.50: Consider $\alpha_n := \mathrm{id} \in F^{(n)}$. By part (ii) of Proposition II.50 and the definition of $F^{(n+1)}$, there is a non-trivial $\alpha_{n+1} \in F^{(n+1)}$ with $\pi_n\alpha_{n+1} = \alpha_n$. Applying part (i) of Proposition II.50 repeatedly, we obtain non-trivial elements $\alpha_k \in F^{(k)}$ for all $k \geq n+1$ with $\pi_k\alpha_{k+1} = \alpha_k$ for all $k \geq n+1$. Set $\alpha_k := \mathrm{id} \in F^{(k)}$ for all $k \leq n$ and define $h \in \mathrm{Aut}(T_d)_b$ by fixing b and setting $\sigma_k(h, b) := \alpha_k \in F^{(k)}$. Because $F^{(l)} \leq \Phi^l(F^{(k)})$ for all $k \leq l$ we conclude that $h \in \bigcap_{k\in\mathbb{N}} \mathrm{U}_k(F^{(k)}) = H(F)$. □

Proposition II.53. Let F be as above. Then $\mathrm{QZ}(H(F))$ is non-discrete.

Proof. The group $H(F)_{B(b,1)}$ is a subgroup of the group $H(F^+)_b$ which is abelian by part (iii) of Proposition II.50. In other words, $\mathrm{QZ}(H(F))$ contains $H(F)_{B(b,1)}$ and is therefore non-discrete. □

Remark II.54. Without assuming local transitivity one can achieve abelian point stabilizers, following the construction of the previous section. This cannot happen for non-discrete locally transitive groups $H \leq \mathrm{Aut}(T_d)$ which are vertex-transitive as the following argument shows: By Proposition I.14, the group H is contained in $\mathrm{U}(F)$ where $F \leq \mathrm{Sym}(\Omega)$ is the local action of H. If H_b is abelian, then so is F. Since any transitive abelian permutation group is regular we conclude that $\mathrm{U}(F)$ and hence H are discrete. In this sense, the construction of this section is efficient.

7.3. Theorem II.41 (iii). For certain semiprimitive $F \leq \mathrm{Sym}(\Omega)$ we construct a group $H(F) \leq \mathrm{Aut}(T_d)$ which is closed, non-discrete, compactly generated, vertex-transitive, locally acts like F and whose quasi-center contains a non-trivial elliptic element.

Let $F \leq \mathrm{Sym}(\Omega)$ be semiprimitive. Assume that F preserves a non-trivial partition $\mathcal{P} : \Omega = \bigsqcup_{i\in I} \Omega_i$ of Ω. Further, suppose that $F_{\Omega_i} \neq \{\mathrm{id}\}$ for all $i \in I$ and that F contains a non-trivial central element τ which preserves $\mathcal{P}$ setwise.

Example II.55. Using the GAP library of small transitive groups [**GAP17**], consider e.g. $\mathrm{Tr}(8, 23) \cong \mathrm{GL}(2, 3)$ with block system $\{\{1, 5\}, \{2, 6\}, \{3, 7\}, \{4, 8\}\}$ and center $\langle(1, 5)(2, 6)(3, 7)(4, 8)\rangle$. It is semiprimitive and has non-trivial block fixators.

Example II.56. Transitive F satisfying the above assumptions can be constructed as follows. Let $F' \leq \mathrm{Sym}(\Omega')$ be transitive, non-regular with $Z(F') \neq \{\mathrm{id}\}$ and $P \leq \mathrm{Sym}(\Lambda)$ transitive for $|\Lambda| \geq 2$. Then $F := F' \wr P \leq \mathrm{Sym}(\Omega' \times \Lambda)$ preserves the partition $\Omega := \Omega' \times \Lambda = \bigsqcup_{\lambda\in\Lambda} \Omega'$ and any diagonal element with entry from $Z(F')$ does so setwise. The rest follows from the assumptions on F' and P.

Define actions $F^{(k)} \leq \mathrm{Aut}(B_{d,k})$ for $k \in \mathbb{N}$ inductively by $F^{(1)} := F$ and

$$F^{(k+1)} := \{(\alpha, (\alpha_\omega)_\omega) \mid \alpha \in F^{(k)},\ \alpha_\omega \in C_{F^{(k)}}(\alpha, \omega) \text{ constant w.r.t } \mathcal{P}\}$$

for all $k \in \mathbb{N}$. Then we have the following.

Proposition II.57. The actions $F^{(k)} \leq \mathrm{Aut}(B_{d,k})$ $(k \in \mathbb{N})$ defined above satisfy:

(i) The compatibility set $C_{F^{(k)}}(\alpha, \Omega_i)$ is non-empty for all $\alpha \in F^{(k)}$ and $i \in I$. In particular, the group $F^{(k)}$ satisfies (C).
(ii) The compatibility set $C_{F^{(k)}}(\mathrm{id}, \Omega_i)$ is non-trivial for all $i \in I$. In particular, the group $F^{(k)}$ does not satisfy (D).
(iii) The element $\gamma_k(\tau) \in \mathrm{Aut}(B_{d,k})$ is central in $F^{(k)}$.

Proof. We prove all three properties simultaneously by induction: For $k = 1$, assertion (i) is trivial whereas (iii) is an assumption. The second translates to F_{Ω_i} being non-trivial for all $i \in I$ which is an assumption. Now, assume that all properties hold for $F^{(k)}$. Then the definition of $F^{(k+1)}$ is meaningful because of (i) and it is a subgroup of $\mathrm{Aut}(B_{d,k+1})$ because F preserves $\mathcal{P}$. Statement (ii) carries over from $F^{(k)}$ to $F^{(k+1)}$. Finally, (iii) follows inductively because τ and hence τ^{-1} preserves $\mathcal{P}$ setwise: For $\widetilde{\alpha} = (\alpha, (\alpha_\omega)_\omega) \in F^{(k+1)}$ we have

$$\gamma_{k+1}(\tau)\widetilde{\alpha}\gamma_{k+1}(\tau)^{-1} = (\gamma_k(\tau)\alpha\gamma_k(\tau)^{-1}, (\gamma_k(\tau)\alpha_{\tau^{-1}(\omega)}\gamma_k(\tau)^{-1})_\omega). \qquad \square$$

Definition II.58. Retain the above notation. Define $H(F) := \bigcap_{k \in \mathbb{N}} \mathrm{U}_k(F^{(k)})$.

The group $H(F)$ is vertex-transitive, compactly generated and contains an involutive inversion because $\mathrm{U}_1(\{\mathrm{id}\}) \leq H(F)$. Also, $H(F)$ is closed as an intersection of closed sets. The 1-local action of H is given by $F = F^{(1)}$ because $\mathrm{D}(F) \leq H(F)$.

Lemma II.59. Let F be as above. Then $H(F)$ is non-discrete.

Proof. A non-trivial element $h \in H(F)$ fixing $B(b, n)$ for a given $n \in \mathbb{N}$ is readily constructed using Proposition II.57: Consider $\alpha_n := \mathrm{id} \in F^{(n)}$. By part (ii) of Proposition II.57 and the definition of $F^{(n+1)}$, there is a non-trivial $\alpha_{n+1} \in F^{(n+1)}$ with $\pi_n \alpha_{n+1} = \alpha_n$. Applying part (i) of Proposition II.57 repeatedly, we obtain non-trivial elements $\alpha_k \in F^{(k)}$ for all $k \geq n+1$ with $\pi_k \alpha_{k+1} = \alpha_k$ for all $k \geq n+1$. Set $\alpha_k := \mathrm{id} \in F^{(k)}$ for all $k \leq n$ and define $h \in \mathrm{Aut}(T_d)_b$ by fixing b and setting $\sigma_k(h, b) := \alpha_k \in F^{(k)}$. Because $F^{(l)} \leq \Phi^l(F^{(k)})$ for all $k \leq l$ we conclude that $h \in \bigcap_{k \in \mathbb{N}} \mathrm{U}_k(F^{(k)}) = H(F)$. $\square$

Proposition II.60. Retain the above notation. Then $\mathrm{QZ}(H(F))$ contains a non-trivial elliptic element.

Proof. By Proposition II.57, the element $d(\tau)$ which fixes b and whose 1-local action is τ everywhere commutes with $H(F)_b$. Hence $d(\tau) \in \mathrm{QZ}(H(F))$. $\square$

Remark II.61. We remark that the argument presented in this section cannot be made work in the quasiprimitive case because a quasiprimitive group $F \leq \mathrm{Sym}(\Omega)$ with non-trivial center necessarily equals its center and is regular: Recall that $Z(F) \trianglelefteq F$. Hence $Z(F)$ is transitive as soon as it is non-trivial by quasiprimitivity. It now suffices to show that F_ω is trivial for all $\omega \in \Omega$: Suppose $a \in F_\omega$ moves $\omega' \in \Omega$ and let $z \in Z(F)$ be such that $z(\omega) = \omega'$. Then $za(\omega) = \omega' \neq az(\omega)$, contradicting the assumption that $z \in Z(F)$.

7.4. Theorem II.41 (iv) (a). For certain intransitive $F \leq \mathrm{Sym}(\Omega)$ we construct a group $H(F) \leq \mathrm{Aut}(T_d)$ which is closed, non-discrete, compactly generated, vertex-transitive, locally acts like F and contains a quasi-central hyperbolic element of length 1.

Let $F \leq \mathrm{Sym}(\Omega)$. Assume that the partition $F\backslash\Omega = \sqcup_{i \in I} \Omega_i$ of Ω has at least three elements and $Z(F) \neq \{\mathrm{id}\}$. Choose a non-trivial element $\tau \in Z(F)$ and $\omega_0 \in \Omega_0$ with $\tau(\omega_0) \neq \omega_0$. Assume further that $F_{\Omega_i} \neq \{\mathrm{id}\}$ for all $\Omega_i \neq \Omega_0$.

Define actions $F^{(k)} \le \mathrm{Aut}(B_{d,k})$ for $k \in \mathbb{N}$ inductively by $F^{(1)} := F$ and

$$F^{(k+1)} := \{(\alpha, (\alpha_\omega)_\omega) \mid \alpha \in F^{(k)},\ \alpha_\omega \in C_{F^{(k)}}(\alpha, \omega) \text{ is constant w.r.t. } F\backslash\Omega,\ \alpha_{\omega_0} = \alpha\}.$$

Proposition II.62. The actions $F^{(k)} \le \mathrm{Aut}(B_{d,k})$ ($k \in \mathbb{N}$) defined above satisfy:

(i) Every $\alpha \in F^{(k)}$ is self-compatible in directions from Ω_0.

(ii) The compatibility set $C_{F^{(k)}}(\alpha, \Omega_i)$ is non-empty for all $\alpha \in F^{(k)}$ and $i \in I$. In particular, the group $F^{(k)}$ satisfies (C).

(iii) The compatibility set $C_{F^{(k)}}(\mathrm{id}, \Omega_i)$ is non-trivial for all $\Omega_i \neq \Omega_0$. In particular, the group $F^{(k)}$ does not satisfy (D).

(iv) The element $\gamma_k(\tau) \in \mathrm{Aut}(B_{d,k})$ is central in $F^{(k)}$.

Proof. We prove all four properties simultaneously by induction: For $k = 1$, the assertions (i) and (ii) are trivial. The third translates to F_{Ω_i} being non-trivial for all $\Omega_i \neq \Omega_0$ which is an assumption, as is commutativity. Now, assume that all properties hold for $F^{(k)}$. Then the definition of $F^{(k+1)}$ is meaningful because of (i) and it is a subgroup of $\mathrm{Aut}(B_{d,k})$ because F preserves $F\backslash\Omega$. Assertion (i) is now evident. Statements (ii), (iii) and (iv) readily carry over from $F^{(k)}$ to $F^{(k+1)}$. $\square$

Definition II.63. Retain the above notation. Define $H(F) := \bigcap_{k\in\mathbb{N}} \mathrm{U}_k(F^{(k)})$.

The group $H(F)$ is vertex-transitive, compactly generated and contains an involutive inversion because $\mathrm{U}_1(\{\mathrm{id}\}) \le H(F)$. Also, $H(F)$ is closed as the intersection of all its k-closures. The 1-local action of H is given by $F = F^{(1)}$ as $\mathrm{D}(F) \le H$.

Lemma II.64. Let F be as above. Then $H(F)$ is non-discrete.

Proof. A non-trivial element $h \in H(F)$ fixing $B(b,n)$ for a given $n \in \mathbb{N}$ is readily constructed using Proposition II.62: Consider $\alpha_n := \mathrm{id} \in F^{(n)}$. By parts (i) and (iii) of Proposition II.62 as well as the definition of $F^{(n+1)}$, there is a non-trivial element $\alpha_{n+1} \in F^{(n+1)}$ with $\pi_n \alpha_{n+1} = \alpha_n$. Applying parts (i) and (ii) of Proposition II.62 repeatedly, we obtain non-trivial elements $\alpha_k \in F^{(k)}$ for all $k \ge n+1$ with $\pi_k \alpha_{k+1} = \alpha_k$ for all $k \ge n+1$. Set $\alpha_k := \mathrm{id} \in F^{(k)}$ for all $k \le n$ and define $h \in \mathrm{Aut}(T_d)_b$ by fixing b and setting $\sigma_k(h,b) := \alpha_k \in F^{(k)}$. Since $F^{(l)} \le \Phi^l(F^{(k)})$ for all $k \le l$ we conclude that $h \in \bigcap_{k\in\mathbb{N}} \mathrm{U}_k(F^{(k)}) = H(F)$. $\square$

Proposition II.65. Let $F \le \mathrm{Sym}(\Omega)$ be as above. Then $\mathrm{QZ}(H(F))$ contains a hyperbolic element of length 1.

Proof. Fix $b \in V(T_d)$ and let τ be as above. Consider the line L through b with edge labels

$$\ldots, \tau^{-2}\omega_0, \tau^{-1}\omega_0, \omega_0, \tau\omega_0, \tau^2\omega_0, \ldots$$

Define $t \in \mathrm{D}(F)$ by $t(b) = b_{\omega_0}$ and $\sigma_1(t,x) = \tau$ for all $x \in V(T_d)$. Then t is a translation of length 1 along L. Furthermore, t commutes with $H(F)_{B(b,1)}$: Indeed, let $g \in H(F)_{B(b,1)}$. Then $(gt)(b) = t(b) = (tg)(b)$ and

$$\sigma_k(gt,b) = \sigma_k(g,tb)\sigma_k(t,b) = \sigma_k(t,b)\sigma_k(g,b) = \sigma_k(t,gb)\sigma_k(g,b) = \sigma_k(tg,b)$$

for all $k \in \mathbb{N}$ because $\sigma_k(t,b) = \gamma_k(\tau) \in Z(F^{(k)})$ and $g \in \mathrm{U}_{k+1}(F^{(k+1)})_{B(b,1)}$. $\square$

7.5. Theorem II.41 (iv) (b). For certain quasiprimitive $F \le \mathrm{Sym}(\Omega)$ we construct a group $H(F) \le \mathrm{Aut}(T_d)$ which is closed, non-discrete, compactly generated, locally acts like F and whose quasi-center contains a hyperbolic element of length 2.

Let $F \le \mathrm{Sym}(\Omega)$ be quasiprimitive. Assume that F preserves a non-trivial partition $\mathcal{P} : \Omega = \bigsqcup_{i\in I} \Omega_i$. Further, suppose that $F_{\Omega_i} \neq \{\mathrm{id}\}$ and $F_{\omega_i} \curvearrowright \Omega_i\backslash\{\omega_i\}$ is transitive for all $i \in I$ and $\omega_i \in \Omega_i$.

Example II.66. Using the GAP library of small transitive groups [**GAP17**], consider e.g. $\mathrm{Tr}(12,33) \cong A_5$, $\mathrm{Tr}(14,10) \cong \mathrm{PSL}(3,2)$ or $\mathrm{Tr}(15,10) \cong S_5$, all of which are quasiprimitive. The former two have blocks of size 2, the latter has blocks of size 3. Its point stabilizers act transitively on the remainder of the respective block.

An orbit for the action of $\Phi(F)$ on $S(b,2) \cong \Omega^{(2)}$ is given by

$$\Omega_0^{(2)} := \{(\omega_1,\omega_2) \mid \exists i \in I: \ \omega_1,\omega_2 \in \Omega_i\} \subseteq \Omega^{(2)}.$$

Indeed, let $\alpha = (a,(a_\omega)_\omega) \in \Phi(F)$ and $(\omega_1,\omega_2) \in \Omega_0^{(2)}$. Then $\alpha(\omega_1,\omega_2) = (a\omega_1, a_{\omega_1}\omega_2)$ is in $\Omega_0^{(2)}$ because a and a_{ω_1} agree on ω_1. Note that if $w = (\omega_1,\omega_2) \in \Omega_0^{(2)}$ then so is $\overline{w} := (\omega_2,\omega_1)$. The subgroup of $\Phi(F)$ consisting of those elements which are self-compatible with respect $\Omega_0^{(2)}$ is given by

$$F^{(2)} := \{(a,(a_\omega)_\omega) \mid a \in F, a_\omega \in C_F(a,\omega) \text{ constant w.r.t. } \mathcal{P}\}.$$

Then define inductively for $k \in \mathbb{N}$:

$$F^{(2(k+1))} := \{(\alpha,(\alpha_w)_w) \mid \alpha \in F^{(2k)}, \alpha_w \in C_F(\alpha,w), \ \forall w \in \Omega_0^{(2)}: \ \alpha_w = \alpha\}$$

Proposition II.67. The actions $F^{(2k)} \leq \mathrm{Aut}(B_{d,2k})$ ($k \in \mathbb{N}$) defined above satisfy:

(i) Every $\alpha \in F^{(2k)}$ is self-compatible in directions from $\Omega_0^{(2)}$.
(ii) The compatibility set $C_{F^{(2k)}}(\alpha,w)$ is non-empty for all $\alpha \in F^{(2k)}$ and $w \in \Omega^{(2)}$. In particular, the group $F^{(2k)}$ satisfies (C).
(iii) The compatibility set $C_{F^{(2k)}}(\mathrm{id},w)$ is non-trivial for all $w \in \Omega^{(2)}$. In particular, the group $F^{(2k)}$ does not satisfy (D).

Proof. We prove all three properties simultaneously by induction: For $k=1$, assertion (i) holds by construction of $F^{(2)}$, as do (ii) and (iii). Now assume that all properties hold for $F^{(2k)}$. Then the definition of $F^{(2(k+1))}$ is meaningful because of (i) and it is a subgroup because $F^{(2)}$ preserves $\Omega_0^{(2)}$. Also, $F^{(2(k+1))}$ satisfies (i) because $\Omega_0^{(2)}$ is inversion-closed and statements (ii), (iii) carry over from $F^{(2k)}$. □

Definition II.68. Retain the above notation. Define $H(F) := \bigcap_{k\in\mathbb{N}} \mathrm{BU}_{2k}^{(l)}(F^{(2k)})$.

The group $H(F)$ is closed as an intersection of closed sets and compactly generated by $H(F)_b$ and a finite generating set of $\mathrm{BU}_2(\{\mathrm{id}\})^+$, see Lemma II.39. For vertices in V_1, the 1-local action is F because $\Gamma^{2k}(F) \leq F^{(2k)}$. For vertices in V_2 the 1-local action is $F^+ = F$ as $\Gamma^2(F) \leq F^{(2)}$.

Lemma II.69. Let F be as above. Then $H(F)$ is non-discrete.

Proof. A non-trivial element $h \in H(F)$ fixing $B(b,2n)$ for a given $n \in \mathbb{N}$ is readily constructed using Proposition II.45: Consider $\alpha_{2n} := \mathrm{id} \in F^{(2n)}$. By parts (i) and (iii) of Proposition II.45 and the definition of $F^{(2(n+1))}$, there is a non-trivial element $\alpha_{2(n+1)} \in F^{(2(n+1))}$ with $\pi_{2n}\alpha_{2(n+1)} = \alpha_{2n}$. Applying parts (i) and (ii) of Proposition II.67 repeatedly, we obtain non-trivial elements $\alpha_{2k} \in F^{(2k)}$ for all $k \geq n+1$ with $\pi_{2k}\alpha^{2(k+1)} = \alpha_{2k}$ for all $k \geq n+1$. Set $\alpha_{2k} := \mathrm{id} \in F^{(2k)}$ for all $k \leq n$ and define $h \in \mathrm{Aut}(T_d)_b$ by fixing b and setting $\sigma_{2k}(h,b) := \alpha_{2k} \in F^{(2k)}$. Since $F^{(2l)} \leq \Psi^{2l}(F^{(2k)})$ for all $k \leq l$ we conclude that $h \in \bigcap_{k\in\mathbb{N}} \mathrm{BU}_{2k}(F^{(2k)}) = H(F)$. □

Proposition II.70. Let F be as above. Then $\mathrm{QZ}(H(F))$ contains a hyperbolic element of length 2.

Proof. Fix $b \in V(T_d)$ and let $w = (\omega_1,\omega_2) \in \Omega_0^{(2)}$. Consider the line L through b with edge labels $\ldots,\omega_1,\omega_2,\omega_1,\omega_2,\ldots$. Define $t \in \mathrm{D}(F)$ by $t(b) = b_w$ and $\sigma_1(t,x) = \mathrm{id}$ for all $x \in V(T_d)$. Then t is a translation of length 2 along L. Furthermore, t commutes with $H(F)_{B(b,2)}$: Indeed, let $g \in H(F)_{B(b,2)}$. Then $gt(b) = t(b) = tg(b)$

and for all $k \in \mathbb{N}$:

$$\begin{aligned}\sigma_{2k}(gt,b) &= \sigma_{2k}(g,tb)\sigma_{2k}(t,b) = \sigma_{2k}(g,b_w)\\ &= \sigma_{2k}(g,b) = \sigma_{2k}(t,gb)\sigma_{2k}(g,b) = \sigma_{2k}(tg,b)\end{aligned}$$

as $\sigma_l(t,x) = \mathrm{id}$ for all $l \in \mathbb{N}$ and $x \in V(T_d)$, and $g \in \mathrm{BU}_{2(k+1)}(F^{(2(k+1))})_{B(b,2)}$. □

7.6. Limitations. We argue that the construction of Section 7.5 does not easily carry over to primitive local actions. Recall that for a transitive permutation group $F \leq \mathrm{Sym}(\Omega)$ one defines $\mathrm{rank}(F) := |F\backslash\Omega^2|$, where F acts on Ω^2 diagonally, and that $\mathrm{rank}(F) = 2$ if and only if F is 2-transitive.

Lemma II.71. Let $F \leq \mathrm{Sym}(\Omega)$. Then $|\Phi(F)\backslash\Omega^{(2)}| = \mathrm{rank}(F) - 1$.

Proof. Notice that $\Omega^{(2)} = \Omega^2\backslash\Delta$ where Δ denotes the diagonal in Ω^2. Given that $\Gamma(F) \leq \Phi(F)$ we therefore conclude $|\Phi(F)\backslash\Omega^{(2)}| \leq |\Gamma(F)\backslash\Omega^{(2)}| = \mathrm{rank}(F) - 1$. The orbits of $\Gamma(F)$ and $\Phi(F)$ are in fact the same: Let $\alpha := (a,(a_\omega)_{\omega\in\Omega}) \in \Phi(F)$. Then we have $\alpha(\omega_1,\omega_2) = (a\omega_1, a_{\omega_1}\omega_2) \in \{(a\omega_1, aF_{\omega_1}\omega_2)\} \subseteq \Gamma(F)(\omega_1,\omega_2)$. □

In particular, a permutation group has to have rank at least 3 in order to be eligible for the construction of the previous section. The smallest non-regular primitive permutation group of rank 3 is $D_5 \leq S_5$. However, we also have the following obstruction to non-discreteness.

Proposition II.72. Let $F \leq \mathrm{Sym}(\Omega)$ be primitive and let $\Omega_0^{(2)}$ be an orbit for the action of $\Phi(F)$ on $\Omega^{(2)} \cong S(b,2)$. The subgroup of elements in $\Phi(F)$ which are self-compatible in directions from $\Omega_0^{(2)}$ is precisely $\Gamma(F)$.

Proof. Every element of $\Gamma(F)$ is self-compatible in every direction from $\Omega^{(2)}$. Conversely, assume that $(a,(a_\omega)_\omega) \in \Phi(F)$ is self-compatible in all directions from $\Omega_0^{(2)}$. Then $a_{\omega_1} = a_{\omega_2}$ whenever $w := (\omega_1,\omega_2) \in \Omega_0^{(2)}$. This induces a non-trivial equivalence relation on Ω which is F-invariant because $\Gamma(F) \leq \Phi(F)$: If $(\omega_1,\omega_2) \in \Omega_0^{(2)}$ then $\gamma(a)(\omega_1,\omega_2) = (a\omega_1, a\omega_2) \in \Omega_0^{(2)}$ for all $a \in F$. Since F is primitive, it is the universal relation, i.e. all a_ω $(\omega \in \Omega)$ coincide. Hence $(a,(a_\omega)_\omega) \in \Gamma(F)$. □

7.7. Groups with Infinitely Many Distinct k-closures. Given a prime p, Banks–Elder–Willis list $\mathrm{PGL}(2,\mathbb{Q}_p) \leq \mathrm{Aut}(T_{p+1})$ as an example of a group with infinitely many distinct k-closures, see [**BEW15**]. Whereas $\mathrm{PGL}(2,\mathbb{Q}_p)$ has trivial quasi-center because it is simple, the groups constructed in the proof of Theorem II.41 provide examples with non-trivial quasi-center. Indeed, we have the following.

Proposition II.73. Let $H \leq \mathrm{Aut}(T_d)$ be closed, non-discrete, locally transitive and contain an involutive inversion. Then $H^{(k)} = \mathrm{U}_k(F^{(k)})$ and $H = \bigcap_{k\in\mathbb{N}} \mathrm{U}_k(F^{(k)})$, where $F^{(k)} \leq \mathrm{Aut}(B_{d,k})$ is action-isomorphic to the action of H on balls of radius k. If, in addition, $\mathrm{QZ}(H) \neq \{\mathrm{id}\}$ then H has infinitely many distinct k-closures.

Proof. We have $H^{(k)} = \mathrm{U}_k(F^{(k)})$ by Theorem II.23. Then $H = \bigcap_{k\in\mathbb{N}} \mathrm{U}_k(F^{(k)})$ by [**BEW15**, Proposition 3.4]. Hence, if H had only finitely many distinct k-closures, the sequence $(H^{(k)})_{k\in\mathbb{N}}$ of subgroups of $\mathrm{Aut}(T_d)$ is eventually constant equal to, say, $H^{(n)} = \mathrm{U}_n(F^{(n)}) \geq H$ which is non-discrete because H is and therefore has trivial quasi-center by Proposition II.16. □

CHAPTER III

Prime Localizations of Burger–Mozes-type Groups

This section is based on [**Tor17**]. We determine the p-localization of Burger–Mozes-type groups, i.e. the groups $\mathrm{U}(F)$, $\mathrm{G}(F,F')$ and $\mathrm{N}(F)$ discussed in Chapter I, for a large class of permutation groups $F \leq F' \leq \mathrm{Sym}(\Omega)$ and primes p.

The concept of prime localization of a totally disconnected locally compact group G was introduced by Reid in [**Rei13**]: Let p be prime. A *local p-Sylow subgroup* of G is a maximal pro-p subgroup of a compact open subgroup of G. The *p-localization* $G_{(p)}$ of G is defined as the commensurator $\mathrm{Comm}_G(S)$ of a local p-Sylow subgroup S of G, equipped with the unique group topology which makes the inclusion of S into $G_{(p)} = \mathrm{Comm}_G(S)$ continuous and open. We refer the reader to [**Rei13**] for general properties of prime localization and its applications, of which we highlight the scale function introduced by Willis in [**Wil94**].

1. Local Sylow Subgroups

This section is concerned with determining local Sylow subgroups of the Burger–Mozes-type groups. Throughout, Ω denotes a set of cardinality $d \in \mathbb{N}_{\geq 3}$ and p is a prime. We consider the d-regular tree $T_d = (V,E)$ with a fixed labelling and base vertex $b \in V$. Furthermore, T denotes a finite subtree of T_d.

Note that it suffices to consider $\mathrm{U}(F)$: Any local Sylow subgroup of $\mathrm{U}(F)$ is also a local Sylow subgroup of $\mathrm{G}(F,F')$ and $\mathrm{N}(F)$ by definition of the topologies.

In a sense, the following proposition provides local p-Sylow subgroups of $\mathrm{U}(F)$ in the case where the operations of taking a p-Sylow subgroup and taking point stabilizers commute for F. It is the basis of all subsequent statements about the p-localization of Burger–Mozes-type groups and amends [**Rei13**, Lemma 4.2].

Proposition III.1. Let $F \leq \mathrm{Sym}(\Omega)$ and $F(p) \leq F$ a p-Sylow subgroup. Then $\mathrm{U}(F(p))_T$ is a p-Sylow subgroup of $\mathrm{U}(F)_T$ if and only if so is $F(p)_\omega \leq F_\omega$ for all $\omega \in \Omega$.

Proof. First, assume that T consists of a single vertex $b \in V$. The sphere $S(b,k) \subset V$ of radius k around $b \in V$ is, via the given labelling, in natural bijection with

$$P_k := \{w = (\omega_1, \dots, \omega_k) \in \Omega^k \mid \forall i \in \{1, \dots, k-1\} : \ \omega_{i+1} \neq \omega_i\}.$$

The restriction of $\mathrm{U}(F)$ to $S(b,k)$ yields a subgroup of $\mathrm{Sym}(S(b,k))$ of cardinality given by $\left|\mathrm{U}(F)_b|_{S(b,1)}\right| = |F|$ and $\left|\mathrm{U}(F)_b|_{S(b,k+1)}\right| = \left|\mathrm{U}(F)_b|_{S(b,k)}\right| \cdot \prod_{w \in P_k} |F_{\omega_k}|$. The maximal powers of p dividing $\left|\mathrm{U}(F)_b|_{S(b,k)}\right|$ and $\left|\mathrm{U}(F(p))_b|_{S(b,k)}\right|$ are hence equal for all $k \in \mathbb{N}_0$ if and only if $F(p)_\omega \leq F_\omega$ is a p-Sylow subgroup for all $\omega \in \Omega$.

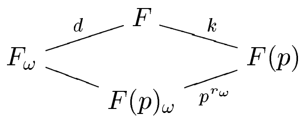
Similarly, when T is not a single vertex, the size of the restriction of $\mathrm{U}(F)_T$ to a sufficiently larger subtree is a product of the $|F_\omega|$ involving *all* $\omega \in \Omega$. □

For transitive $F \leq \mathrm{Sym}(\Omega)$, it suffices to check the above criterion for one choice of a p-Sylow subgroup $F(p)$ of F and all $\omega \in \Omega$. We now identify classes of permutation group and values of p to which Proposition III.1 applies. For the symmetric and alternating groups we have the following, complete description.

Proposition III.2. Let $F = \mathrm{Sym}(\Omega)$ or $F = \mathrm{Alt}(\Omega)$ and $F(p) \leq F$ a p-Sylow subgroup. Further, let p^s ($s \in \mathbb{N}_0$) be the maximal power of p dividing d. Then $F(p)_\omega \leq F_\omega$ is a p-Sylow subgroup for all $\omega \in \Omega$ if and only if either

(i) $p > d$, or
(ii) $s \geq 1$ and $p^{s+1} > d$, or
(iii) $F = \mathrm{Alt}(\Omega)$ and $(d,p) = (3,2)$.

Proof. If $p > d$ then $F(p)$ is trivial and so is any p-Sylow subgroup of F_ω. Now assume $p \leq d$ and consider the following diagram of subgroups of F and indices.



For every $\omega \in \Omega$ we have $[F : F_\omega] = |F \cdot \omega| = d$ and $[F(p) : F(p)_\omega] = |F(p) \cdot \omega| = p^{r_\omega}$ for some $r_\omega \in \mathbb{N}_0$. Note that $p \nmid k$ by definition. Now examine the equation $d \cdot [F_\omega : F(p)_\omega] = k \cdot p^{r_\omega}$. If $F(p)$ is trivial then $F = \mathrm{Alt}(\Omega)$ and p is even, hence (iii). Now assume that $F(p)$ is non-trivial. Then there is $\omega \in \Omega$ such that $r_\omega \geq 1$. Thus, if $p \nmid d$, then $p \mid [F_\omega : F(p)_\omega]$ and hence $F(p)_\omega$ is not a p-Sylow subgroup of F_ω. We conclude that the condition $s \geq 1$ is necessary. Note that the biggest p^{r_ω} ($\omega \in \Omega$) which occurs is given by the biggest power of p which is smaller than or equal to d due to the iterated wreath product structure of $F(p)$. As $p \nmid k$ we conclude (ii).

Conversely, suppose $s \geq 1$ and $p^{s+1} \geq d$. If p is odd, or $F = \mathrm{Sym}(\Omega)$ and p is even, then $F(p)$ is a direct product of s-fold iterated wreath products and the maximum power of p dividing $[F(p) : F(p)_\omega]$ and $[F : F_\omega]$ is p^s in both cases. The same index assertions hold for $F = \mathrm{Alt}(\Omega)$ and p even. $\square$

For a general permutation group $F \leq \mathrm{Sym}(\Omega)$ and $\omega \in \Omega$ we have

$$|F(p) \cdot \omega| = \frac{|F(p)|}{|F(p)_\omega|} = \frac{|F(p)| \cdot [F_\omega : F(p)_\omega]}{|F_\omega|} = \frac{[F_\omega : F(p)_\omega]}{[F : F(p)]} \cdot |F \cdot \omega|.$$

by the orbit-stabilizer theorem. In particular, we conclude the following.

Proposition III.3. Let $F \leq \mathrm{Sym}(\Omega)$ and $F(p) \leq F$ a p-Sylow subgroup. Assume that $F\backslash\Omega = F(p)\backslash\Omega$. Then $F(p)_\omega \leq F_\omega$ is a p-Sylow subgroup for all $\omega \in \Omega$. $\square$

Proposition III.4. Let $|\Omega| = p^n$ and $F \leq \mathrm{Sym}(\Omega)$ transitive. Also, let $F(p) \leq F$ be a p-Sylow subgroup. Then so is $F(p)_\omega \leq F_\omega$ for all $\omega \in \Omega$ and $F(p)$ is transitive.

Proof. In this case, the above equation is $|F(p) \cdot \omega| = ([F_\omega : F(p)_\omega]/[F : F(p)]) \cdot p^n$. As always, $|F(p) \cdot \omega|$ is a power of p and bounded by $|\Omega| = p^n$. Since p does not divide $[F : F(p)]$ the above implies that p does not divide $[F_\omega : F(p)_\omega]$. $\square$

2. Prime Localizations

This section is concerned with the p-localizations of Burger–Mozes-type groups. Recall that for groups $H \leq G$ one defines the *commensurator of H in G* by

$$\mathrm{Comm}_G(H) := \{g \in G \mid [H : H \cap gHg^{-1}] < \infty \text{ and } [gHg^{-1} : gHg^{-1} \cap H] < \infty\}.$$

The *p-localization* of a totally disconnected locally compact group G is defined as the commensurator $\mathrm{Comm}_G(S)$ of a local p-Sylow subgroup S of G, equipped with the unique group topology that makes the inclusion of S into $G_{(p)} := \mathrm{Comm}_G(S)$ continuous and open. Then the inclusion $\mathrm{Comm}_G(S) \to G$ is continuous.

The following lemma due to Caprace–Monod [**CM11**, Section 4] and Caprace–Reid–Willis [**CRW17**, Corollary 7.4] is crucial for the subsequent statements of this section. See also [**Wes15**].

Lemma III.5. Let G be residually discrete, locally compact and totally disconnected. Further, let $K \leq G$ be compact. Then $\mathrm{Comm}_G(K) = \bigcup_{L \leq_o K} N_G(L)$.

Proof. Every element of G which normalizes an open subgroup of K commensurates K because open subgroups of K have finite index in K given that K is compact.

Conversely, let $g \in \mathrm{Comm}_G(K)$ and consider $H := \langle K, g\rangle$. Then H is a compactly generated open subgroup of $\mathrm{Comm}_G(K)$ and hence a compactly generated, totally disconnected locally compact group in its own right. It inherits residual discreteness from $\mathrm{Comm}_G(K)$ which injects continuously into the residually discrete group G. By [**CM11**, Corollary 4.1], H has an identity neighbourhood basis of compact open normal subgroups. Hence g normalizes an open subgroup of K. $\square$

Now, let $F \le F' \le \widehat{F} \le \mathrm{Sym}(\Omega)$. In the case of Proposition III.1, the following proposition identifes certain subsets of the p-localization of $\mathrm{G}(F, F')$ and thereby expands [**Rei13**, Lemma 4.2] given that $\mathrm{U}(F) = \mathrm{G}(F, F)$. We establish the following notation: Given partitions $\mathcal{P} := (P_i)_{i\in I}$ of V and $\mathcal{H} = (H_j)_{j\in J}$ of $H \le \mathrm{Sym}(\Omega)$, let

$$\Gamma_{\mathcal{P}}(\mathcal{H}) := \{g \in \mathrm{Aut}(T_d) \mid \forall i \in I : \ \exists j \in J : \ \forall v \in P_i : \ \sigma(g, v) \in H_j\}$$

denote the set of automorphisms of T_d whose local permutations at the vertices of a given element of $\mathcal{P}$ all come from the same element of $\mathcal{H}$.

Proposition III.6. Let $F \le F' \le \widehat{F} \le \mathrm{Sym}(\Omega)$ and $F(p) \le F$ a p-Sylow subgroup such that $F(p)_\omega \le F_\omega$ is a p-Sylow subgroup for all $\omega \in \Omega$. Set $S := \mathrm{U}(F(p))_b$. Then

$$\begin{aligned}\mathrm{Comm}_{\mathrm{G}(F,F')}(S) &= \langle \mathrm{U}(\{\mathrm{id}\}), \mathrm{Comm}_{\mathrm{G}(F,F')_b}(S)\rangle \\ &\ge \langle \mathrm{G}(F(p), F'), \{\Gamma_{V/L}(N_F(F(p))/F(p)) \mid L \le S \text{ open}\}\rangle.\end{aligned}$$

Proof. By Proposition III.1, the group S is a local p-Sylow subgroup of $\mathrm{U}(F)$ and hence of $\mathrm{G}(F, F')$. We first show that $\mathrm{G}(F, F')_{(p)}$ contains $\mathrm{U}(\{\mathrm{id}\})$. Indeed, given $g \in \mathrm{U}(\{\mathrm{id}\})$ we have $gSg^{-1} = \mathrm{U}(F(p))_{g(b)}$. Thus $S \cap gSg^{-1} = \mathrm{U}(F(p))_{(b,g(b))}$ which has finite index in both $S = \mathrm{U}(F)_b$ and $gSg^{-1} = \mathrm{U}(F(p))_{g(b)}$ by the orbit-stabilizer theorem. Since $\mathrm{U}(\{\mathrm{id}\})$ acts vertex-transitively on T_d we conclude

$$\mathrm{Comm}_{\mathrm{G}(F,F')}(S) = \langle \mathrm{U}(\{\mathrm{id}\}), \mathrm{Comm}_{\mathrm{G}(F,F')_b}(S)\rangle.$$

Now, the vertex stabilizer $\mathrm{G}(F, F')_b$ is residually discrete by Proposition I.18. Hence, by Lemma III.5, the commensurator $\mathrm{Comm}_{\mathrm{G}(F,F')_b}(S)$ is the union of the normalizers in $\mathrm{G}(F, F')_b$ of open subgroups of $S = \mathrm{U}(F(p))_b$. For example, we may consider $L_n := \mathrm{U}(F(p))_{B(b,n)} \le_o S$ for every $n \in \mathbb{N}$. The normalizer of L_n in $\mathrm{G}(F, F')_b$ contains those elements of $\mathrm{G}(F(p), F')_b$ all of whose singularities are contained in $B(b, n)$. Taking the union over all $n \in \mathbb{N}$ and using vertex-transitivity of $\mathrm{G}(F(p), F')$ in the sense that $\mathrm{G}(F(p), F') = \langle \mathrm{G}(F(p), F')_b, \mathrm{U}(\{\mathrm{id}\})\rangle$ we conclude that $\mathrm{Comm}_{\mathrm{G}(F,F')}(S)$ contains $\mathrm{G}(F(p), F')$ as a topological subgroup. Alternatively, use [**Bou16**, Lemma 3.2]. Now, note that for all $g, s \in \mathrm{Aut}(T_d)$ and $v \in V$ we have

$$\begin{aligned}\sigma(gsg^{-1}, v) &= \sigma(g, sg^{-1}v)\sigma(s, g^{-1}v)\sigma(g^{-1}, v) \\ &= \sigma(g, sg^{-1}v)\sigma(s, g^{-1}v)\sigma(g, g^{-1}v)^{-1}.\end{aligned}$$

Hence if $g \in \Gamma_{V/L}(N_F(F(p))/F(p))$, i.e. the coset $\sigma(g, v)F(p) \subseteq N_F(F(p))$ is constant on L-orbits, then $gLg^{-1} \subseteq \mathrm{U}(F(p))$ whence $g \in \mathrm{Comm}_{\mathrm{G}(F,F')}(S)$. $\square$

Remark III.7. Whereas the next result provides conditions on $F \le \mathrm{Sym}(\Omega)$ which ensure $\mathrm{U}(F)_{(p)} = \mathrm{G}(F(p), F)$ and we have $\mathrm{U}(F)_{(p)} = \mathrm{U}(F)$ for semiregular F by Proposition I.12, it may happen that $\mathrm{G}(F(p), F) \lneq \mathrm{U}(F)_{(p)} \lneq \mathrm{U}(F)$. Indeed, if for every $\omega \in \Omega$ there is an element $a_\omega \in F_\omega$ such that for all $\lambda \in \Omega$ we have $F(p)_\lambda \cap a_\omega F(p)_\lambda a_\omega^{-1} = \{\mathrm{id}\}$ then there is an element $g \in \mathrm{U}(F)_{B(b,1)}$ such that for $S := \mathrm{U}(F(p))_{B(b,1)}$ we have $S \cap gSg^{-1} = \{\mathrm{id}\}$ and therefore $g \notin \mathrm{U}(F)_{(p)}$: Choose the local permutation of g at $v \in V(T_d)$ to be a_ω whenever $d(v, b) = d(v, b_\omega) + 1$. If in addition $\mathrm{N}_F(F(p)) \gneq F(p)$ then the assertion holds by virtue of Proposition III.6. For instance, these assumptions are satisfied for $F = S_6$ and $p = 3$.

Theorem III.8. Let $F \leq F' \leq \widehat{F} \leq \mathrm{Sym}(\Omega)$ and $F(p) \leq F$ a p-Sylow subgroup of F. Assume that we have $F\backslash\Omega = F(p)\backslash\Omega$ and $N_{F'_\omega}(F(p)_\omega) = F(p)_\omega$ for all $\omega \in \Omega$. Then $\mathrm{G}(F, F')_{(p)} = \mathrm{G}(F(p), F')$.

If F does not fix a point of Ω and $F\backslash\Omega = F(p)\backslash\Omega$ then p divides $|\Omega|$. By Proposition III.3 the same assumption implies that the point stabilizers in $F(p)$ are p-Sylow subgroups of the respective point stabilizers in F. In the case $F = F'$, the theorem asks that these be self-normalizing.

Proof. (Theorem III.8). By Proposition III.1 and Proposition III.6 it suffices to show that $\mathrm{Comm}_{\mathrm{G}(F,F')_b}(\mathrm{U}(F(p))_b) = \mathrm{G}(F(p), F')_b$. By Proposition III.6, the group $\mathrm{G}(F(p), F')_b$ is a subgroup of said commensurator.

Now suppose $g \in \mathrm{Comm}_{\mathrm{G}(F,F')_b}(\mathrm{U}(F(p))_b) \leq \mathrm{G}(F, F')_b$. Given that $\mathrm{G}(F, F')_b$ is residually discrete by Proposition I.18, the element g normalizes an open subgroup $L \leq \mathrm{U}(F(p))_b$ by virtue of Lemma III.5. If g has only finitely many local permutations in $F'\backslash F(p)$ then $g \in G(F(p), F')_b$. Otherwise, the above implies that there is $n \in \mathbb{N}$ such that $g\mathrm{U}(F(p))_{B(b,n)}g^{-1} \subseteq L \subseteq \mathrm{U}(F(p))_b$ and g has a local permutation in $F'\backslash F(p)$ on $S(b, n)$. Then construct $h \in \mathrm{G}(F(p), F')$ with local permutations in $F(p)$ on spheres of radius at least n and such that $h^{-1}g$ fixes $B(b, n)$ pointwise as follows: Set $h|_{B(b,n-1)} := g$ and use the assumption $F'\backslash\Omega = F\backslash\Omega = F(p)\backslash\Omega$ to extend h to all T_d using $F(p)$ only. Then $h^{-1}g$ has a local permutation in $F'_\omega\backslash F(p)_\omega$ for some $\omega \in \Omega$ on $S(b, n)$ and $(h^{-1}g)\mathrm{U}(F(p))_{B(b,n)}(h^{-1}g)^{-1} \subseteq L \subseteq \mathrm{U}(F(p))_b$. However, this contradicts the assumption $N_{F'_\omega}(F(p)_\omega) = F(p)_\omega$ for all $\omega \in \Omega$. □

Theorem III.8 can be used to determine the p-localization of Lederle's coloured Neretin group $\mathrm{N}(F)$ under similar assumptions.

Theorem III.9. Let $F \leq \mathrm{Sym}(\Omega)$ and $F(p) \leq F$ a p-Sylow subgroup. If $F\backslash\Omega = F(p)\backslash\Omega$ and $N_{\widehat{F}_\omega}(F(p)_\omega) = F(p)_\omega$ for all $\omega \in \Omega$ then $\mathrm{N}(F)_{(p)} = \mathrm{N}(F(p))$.

Proof. By Proposition III.1, the group $S := \mathrm{U}(F(p))_b$ is a local Sylow subgroup of $\mathrm{N}(F)$. Also, by [**Led17**, Proposition 2.24], we have $\mathrm{N}(F(p)) \leq \mathrm{Comm}_{\mathrm{N}(F)}(S)$. Now, let $g \in \mathrm{Comm}_{\mathrm{N}(F)}(S)$ and let $g : T_d\backslash T \to T_d\backslash T'$ be a representative of g as an $\mathrm{U}(F)$-honest almost automorphism. Given that $F\backslash\Omega = F(p)\backslash\Omega$ there is a $\mathrm{U}(F(p))$-honest almost automorphism $h \in \mathrm{N}(F(p)) \leq \mathrm{Comm}_{\mathrm{N}(F)}(S)$ with representative $h : T_d\backslash T' \to T_d\backslash T$ such that $hg : T_d\backslash T \to T_d\backslash T$ fixes the leaves of T and therefore extends to an autormorphism of T_d fixing T. Furthermore, on each connected component of $T_d\backslash T$, the automorphism $hg \in \mathrm{N}(F) \cap \mathrm{Aut}(T_d)$ coincides with an element of $\mathrm{U}(F)$. Hence, using Proposition II.7, we have $hg \in \mathrm{U}(F)$ whence

$$hg \in \mathrm{Comm}_{\mathrm{N}(F)\cap\mathrm{Aut}(T_d)}(S) = \mathrm{Comm}_{\mathrm{G}(F)}(S) = \mathrm{G}(F)_{(p)} = \mathrm{G}(F(p)) \leq \mathrm{N}(F(p)).$$

by Theorem III.8. Given that $h \in \mathrm{N}(F(p))$ we conclude $g \in \mathrm{N}(F(p))$ as required. □

Proposition III.6 suggests that Theorem III.8 might hold as soon as $F(p)$ is self-normalizing in F'. This is not the case as the following remark shows.

Remark III.10. Theorem III.8 does not hold if the condition $N_{F'_\omega}(F(p)_\omega) = F(p)_\omega$ for all $\omega \in \Omega$ is replaced with $N_{F'}(F(p)) = F(p)$: There are transitive, non-regular permutation groups $F \leq \mathrm{Sym}(\Omega)$ and primes p such that $F\backslash\Omega = F(p)\backslash\Omega$ and $N_F(F(p)) = F(p)$ for which $F(p)$ is regular. In particular, $N_{F_\omega}(F(p)_\omega) \gneq F(p)_\omega$. In this case, $\mathrm{U}(F(p))_b$ is a local p-Sylow subgroup of $\mathrm{U}(F)$ by Proposition III.3. However, $\mathrm{U}(F(p))_b \cong F(p)$ is finite and hence $\mathrm{U}(F)_{(p)} = \mathrm{U}(F) \gneq \mathrm{G}(F(p), F)$.

A small example of this situation is a certain $F \cong S_4 \leq S_8$ and the prime $p = 2$, namely put $F := \langle (123)(456), (14)(25)(37)(68) \rangle$. Here, $F(2)$ is regular and self-normalizing in F of order 8.

Part 2

Contributions to Willis Theory

CHAPTER IV

Preliminaries

1. Willis Theory

In this chapter we recall central definitions of Willis theory and collect results around them. Let G be a t.d.l.c. group. In [**Wil94**], Willis introduced the notions of *scale* of an automorphism of G and *tidiness* of a compact open subgroup of G for a given automorphism of G.

Searching for the most general natural setting of tidiness and the scale, the definitions were generalized to endomorphisms in [**Wil15**]: Let G be a t.d.l.c. group and $\alpha \in \mathrm{End}(G)$. Note that $[\alpha(U) : \alpha(U) \cap U] \in \mathbb{N}$ for every compact open subgroup $U \leq G$ because $\alpha(U)$ is compact and $\alpha(U) \cap U$ is open in $\alpha(U)$. The *scale* of α is

$$s(\alpha) = \min \big\{ [\alpha(U) : \alpha(U) \cap U] \mid U \leq G \text{ compact open} \big\}.$$

A compact open subgroup $U \leq G$ is *minimizing* if $[\alpha(U) : \alpha(U) \cap U] = s(\alpha)$.

It is a cornerstone of Willis theory that a compact open subgroup of G is minimizing for α if and only if it has a certain structure. This structure is phrased in terms of the following subgroups of G, see [**Wil94**] and [**Wil15**] for more context. Put $U_0 := U$. For $n \in \mathbb{N}_0$, we define $U_{-n} = \bigcap_{k=0}^{n} \alpha^{-k}(U)$ and, inductively, the groups $U_{n+1} := U \cap \alpha(U_n)$. Now set

$$U_+ := \bigcap_{n\in\mathbb{N}_0} U_n, \quad U_- := \bigcap_{n\in\mathbb{N}_0} U_{-n} = \bigcap_{k=0}^{\infty} \alpha^{-k}(U),$$
$$U_{++} := \bigcup_{n\in\mathbb{N}_0} \alpha^n(U_+) \quad \text{and} \quad U_{--} := \bigcup_{n\in\mathbb{N}_0} \alpha^{-n}(U_-).$$

Both from a theoretical and mnemonic point of view, the following descriptions of the above subgroups are important: Let $x \in G$. The *α-trajectory* of x is the sequence $(\alpha^n(x))_{n\in\mathbb{N}_0}$ in G. An *α-regressive trajectory* of x is a sequence $(x_n)_{n\in\mathbb{N}_0}$ in G such that $x_0 = x$ and $\alpha(x_n) = x_{n-1}$ for all $n \in \mathbb{N}$. Consequently, we have the following verbal descriptions of the subgroups defined above.

$$U_- = \left\{ \begin{array}{c} \text{elements of } U \text{ whose} \\ \alpha\text{-trajectory is contained in } U \end{array} \right\},$$
$$U_+ = \left\{ \begin{array}{c} \text{elements of } U \text{ which admit an} \\ \alpha\text{-regressive trajectory contained in } U \end{array} \right\},$$
$$U_{--} = \left\{ \begin{array}{c} \text{elements of } G \text{ whose } \alpha\text{-trajectory} \\ \text{is eventually contained in } U \end{array} \right\}.$$
$$U_{++} = \left\{ \begin{array}{c} \text{elements of } G \text{ which admit an } \alpha\text{-regressive} \\ \text{trajectory eventually contained in } U \end{array} \right\},$$

The subgroup U is *tidy above* for α if $U = U_+U_-$, and *tidy below* for α if U_{--} is closed. It is *tidy* for α if it is both tidy above and tidy below for α. Note that this definition of being tidy below deviates from [**Wil15**, Definition 9] but turns out to be equivalent in the case of tidy above subgroups, see [**Wil15**, Proposition 9].

The announced cornerstone of Willis theory now reads as follows.

Theorem IV.1 ([**Wil15**, Theorem 2]). Let G be a t.d.l.c. group, $\alpha \in \mathrm{End}(G)$ and $U \leq G$ compact open. Then U is minimizing for α if and only if it is tidy for α.

We have $\alpha(U_+) \geq U_+$ and $\alpha(U_-) \leq U_-$. It can be shown that $s(\alpha) = [\alpha(U_+) : U_+]$ if $U \leq G$ is tidy for $\alpha \in \mathrm{End}(G)$, and $[U_- : \alpha(U_-)] = s(\alpha^{-1})$ in case $\alpha \in \mathrm{Aut}(G)$.

For future reference, we include the following result which constitutes an endomorphism version of the equality

$$\alpha^k\left(\bigcap_{i=m}^{n} \alpha^i(U)\right) = \bigcap_{i=m+k}^{n+k} \alpha^i(U)$$

which holds for an automorphism $\alpha \in \mathrm{Aut}(G)$, $U \leq G$ compact open and $m, n, k \in \mathbb{Z}$.

Lemma IV.2 ([**Wil15**, Lemma 2]). Retain the above notation. For all $n, m \in \mathbb{N}$:

(i) $U_{-n-m} = (U_{-n})_{-m}$, and
(ii) $\alpha^k(U_{-n}) = \begin{cases} U_k \cap U_{k-n} & 0 \leq k \leq n \\ \alpha^{k-n}(U_n) & k \geq n \end{cases}$, and
(iii) $(U_{-n})_k = U_k \cap U_{-n}$ for all $k \geq 0$ and $(U_{-n})_+ = U_+ \cap U_{-n}$.

Complementing Theorem IV.1, Willis provides an algorithm, the *tidying procedure*, which, starting from an arbitrary compact open subgroup of $U \leq G$, produces a compact open subgroup of G which is tidy for α.

Algorithm IV.3 ([**Wil15**, Section 7]). Let $U \leq G$ be compact open and $\alpha \in \mathrm{End}(G)$.

(i) There exists $n \in \mathbb{N}$ such that U_{-n} is tidy above for α.
Replacing U with U_{-n} we may assume that U is tidy above for α.
(ii) Define $\mathfrak{L}_U := U_{++} \cap U_{--}$ and $L_U := \overline{\mathfrak{L}_U}$.
(iii) Set $\widetilde{U} := \{x \in U : xL_U \subseteq L_U U\}$.
(iv) Then $\widetilde{U} L_U$ is a compact open subgroup of G which is tidy for α.

If, in Algorithm IV.3, the subgroup $U \leq G$ is already tidy for α, then $\widetilde{U} L_U = U$. We remark that $\mathfrak{L}_U$ of Algorithm IV.3 is given by

$$\mathfrak{L}_U = \{x \in G \mid \exists y \in U_+ \ \exists m, n \in \mathbb{N} \text{ with } \alpha^m(y) = x \text{ and } \alpha^n(x) \in U_-\}.$$

We continue with the introduction of further relevant subgroups of G associated to an endomorphism $\alpha \in \mathrm{End}(G)$. The identity element of G is denoted by e.

(a) The *nub* of α is given by

$$\mathrm{nub}(\alpha) := \bigcap\{U \leq G \mid U \text{ is compact open and tidy for } \alpha\}.$$

It is a compact subgroup of G which by [**Wil15**, Proposition 12] captures the obstruction for there to be an identity neighbourhood basis of tidy subgroups.

(b) The *contraction groups*

$$\mathrm{con}(\alpha) := \{x \in G \mid \lim_{n\to\infty} \alpha^n(x) = e \in G\} \text{ and}$$
$$\mathrm{con}^-(\alpha) := \{x \in G \mid \exists (x_n)_{n\in\mathbb{N}_0} \ \alpha\text{-regressive for } x \text{ with } \lim_{n\to\infty} x_n = e \in G\}.$$

play a particularly important role in the general theory of t.d.l.c. groups, see e.g. [**BW04**], [**BGT16**] and [**CRW17**]. They are α-invariant subgroups of G but not necessarily closed in G.

(c) The relevance of the *parabolic subgroups*

$$\mathrm{par}(\alpha) := \{x \in G \mid \{\alpha^n(x) \mid n \in \mathbb{N}_0\} \text{ is precompact}\} \text{ and}$$
$$\mathrm{par}^-(\alpha) := \{x \in G \mid x \text{ admits a precompact } \alpha\text{-regressive trajectory}\}$$

stems from the fact that $\mathrm{par}^{-}(\alpha)$ admits a quotient on which α induces an automorphism, see [**Wil15**, Proposition 20]. They are closed and α-invariant subgroups of G. Note that $\mathrm{con}(\alpha) \leq \mathrm{par}(\alpha)$ and $\mathrm{con}^{-}(\alpha) \leq \mathrm{par}^{-}(\alpha)$.

(d) The normal subgroup of said quotient is the *bounded iterated kernel*

$$\mathrm{bik}(\alpha) := \overline{\{x \in \mathrm{par}^{-}(\alpha) \mid \alpha^n(x) = e \text{ for some } n \in \mathbb{N}\}}.$$

It is a consequence of [**Wil15**, Proposition 20] that any two α-regressive trajectories of elements of $\mathrm{par}^{-}(\alpha)$ differ only by elements of $\mathrm{bik}(\alpha)$: Let $x \in \mathrm{par}^{-}(\alpha)$ and suppose that $(x_n)_{n\in\mathbb{N}_0}$ and $(x'_n)_{n\in\mathbb{N}_0}$ are α-regressive trajectories of x. Then $x'_n x_n^{-1} \in \mathrm{bik}(\alpha)$ for all $n \in \mathbb{N}_0$.

We remark that $\mathrm{bik}(\alpha) \leq \mathrm{nub}(\alpha) \leq \mathrm{par}(\alpha) \cap \mathrm{par}^{-}(\alpha)$ by [**Wil15**, Proposition 20].

2. Directed Graphs

Chapter VI makes use of the permutation topology introduced in Section I.1.2 as well as directed graphs. Here, we recall notation around the latter, largely following Möller [**Möl02**].

A *directed graph* Γ is a tuple $(V(\Gamma), E(\Gamma))$ consisting of a *vertex set* $V(\Gamma)$ and an *edge set* $E(\Gamma) \subseteq V(\Gamma) \times V(\Gamma) \setminus \{(u,u) \mid u \in V(\Gamma)\}$. We let $\mathrm{pr}_1, \mathrm{pr}_2 : E(\Gamma) \to V(\Gamma)$ denote the projections onto the first and second factor, the *origin* and *terminus* of an edge. Let Γ be a directed graph. An *arc* of length $k \in \mathbb{N}$ from $v \in V(\Gamma)$ to $v' \in V(\Gamma)$ is a tuple $(v = v_0, \ldots, v_k = v')$ of distinct vertices of Γ such that (v_i, v_{i+1}) in an edge in Γ for all $i \in \{0, \ldots, k-1\}$. Two vertices $v, w \in \Gamma(V)$ are *adjacent* if either $(v,w) \in E(\Gamma)$ or $(w,v) \in E(\Gamma)$. A *path* of length $k \in \mathbb{N}$ from $v \in V(\Gamma)$ to $v' \in V(\Gamma)$ is a tuple $(v = v_0, \ldots, v_k = v')$ of distinct vertices of Γ such that either (v_i, v_{i+1}) or (v_{i+1}, v_i) is an edge in Γ for all $i \in \{0, \ldots, k-1\}$. The directed graph Γ is connected if for all $v, w \in V(\Gamma)$ there is a path from v to w. It is a *tree* if it is connected and has no non-trivial cycles, i.e. tuples $(v_0, \ldots, v_k)$ with $k \geq 3$ and such that $(v_0, \ldots, v_{k-1})$ and $(v_{k-1}, v_k) \in E(\Gamma)$ are both paths and $v_k = v_0$. Two infinite paths in Γ are *equivalent* if they intersect in an infinite path. When Γ is a tree, this is an equivalence relation on infinite paths and the *boundary* $\partial\Gamma$ of Γ is the set of these equivalence classes.

For the following, let $v \in V(\Gamma)$. Set $\mathrm{in}_\Gamma(v) := \{w \in V(\Gamma) \mid (w,v) \in E(\Gamma)\}$ and $\mathrm{out}_\Gamma(v) := \{w \in V(\Gamma) \mid (v,w) \in E(\Gamma)\}$. The *in-valency* of $v \in V(\Gamma)$ is the cardinality of $\mathrm{in}_\Gamma(v)$ and the *out-valency* of $v \in V(\Gamma)$ is the cardinality of $\mathrm{out}_\Gamma(v)$. The directed graph Γ is *locally finite* if all its vertices have finite in- and out-valency.

A *directed line* in Γ is a sequence $(v_i)_{i\in\mathbb{Z}}$ of distinct vertices such that either (v_i, v_{i+1}) is an edge for every $i \in \mathbb{Z}$, or (v_i, v_{i-1}) is an edge for every $i \in \mathbb{Z}$.

For a subset $A \subseteq V(\Gamma)$, the *subgraph of* Γ *spanned by* A is the directed graph with vertex set A and edge set $\{(v,w) \in E(\Gamma) \mid v, w \in A\}$.

The *set of descendants* of $v \in V(\Gamma)$ is $\mathrm{desc}_\Gamma(v) := \{w \in V(\Gamma) \mid \exists \text{ arc from } v \text{ to } w\}$. For $A \subseteq V(\Gamma)$, set $\mathrm{desc}_\Gamma(A) := \bigcup_{v\in A} \mathrm{desc}_\Gamma(v)$. A directed tree Γ is *rooted* at $v_0 \in V(\Gamma)$ if $\Gamma = \mathrm{desc}(v_0)$, in which case $|\mathrm{in}_\Gamma(v)| = 1$ for all vertices $v \neq v_0$ and $|\mathrm{in}_\Gamma(v_0)| = 0$. The definition of being regular is altered for rooted trees: A directed tree rooted at v_0 is *regular* if $|\mathrm{out}(v)|$ is constant for $v \in V(\Gamma)$.

A *morphism* between directed graphs $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$ is a pair (α_V, α_E) of maps $\alpha_V : V_1 \to V_2$ and $\alpha_E : E_1 \to E_2$ preserving the graph structure, i.e. $\alpha_V(\mathrm{pr}_1(e)) = \mathrm{pr}_1 \alpha_E(e)$ and $\alpha_V(\mathrm{pr}_2(e)) = \mathrm{pr}_2 \alpha_E(e)$ for all $e \in E_1$. An *automorphism* of a directed graph $\Gamma = (V, E)$ is a morphism $\alpha = (\alpha_V, \alpha_E)$ from Γ to itself such that α_V and α_E are bijective and α admits an inverse morphism.

CHAPTER V

Tidiness and Scale for Subgroups and Quotients

This section contains joint work with T. Bywaters and H. Glöckner, namely [**BGT16**, Section 8]. We generalize several results of [**Wil01**] about how tidy subgroups and the scale behave with respect to taking subgroups and quotients from automorphisms to endomorphisms. This can be seen as a parallel to the study of topological entropy given in [**BV16**]. Generally speaking, the proofs follow the same basic structure as those for automorphisms but changes need to be made to accommodate for the additional complications that arise in the case of endomorphisms.

1. Subgroups

We first explore the effect of taking subgroups on tidiness and the scale. The following two lemmas show that tidy subgroups behave well when passing to subgroups. Lemma V.2 is applied in Theorem V.3 which concerns the scale.

Lemma V.1. Let G be a t.d.l.c. group, $\alpha \in \mathrm{End}(G)$ and $H \leq G$ closed with $\alpha(H) \leq H$. Further, let $W \leq G$ be compact open. Then there exists $N \in \mathbb{N}_0$ such that $W_{-n} \cap H$ is tidy above for $\alpha|_H$, for all $n \geq N$.

Proof. Since $\alpha(H) \leq H$ we conclude that $H \cap W_{-n}$ equals

$$H \cap \bigcap_{k=0}^{n} \alpha^{-k}(W) = \{w \in H \,|\, \forall k \in \{1,\ldots,n\} : \alpha^k(w) \in W\}$$
$$= \{w \in H \,|\, \forall k \in \{1,\ldots,n\} : \alpha^k(w) \in W \cap H\} = \bigcap_{k=0}^{n} (\alpha|_H)^{-k}(H \cap W).$$

which is tidy above for $\alpha|_H$ by [**Wil15**, Proposition 3] for large n. □

Lemma V.2. Let G be a t.d.l.c. group, $\alpha \in \mathrm{End}(G)$ and $H \leq G$ closed with $\alpha(H) \leq H$. Further, let $U \leq G$ be compact open and tidy for α. Set $V := U \cap H$. Then there is $N \in \mathbb{N}$ such that V_{-N} is tidy for $\alpha|_H$.

Proof. Note that V is a compact open subgroup of H. By [**Wil15**, Proposition 3] there is $N \in \mathbb{N}$ such that V_{-N} is tidy above for $\alpha|_H$. Since U is minimizing, the same proposition implies that U_{-N} is tidy for α. By Lemma V.1, replacing U by U_{-N}, we may assume that V is tidy above for $\alpha|_H$. To see that this V is tidy, we show that $\mathcal{L}_V \leq V$ where $\mathcal{L}_V$ is given in Algorithm IV.3. Since $V \leq H$ is closed this implies that $L_V = \overline{\mathcal{L}_V} \leq V$ and hence V is tidy below and therefore tidy for $\alpha|_H$ by [**Wil15**, Proposition 8]. First, note that

$$V_- = \bigcap_{n \geq 0} V_{-n} = U_- \cap H$$

Also, since V_+ is the collection of all elements in V that admit an α-regressive trajectory in $V = U \cap H$, it follows that $V_+ \leq U_+ \cap H$. Now, suppose that $x \in \mathcal{L}_V$. Then $x \in H$ and there are $y \in V_+$ and $m, n \in \mathbb{N}$ such that $\alpha^m(y) = x$ and $\alpha^n(y) \in V_-$. By the above, $y \in U_+$ and $\alpha^n(y) \in U_-$. Therefore, $x \in \mathcal{L}_U \cap H$. Since U is tidy for α we have $\mathcal{L}_U \leq U$ and thus conclude $x \in U \cap H = V$. This shows $\mathcal{L}_V \leq V$ as required. □

Theorem V.3. Let G be a t.d.l.c. group and $\alpha \in \mathrm{End}(G)$. Furher, let $H \leq G$ be closed with $\alpha(H) \leq H$. Then $s_H(\alpha|_H) \leq s_G(\alpha)$. Furthermore, if $H \trianglelefteq G$ and $U \leq G$ is compact open and tidy for α such that $U \cap H$ is tidy for $\alpha|_H$, then $\alpha((U \cap H)_+)U_+$ is a subgroup of G and $s_H(\alpha|_H) = [\alpha((U \cap H)_+)U_+ : U_+]$.

Proof. By Lemma V.2 there is a compact open subgroup $U \leq G$ which is tidy for α and such that $V := U \cap H$ is tidy for $\alpha|_H$. In particular, $s_H(\alpha|_H) = [\alpha(V_+) : V_+]$ and $s_G(\alpha) = [\alpha(U_+) : U_+]$. Define $\varphi : \alpha(V_+)/V_+ \to \alpha(U_+)/U_+$ by $\varphi(uV_+) := uU_+$ for all $uV_+ \in \alpha(V_+)/V_+$. Then φ is well-defined as $V_+ \leq U_+$. For the first claim it suffices to show that φ is injective. Indeed, assume that $\varphi(uV_+) = \varphi(vV_+)$ for some $uV_+, vV_+ \in \alpha(V_+)/V_+$. Then it follows that $x := v^{-1}u \in \alpha(V_+) \cap U_+$ where $\alpha(V_+) = \alpha((U \cap H)_+) \leq H$. It is now a consequence of [**Wil15**, Lemma 1] that $x \in U \cap H \cap \alpha(V_+) = V \cap \alpha(V_+) = V_+$.

For the second claim, suppose that H is normal in G. It suffices to show that $\alpha((U \cap H)_+)U_+ = U_+\alpha((U \cap H)_+)$: Indeed, this implies that $\alpha((U \cap H)_+)U_+$ is a group in which case the assertion follows from the previous paragraph. Now, $(U \cap H)_0 := U \cap H$ is normal in $U_0 := U$ and $(U \cap H)_{n+1} := \alpha((U \cap H)_n) \cap U \cap H$ is normal in $U_{n+1} := \alpha(U_n) \cap U$ for each $n \in \mathbb{N}_0$ by the following inductive argument: By the inductive hypothesis, $(U \cap H)_n$ is normal in U_n. Hence $\alpha((U \cap H)_n)$ is normal in $\alpha(U_n)$. Since $U \cap H$ is normal in U, it follows that $\alpha((U \cap H)_n) \cap U \cap H$ is normal in $\alpha(U_n) \cap U$ which completes the induction. As a consequence,

$$(U \cap H)_+ := \bigcap_{n \in \mathbb{N}_0} (U \cap H)_n \quad \text{is normal in} \quad U_+ := \bigcap_{n \in \mathbb{N}_0} U_n.$$

Let $u \in U_+$. Pick $w \in U_+$ with $\alpha(w) = u$. Applying α to $(U \cap H)_+ w = w(U \cap H)_+$, we deduce that $\alpha((U \cap H)_+)u = u\alpha((U \cap H)_+)$. □

2. Quotients

We now turn our attention to quotients. Again, we first consider tidy subgroups and then apply our findings to gain insight into the scale. Our first lemma provides control over α-regressive trajectories. We let L_U and $\widetilde{U}$ be as in Algorithm IV.3.

Lemma V.4. Let G be a t.d.l.c. group, $\alpha \in \mathrm{End}(G)$ and $U \leq G$ compact open as well as tidy above for α. Then $U \cap \widetilde{U} L_U = \widetilde{U}$.

Proof. By definition $\widetilde{U} \leq U \cap \widetilde{U} L_U$ as $\widetilde{U} \leq U$ and $\widetilde{U} \leq \widetilde{U} L_U$. Now, let $x \in U \cap \widetilde{U} L_U$. We need to show $xL_U \leq L_U U$. Indeed, $xL_U \leq \widetilde{U} L_U L_U = \widetilde{U} L_U \leq L_U U$. □

There are examples of automorphisms [**Wil01**] and associated tidy below subgroups which do not behave well when passing to quotients. Lemma V.6 shows that although we cannot expect a tidy below subgroup to be tidy below when passing to a quotient, the original subgroup can be chosen so that the quotient is as close as possible to being tidy below using Algorithm IV.3. The proof of LemmaV.6 relies on the following result which is immediate from the proof of [**Wil15**, Lemma 16].

Lemma V.5. Let G be a t.d.l.c. group, $\alpha \in \mathrm{End}(G)$ and $U \leq G$ compact open as well as tidy above for α. Let $u \in \widetilde{U}$. Then $u_\pm \in \widetilde{U}_\pm$ for any $u_\pm \in U_\pm$ with $u = u_+ u_-$.

Lemma V.6. Let G be a t.d.l.c. group, $\alpha \in \mathrm{End}(G)$ and $H \trianglelefteq G$ closed with $\alpha(H) \leq H$. Denote by $\overline{\alpha}$ the endomorphism induced by α on G/H and by $q\colon G \to G/H$ the quotient map. Then there is a compact open subgroup U of G such that

(i) U tidy for α,
(ii) $U \cap H$ is tidy for $\alpha|_H$, and
(iii) $q(U)$ is tidy above for $\overline{\alpha}$, and $L_{q(U)}q(U) = q(U)L_{q(U)}$.

Proof. Applying Lemma V.2, choose $V \leq G$ compact open and tidy for α and such that $V \cap H$ is tidy for $\alpha|_H$. Then $q(V)$ is tidy above for $\overline{\alpha}$: On the one hand

$$q(V_-) = q\left(\bigcap_{n\geq 0} \alpha^{-n}(V)\right) \subseteq \bigcap_{n\geq 0} q(\alpha^{-n}(V)) = \bigcap_{n\geq 0} \overline{\alpha}^{-n}(q(V)) = q(V)_-.$$

Also, $V_+ = \{x \in V \,|\, x$ admits an α-regressive trajectory in $V\}$. Thus $q(V_+) \subseteq q(V)_+$ as α-regressive trajectories descend to the quotient. Combined, we conclude

$$q(V) = q(V_+V_-) = q(V_+)q(V_-) \subseteq q(V)_+q(V)_-.$$

That is, $q(V)$ is tidy above for $\overline{\alpha}$. Now define $U := V \cap q^{-1}(q(V)^{\widetilde{}})$, where $q(V)^{\widetilde{}}$ is as in Algorithm IV.3. Then $q(U) = q(V)^{\widetilde{}}$ and hence $q(U)$ is tidy above for $\overline{\alpha}$ by [**Wil15**, Lemma 16]. In addition, by applying [**Wil15**, Proposition 6 (3)] we see that $L_{q(U)} = L_{q(V)^{\widetilde{}}} = L_{q(V)}$. It follows from [**Wil15**, Lemma 13] and $q(U) = q(V)^{\widetilde{}}$ that $q(U)L_{q(U)} = L_{q(U)}q(U)$. Furthermore, $V \cap H \subseteq \ker q \subseteq q^{-1}(q(V)^{\widetilde{}})$. Hence

$$U \cap H = V \cap H \cap q^{-1}(q(V)^{\widetilde{}}) = V \cap H$$

is tidy for $\alpha|_H$.

It remains to show that U is tidy for α. We begin by proving that U is tidy above for α. Let $u \in U$. Then since V is tidy above, $u = v_+v_-$ for some $v_\pm \in V_\pm$ and we aim to show that $v_\pm \in U_\pm$. Note that $q(u) = q(v_+)q(v_-)$ with $q(v_\pm) \in q(V_\pm) \subseteq q(V)_\pm$. Since $q(u) \in q(V)^{\widetilde{}}$, we deduce $q(v_\pm) \in (q(V)^{\widetilde{}})_\pm$ by Lemma V.5. Since $\alpha^n(v_-) \in V_-$ and $\overline{\alpha}^n(q(v_-)) \in (q(V)^{\widetilde{}})_-$ for all $n \geq 0$ we have $q(\alpha^n(v_-)) \in (q(V)^{\widetilde{}})_-$. Therefore, the orbit of $v_- \in V \cap q^{-1}(q(V)^{\widetilde{}}) = U$ stays in U and we conclude $v_- \in U_-$.

As to v_+, choose an α-regressive trajectory $(v_i)_{i\in\mathbb{N}_0}$ for v_+ contained in V_+. We show that this sequence is contained within U. It is clear that $v_0 = v_+ \in U$. Suppose for the purpose of induction that $v_n \in U$. Applying [**Wil15**, Lemma 15] we see that $q(v_n) \in q(U) \cap q(V_+) \subseteq q(V)^{\widetilde{}} \cap q(V)_+ = (q(V)^{\widetilde{}})_+$. There exists $w \in (q(V)^{\widetilde{}})_+$ with

$$\overline{\alpha}(w) = q(v_n) = \overline{\alpha}(q(v_{n+1})).$$

Now w, $q(v_n)$ and $q(v_{n+1})$ are elements of $\mathrm{par}^-(\overline{\alpha})$. By [**Wil15**, Proposition 20], there is $b \in \mathrm{bik}(\overline{\alpha})$ such that $q(v_{n+1}) = wb$. Since $q(V)^{\widetilde{}}L_{q(V)}$ is tidy, $b \in q(V)^{\widetilde{}}L_{q(V)}$. Hence $q(v_{n+1}) \in q(V)^{\widetilde{}}L_{q(V)}$. By Lemma V.4, $q(v_{n+1}) \in q(V)^{\widetilde{}}$ whence $v_{n+1} \in U$. Inductively, $v_i \in U$ for all $i \in \mathbb{N}_0$ and so $v_+ \in U_+$.

To see that U is tidy below, note that V is tidy below and $U \subseteq V$. Hence $L_U \subseteq V_+ \cap V_-$. Clearly, $q(V_+ \cap V_-) \subseteq L_{q(V)}$ and so $q(V_+ \cap V_-) \subseteq q(V)^{\widetilde{}}$. Hence $V_+ \cap V_- \subseteq U$. As a consequence, $L_U \subseteq U$ which implies that U is tidy below, see [**Wil15**, Proposition 8]. □

In the following lemma, we factor the subgroup used to calculate the scale. Later on, we turn this into a factorization of the scale itself.

Lemma V.7. Let G be a t.d.l.c. group, $\alpha \in \mathrm{End}(G)$ and $H \trianglelefteq G$ closed with $\alpha(H) \leq H$. Denote by $\overline{\alpha}$ the endomorphism induced by α on G/H. Then there is a closed subgroup J of G with $\alpha((H \cap U)_+)U_+ \leq J \leq \alpha(U_+)$ and $s_{G/H}(\overline{\alpha}) = [\alpha(U_+) : J]$.

Proof. Let U satisfy the conclusions of Lemma V.6 and let $q\colon G \to G/H$ denote the quotient map. Then $q(U)L_{q(U)}$ is tidy for $\overline{\alpha}$ and

$$s_{G/H}(\overline{\alpha}) = [\overline{\alpha}(q(U)_+)L_{q(U)} : q(U)_+L_{q(U)}]$$

using [**Wil15**, Proposition 4, Proposition 6 (2)]. Now consider the map

$$\overline{\alpha}(q(U)_+)/(\overline{\alpha}(q(U)_+) \cap q(U)_+L_{q(U)}) \to \overline{\alpha}(q(U)_+L_{q(U)})/q(U)_+L_{q(U)}$$

given by

$$g(\overline{\alpha}(q(U)_+) \cap q(U)_+L_{q(U)}) \mapsto g(q(U)_+L_{q(U)}).$$

This map is well-defined as $\overline{\alpha}(q(U)_+) \cap q(U)_+ L_{q(U)}) \le q(U)_+ L_{q(U)}$. It is injective because any two elements in the domain which have the same image have coset representatives which differ by an element in $\overline{\alpha}(q(U)_+) \cap q(U)_+ L_{q(U)}$. To see surjectivity, simply note that $\overline{\alpha}(L_{q(U)}) \le L_{q(U)} \le q(U_+) L_{q(U)}$ by [**Wil15**, Lemma 6]. This shows

$$\begin{aligned} s_{G/H}(\overline{\alpha}) &= [\overline{\alpha}(q(U)_+) L_{q(U)} : q(U)_+ L_{q(U)}] \\ &= [\overline{\alpha}(q(U)_+) : \overline{\alpha}(q(U)_+) \cap q(U)_+ L_{q(U)}]. \end{aligned} \tag{1}$$

We know that $\overline{\alpha}(q(U)_+) \cap q(U)_+ L_{q(U)}$ is closed in G/H because $\overline{\alpha}$ and q are continuous, U is compact and $L_{q(U)}$ is closed. Set

$$J := q^{-1}\left(\overline{\alpha}(q(U)_+) \cap q(U)_+ L_{q(U)}\right) \cap \alpha(U_+).$$

By the above, $J \le \alpha(U_+)$ is closed. To see $\alpha((H \cap U)_+)U_+ \le J$, note that

$$q(\alpha((H \cap U)_+)U_+) = q(U_+) \le q(U)_+ \le \overline{\alpha}(q(U)_+) \cap q(U)_+ L_{q(U)} =: S \tag{2}$$

because $\alpha((H \cap U)_+)U_+ = U_+ \alpha((H \cap U)_+)$ and $\alpha((H \cap U)_+)$ is contained in H. The formula

$$x.(yS) := q(x)yS \quad \text{for } x \in \alpha(U_+) \text{ and } y \in q(U_+)$$

defines a transitive action of $\alpha(U_+)$ on $X := \overline{\alpha}(q(U_+))/S$ as $q(\alpha(U_+)) = \overline{\alpha}(q(U_+))$. Since $S \in X$ has stabilizer $q^{-1}(S) \cap \alpha(U_+) = J$ under the action, the Orbit Stabilizer Theorem (as in [**Rob96**, 1.6.1 (i)]) shows that

$$\alpha(U_+) : J] = |X| = [\overline{\alpha}(q(U_+)) : S].$$

Combining this with (2) and (1) we obtain $s_{G/H}(\overline{\alpha}) = [\alpha(U_+) : J]$. □

Theorem V.8. Let G be a t.d.l.c. group, $\alpha \in \mathrm{End}(G)$ and $H \trianglelefteq G$ closed with $\alpha(H) \le H$. Then $s_H(\alpha|_H) s_{G/H}(\overline{\alpha})$ divides $s_G(\alpha)$.

Proof. Let U satisfy the conclusions of Lemma V.6. By Lemma V.7 there is a closed subgroup J of G such that

$$U_+ \subseteq \alpha((U \cap H)_+)U_+ \subseteq J \subseteq \alpha(U_+).$$

Recall that by Theorem V.3, the set $\alpha((U \cap H)_+)U_+$ is indeed a subgroup of G. Applying Lemma V.7 and Theorem V.3 yields

$$\begin{aligned} s_G(\alpha) &= [\alpha(U_+) : U_+] \\ &= [\alpha(U_+) : J][J : \alpha((U \cap H)_+)U_+][\alpha((U \cap H)_+)U_+ : U_+] \\ &= s_{G/H}(\overline{\alpha})[J : \alpha((U \cap H)_+)U_+] s_H(\alpha|_H). \end{aligned}$$

which completes the proof. □

We end this section by considering the special case of nested subgroups inside $\mathrm{par}^-(\alpha)$ for which we achieve equality in Theorem V.8.

Lemma V.9. Let G be a t.d.l.c. group, $\alpha \in \mathrm{End}(G)$ and $H \le \mathrm{par}^-(\alpha)$ closed such that $\alpha(H) = H$. Then $\mathrm{par}^-(\alpha|_H) = H$.

Proof. Suppose $x \in H$. We can find an α-regressive trajectory $(x = x_0, x_1, \ldots)$ which is contained in some compact set K. Since $\alpha(H) = H$ we can choose another α-regressive trajectory $(x = y_0, y_1, \ldots)$ such that $y_n \in H$ for all $n \in \mathbb{N}$. Therefore $y_n, x_n \in \mathrm{par}^-(\alpha)$ and hence $x_n^{-1} y_n \in \mathrm{bik}(\alpha)$ for all $n \in \mathbb{N}$. Thus $y_n \in x_n \mathrm{bik}(\alpha)$ which is contained in $K \mathrm{bik}(\alpha)$. Since both K and $\mathrm{bik}(\alpha)$ are compact, $K \mathrm{bik}(\alpha)$ is compact and hence $K \mathrm{bik}(\alpha) \cap H$ is a compact subset of H. This shows that $(y_0, y_1, \ldots)$ is bounded and hence $x \in \mathrm{par}^-(\alpha|_H)$. □

The following result is known for automorphisms [**DS91**, Proposition 3.21 (2)]. Its proof utilizes the modular function which is not defined for endomorphisms. Instead we consider the factoring of the scale given by Theorem V.8.

Proposition V.10. Let G be a t.d.l.c. group, $\alpha \in \mathrm{End}(G)$ and $H \leq \mathrm{par}^-(\alpha)$ closed such that $\alpha(H) = H$. Further, let $N \trianglelefteq H$ be closed with $\alpha(N) = N$. Denote by $\overline{\alpha}$ the endomorphism induced by $\alpha|_H$ on H/N. Then

$$s_H(\alpha|_H) = s_{H/N}(\overline{\alpha})s_N(\alpha|_N).$$

Proof. For simplicity, we write α for $\alpha|_H$ as the enveloping group will play no further role. By Lemma V.9, $\mathrm{par}^-(\alpha) = H$ and so if $U \leq H$ is compact open as well as tidy for α, then $U = U_+$ by [**Wil15**, Proposition 11].

By Lemma V.2, we may assume that $U \cap N$ is tidy for $\alpha|_N$. Let $q : H \to H/N$ denote the quotient map. Choose $U \leq H$ compact open and satisfying conditions of Lemma V.6 with respect to N. From the proof of Theorem V.8 we have

$$s_H(\alpha) = s_{H/N}(\overline{\alpha})[J : \alpha((U \cap N)_+)U_+]s_N(\alpha|_N),$$

where J is given in the proof of Lemma V.6 by

$$J = q^{-1}(\overline{\alpha}(q(U)_+) \cap q(U)_+L_{q(U)}) \cap \alpha(U_+).$$

It suffices to show $J \leq \alpha((U \cap N)_+)U_+$. Since $q(U_+) \leq q(U)_+$, as seen in the proof of Lemma V.6, and $U_+ = U$ we have $q(U_+) \leq q(U)_+ \leq q(U) = q(U_+)$, which gives equality throughout. Thus $J = q^{-1}\left(\overline{\alpha}(q(U)) \cap q(U)L_{q(U)}\right) \cap \alpha(U)$. Since $q(U)$ is an open identity neighbourhood, we obtain

$$q(U)L_{q(U)} = q(U)\overline{\mathcal{L}_{q(U)}} = q(U)\mathcal{L}_{q(U)}.$$

Suppose that $x \in q^{-1}(q(U)L_{q(U)})$. Then we can write $x = ul$ for some $u \in U$ and $l \in q^{-1}(\mathcal{L}_{q(U)})$. Consider $q(l) = lN \in \mathcal{L}_{q(U)}$. There exists $n \in \mathbb{N}$ with

$$\overline{\alpha}^n(lN) = \alpha^n(l)N \in q(U).$$

This implies $\alpha^n(l)m \in U$ for some $m \in N$. Then $\alpha^n(l)m$ has an α-regressive trajectory contained in $U = U_+$. Using that fact that N is assumed to satisfy $\alpha(N) = N$, choose $m' \in N$ such that $\alpha^n(m') = m$.

Since [**Wil15**, Proposition 20] implies that any two elements in the preimage of an element of $\mathrm{par}^-(\alpha) = H$ are equal modulo $\mathrm{bik}(\alpha)$, we have $lm' \in U\,\mathrm{bik}(\alpha)$ by comparing $\alpha^n(lm') = \alpha^n(l)m$ with the α-regressive trajectory for $\alpha^n(l)m$ contained in U. But U is tidy and so $\mathrm{bik}(\alpha) \leq U$. Hence $l \in UN$ and thus $x \in UN$. This shows that $J \subset UN \cap \alpha(U)$. Suppose now that $x \in UN \cap \alpha(U)$. Then we can write $x = un$ where $u \in U$ and $n \in N$. Choose α-regressive trajectories

$$(u = u_0, u_1, \ldots), (un = v_0, v_1, \ldots), \text{ and } (n = n_0, n_1, \ldots)$$

such that $u_i, v_{i+1} \in U$ for all $i \geq 0$ and $n_i \in N$ for all $i \in \mathbb{N}$. Now, notice that $(un = u_0n_0, u_1n_1, \ldots)$ is also an α-regressive trajectory. For all $i \geq 1$ we have $u_in_i \in v_i\,\mathrm{bik}(\alpha)$. Noting that $\mathrm{bik}(\alpha) \leq U$, we have $n_i \in U$ for all $i \geq 1$. Then $n_1 \in (U \cap N)_+$ and so $n = n_0 = \alpha(n_1) \in \alpha((U \cap N)_+)$. As $x = un$, this shows $x \in U\alpha((U \cap N)_+) = \alpha((U \cap N)_+)U$ (with equality by Theorem V.3). □

CHAPTER VI

Tidiness and Scale via Graphs

This section contains joint work with T. Bywaters, namely [**BT17**]. We study Willis' theory of totally disconnected locally compact groups and their endomorphisms in a geometric framework using graphs. This leads to new interpretations of tidy subgroups and the scale function. Foremost, we obtain a geometric tidying procedure which applies to endomorphisms as well as a geometric proof of the fact that tidiness is equivalent to being minimizing for a given endomorphism. Our framework also yields an endomorphism version of the Baumgartner-Willis tree representation theorem. We conclude with a construction of new endomorphisms of totally disconnected locally compact groups from old via HNN-extensions.

1. Characterization of Tidy Subgroups

Let G be a totally disconnected, locally compact group and let $\alpha \in \mathrm{End}(G)$. In this section, we characterize the compact open subgroups U of G which are tidy for α in terms of certain directed graphs. In doing so we generalize several results of [**Möl02**] from conjugation automorphisms to general endomorphisms.

Frequently, we restrict to the case where the set $\{\alpha^{-i}(U) \mid i \in \mathbb{N}_0\}$ is infinite and hence all $\alpha^{-i}(U)$ $(i \in \mathbb{N}_0)$ are distinct. The finite case corresponds to Möller's periodicity case [**Möl02**, Lemma 3.1] and is covered by the following lemma.

Lemma VI.1. Let G be a t.d.l.c. group, $\alpha \in \mathrm{End}(G)$ and $U \leq G$ compact open. If $\{\alpha^{-i}(U) \mid i \in \mathbb{N}_0\}$ is finite then there is $N \in \mathbb{N}_0$ such that $V := \bigcap_{k=0}^{N} \alpha^{-k}(U) = U_-$ satisfies $\alpha(V) \leq V$ and is tidy for α.

Proof. If $\{\alpha^{-i}(U) \mid i \in \mathbb{N}_0\}$ is finite, then $U_- = \bigcap_{k\in\mathbb{N}_0} \alpha^{-i}(U)$ is an intersection of finitely many open subgroups. Say $U_- = \bigcap_{k=0}^{N} \alpha^{-k}(U) =: V$. Then $V \leq G$ is compact open and $\alpha(V) \leq V$. We conclude $V = V_-$. Hence V is tidy above for α. Since $V = V_- \leq V_{--}$ we also deduce that V_{--} is open and hence closed. Thus V is also tidy below for α. □

1.1. Tidiness Above.

We recover the fact that for every compact open subgroup $U \leq G$ there is $n \in \mathbb{N}_0$ such that $U_{-n} = \bigcap_{k=0}^{n} \alpha^{-n}(U)$ is tidy above for α.

Consider the graph Γ defined as follows: Set $v_{-i} := \alpha^{-i}(U) \in \mathcal{P}(G)$ for $i \in \mathbb{N}_0$, where $\mathcal{P}(G)$ denotes the power set of G. Now set

$$V(\Gamma) := \{gv_{-i} \mid g \in G,\ i \in \mathbb{N}_0\} \quad \text{and} \quad E(\Gamma) := \{(gv_{-i}, gv_{-i-1}) \mid g \in G,\ i \in \mathbb{N}_0\}.$$

Note that G acts on Γ by automorphisms via left multiplication. For this action, we compute the stabilizer $G_{v_{-i}} = \alpha^{-i}(U)$ $(i \geq 0)$, as well as

$$G_{\{v_{-m} \mid m \geq 0\}} = \bigcap_{m\geq 0} \alpha^{-m}(U) = U_-.$$

We now reprove [**Wil15**, Lemma 4] in terms of the graph Γ.

Lemma VI.2. Retain the above notation. Suppose that $U_N v_{-1} = U_+ v_{-1}$ for some $N \in \mathbb{N}$. Then $U_{-n} v_{-n-1} = (U_{-n})_+ v_{-n-1}$ for all $n \geq N$.

Proof. By definition, $(U_{-n})_+v_{-n-1} \subseteq U_{-n}v_{-n-1}$. Now, let $w \in U_{-n}v_{-n-1}$. Then there is $u \in U_{-n}$ such that $w = uv_{-n-1}$. We obtain $\alpha^n(u) \in \alpha^n(U_{-n})$ which equals U_n by Lemma IV.2 and is contained in U_N since $n \geq N$. Hence, by assumption, there is $u_+ \in U_+$ such that $\alpha^n(u)v_{-1} = u_+v_{-1}$. By definition of U_+, we may pick $u'_+ \in U_+ \cap U_{-n}$ such that $u_+v_{-1} = \alpha^n(u'_+)v_{-1}$. Then $u'_+ \in (U_{-n})_+$ as by Lemma IV.2 we have $U_+ \cap U_{-n} = (U_{-n})_+$. We conclude that $u'_+v_{-n-1} = uv_{-n-1}$ since $u'_+u^{-1} \in U_{-n-1} \leq G_{v_{-n-1}}$ by the following argument: We have $u'_+u^{-1} \in U_{-n}$ by definition and $u'_+u^{-1} \in \alpha^{-n-1}(U)$ by the subsequent computation:

$$\alpha^{n+1}(u'_+u^{-1}) = \alpha^{n+1}(u'_+)\alpha^{n+1}(u^{-1}) = \alpha(u_+\alpha^n(u^{-1})) \in U$$

since, by construction, $u_+\alpha^n(u)^{-1} \in G_{v_{-1}} = \alpha^{-1}(U)$. □

The following Lemma will be used to prove analogues of Theorems 2.1 and 2.3 from [**Möl02**].

Lemma VI.3. Retain the above notation. Fix $N \in \mathbb{N}$ and consider the following:

(i) $U_Nv_{-1} = U_+v_{-1}$.
(ii) For every $u \in U_{-N}$ there is $u_+ \in U_+ \cap U_{-N}$ with $u_+v_i = uv_i$ for all $i \leq 0$.
(iii) The subgroup U_{-N} is tidy above for α.

Then (i) implies (ii), and (ii) implies (iii).

Proof. To see (i) implies (ii) let $u \in U_{-N}$. By induction, we construct a sequence $(u_n)_{n\in\mathbb{N}}$ contained in $U_+ \cap U_{-N}$ such that $u_nv_i = uv_i$ for all $i \in \{-N-n, \dots, 0\}$. Then, as $U_+ \cap U_{-N}$ is compact, $(u_n)_{n\in\mathbb{N}}$ has an accumulation point $u_+ \in U_+ \cap U_{-N}$. We conclude that for any given $n \in \mathbb{N}$, we have

$$u_k^{-1}u_+ \in G_{v_{-n}} = \alpha^{-n}(U)$$

for large enough $k \in \mathbb{N}$ because $\alpha^{-n}(U)$ is open. That is, given $n \in \mathbb{N}$ we have

$$u_+(v_{-n}) = u_k(v_{-n}) = u(v_{-n}).$$

for sufficiently large $k \in \mathbb{N}$.

Now, by (i), Lemma VI.2 and Lemma IV.2, we may pick $u_1 \in U_+ \cap U_{-N}$ such that $u_1v_{-N-1} = uv_{-N-1}$. Next, assume that u_n has been constructed for some $n \in \mathbb{N}$. Then $uu_n^{-1}(v_i) = v_i$ for all $i \in \{-N-n, \dots, 0\}$. That is,

$$u_n^{-1}u \in \bigcap_{i=0}^{n+N} \alpha^{-i}(U) = U_{-N-n}.$$

By Lemma VI.2, there exists $x \in (U_{-N-n})_+$ such that $u_n^{-1}uv_{-N-n-1} = xv_{-N-n-1}$. By assumption, $u_n \in U_+ \cap U_{-N}$ and, by Lemma VI.2, $x \in (U_{-N-n})_+ = U_+ \cap U_{-N-n}$. Hence $u_nx \in U_+ \cap U_{-N}$. Also, $u_nx(v_i) = u(v_i)$ for all $i \in \{-N-n-1, \dots, 0\}$. We may therefore set $u_{n+1} := u_nx$.

To see that (ii) implies (iii) we use that, by assumption, for every $u \in U_{-N}$ there is $u_+ \in U_+ \cap U_{-N}$ such that u and u_+ agree on v_i for all $i \leq 0$. Set $u_- := u_+^{-1}u$. Then $u_-v_i = v_i$ for all $i \leq 0$. Hence $u_- \in G_{\{v_m \mid m \leq 0\}} = U_-$ and

$$U_{-N} = (U_+ \cap U_{-N})U_- = (U_{-N})_+(U_{-N})_-$$

by Lemma IV.2 as required. □

Theorem VI.4. Let G be a t.d.l.c. group, $\alpha \in \mathrm{End}(G)$ and $U \leq G$ compact open. Then there is $N \in \mathbb{N}$ such that $U_Nv_{-1} = U_+v_{-1}$, and U_{-N} is tidy above for α.

Proof. First note that $U_+v_{-1} \subseteq U_mv_{-1} \subseteq U_nv_{-1}$ for all $0 \leq n \leq m$ since the sets U_n $(n \in \mathbb{N}_0)$ are nested. Thus it suffices to show that $U_Nv_{-1} \subset U_+v_{-1}$ for some $N \in \mathbb{N}$. Towards a contradiction, assume that $U_+v_{-1} \subsetneq U_nv_{-1}$ for all $n \in \mathbb{N}$, i.e. there is $w_n \in V(\Gamma)$ such that $w_n \in U_nv_{-1}$ for all $n \in \mathbb{N}$ but $w_n \notin U_+v_{-1}$. Then there is a sequence $(u_n)_{n\in\mathbb{N}}$ contained in U such that $u_n \in U_n$ and $u_n v_{-1} = w_n$. Since U is compact, the sequence $(u_n)_{n\in\mathbb{N}}$ has an accumulation point u_+ in U. This accumulation point has to be contained in U_+: Indeed, pick a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ of $(u_n)_{n\in\mathbb{N}}$ converging to u_+. Then for any given $m \in \mathbb{N}$, we have $u_{n_k} \in U_m$ for almost all k. Since U_m is closed we conclude that $u_+ \in U_m$ for every $m \in \mathbb{N}$. Hence

$$u_+ \in \bigcap_{m\in\mathbb{N}} U_m = U_+.$$

Furthermore, if $u_+ v_{-1} = w$, then because $u_+ u_{n_k}^{-1}$ is contained in the open set $G_{v_{-1}}$ for large enough $k \in \mathbb{N}$ we must have $w = w_k$ for sufficiently large $k \in \mathbb{N}$. We conclude that $w_k \in U_+ v_{-1}$ for sufficiently large $k \in \mathbb{N}$ and thus we have a contradiction. Now, U_{-N} is tidy above for α by Lemma VI.3. □

Theorem VI.5. Let G be a t.d.l.c. group, $\alpha \in \mathrm{End}(G)$ and $U \leq G$ compact open. Then the following statements are equivalent.

(i) $Uv_{-1} = U_+ v_{-1}$.
(ii) For every $u \in U$ there is $u_+ \in U_+$ such that $u_+ v_i = uv_i$ for all $i \leq 0$.
(iii) The subgroup U is tidy above for α.

Proof. Note that (i) implies (ii) and (ii) implies (iii) by Lemma VI.3 for $N = 0$. Now, if (iii) holds, then $Uv_{-1} = U_+ U_- v_{-1} = U_+ v_{-1}$ as $U_- \leq G_{v_{-1}}$. □

Proposition VI.6. Let G be a t.d.l.c. group, $\alpha \in \mathrm{End}(G)$ and $U \leq G$ compact open as well as tidy above for α. Then

$$[U_{-n} : U_{-n-1}] = [U : U_{-1}] = [\alpha^{-n}(U) : \alpha^{-n-1}(U) \cap \alpha^{-n}(U)]$$

for all $n \in \mathbb{N}$.

Proof. Let $u \in U_{-n} \backslash U_{-n-1}$. Then $\alpha^n(u) \in U \backslash U_{-1}$. Hence $[U_{-n} : U_{-n-1}] \leq [U : U_{-1}]$. Conversely, if $u \in U \backslash U_{-1}$ then u admits a representative in U_+ by Theorem VI.5. Let $(u_n)_n$ be an α-regressive sequence of u contained in U. Then $u_n \in U_{-n-1} \backslash U_{-n}$. Hence equality holds. The same argument applies to the right hand equality. □

The following equality is used in Section 4.

Lemma VI.7. Let G be a t.d.l.c. group, $\alpha \in \mathrm{End}(G)$ and $U \leq G$ compact open as well as tidy above for α. Then $[\alpha(U) : U \cap \alpha(U)] = [\alpha(U_+) : U_+]$.

Proof. Note that

$$\alpha(U)(U \cap \alpha(U)) = \alpha(U_+)\alpha(U_-)(U \cap \alpha(U)) = \alpha(U_+)(U \cap \alpha(U))$$

as $\alpha(U_-) \leq U$. Thus

$$[\alpha(U) : U \cap \alpha(U)] = [\alpha(U_+) : U \cap \alpha(U) \cap \alpha(U_+)] = [\alpha(U_+) : U \cap \alpha(U_+)].$$

Since $U \cap \alpha(U_+) = U_+$, the desired equality follows. □

1.2. Tidiness Below. In this section we present a geometric proof for the commonly used criterion that identifies a compact open and tidy above subgroup $U \leq G$ as tidy below if $U_{--} \cap U = U_-$, cf. [**Wil15**, Proposition 8].

First, recall that $U_{++} = \bigcup_{i\in\mathbb{N}_0} \alpha^i(U_+)$ and $U_{--} = \bigcup_{i\in\mathbb{N}_0} \alpha^{-i}(U_-)$. In terms of the graph Γ introduced in Section 1.1, we have

$$U_{--} = \bigcup\nolimits_{n\in\mathbb{N}} G_{\{v_{-m} \mid m \leq -n\}}.$$

Lemma VI.8. Let G be a t.d.l.c. group, $\alpha \in \mathrm{End}(G)$ and $U \leq G$ be compact open as well as tidy above for α. Then

(i) the group $U_{--} \leq G$ is closed if and only if $U_{--} \cap U = U_-$, and
(ii) if U_{--} is closed then $U_{++} \cap U = U_+$.

Proof. For (i), first assume that $U_{--} \cap U = U_-$. Then $U_{--} \cap U$ is closed. Since U is closed, this implies that U_{--} is closed, see [**HR12**, 5.37].

Now suppose that $U_{--} \cap U \neq U_-$. By definition, $U_- \subseteq U_{--} \cap U$. Hence there exists $u \in U = G_{v_0}$ with $u \in G_{\{v_m \mid m \leq -n\}}$ for some $n \in \mathbb{N}$ but $u \notin U_- = G_{\{v_m \mid m \leq 0\}}$. Then there is $l \in \mathbb{N}$ with $0 < l < n$ and such that $uv_{-l} \neq v_{-l}$. Since U is tidy above, we may decompose $u = u_+ u_-$ for some $u_+ \in U_+$ and $u_- \in U_-$. Hence, replacing u with uu_-^{-1}, we may assume that $u \in U_+$.

Choose an α-regressive trajectory $(u_j)_{j \in \mathbb{N}}$ of u contained in U_+. Define a sequence $(x_i)_{i \in \mathbb{N}}$ contained in $U_{--} \cap U_+ \leq U$ as follows: Set $x_1 := u$ and $x_{i+1} := x_i u_{in}$. We collect the relevant properties of the sequences $(u_j)_{j \in \mathbb{N}}$ and $(x_i)_{i \in \mathbb{N}}$ in the following lemma, see below for an illustration of the second sequence.

Lemma VI.9. The sequences $(u_j)_{j \in \mathbb{N}}$ and $(x_i)_{i \in \mathbb{N}}$ have the following properties.

(a) For all $j \in \mathbb{N}$: $u_j \in G_{\{v_m \mid m \leq -n-j\}} \cap G_{\{v_m \mid -j \leq m \leq 0\}} \cap U_+ \leq U_{--} \cap U_+$.
(b) For all $i \in \mathbb{N}$: $x_i \in G_{\{v_m \mid m \leq -in\}} \cap U_+ \leq U_{--} \cap U_+$.
(c) For all $j \in \mathbb{N}$: $u_j \notin G_{v_{-l-j}}$.
(d) For all $i \in \mathbb{N}$ and $0 \leq j \leq i-1$: $x_i \notin G_{v_{-l-jn}}$ and $x_{i+1} v_{-l-jn} = x_i v_{-l-jn}$.

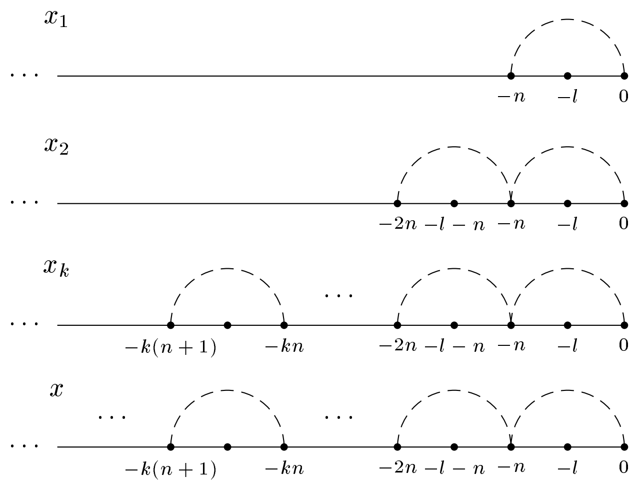
Proof. For (a), note that $\alpha^j(u_j) = u \in G_{\{v_m \mid m \leq -n\}} = \bigcap_{k \geq n} \alpha^{-k}(U)$ by assumption and therefore $u_j \in \alpha^{-j}\big(\bigcap_{k \geq n} \alpha^{-k}(U)\big) = \bigcap_{k \geq n+j} \alpha^{-k}(U) = G_{\{v_m \mid m \leq -n-j\}}$. For the second part, simply recall that $(u_j)_j$ is an α-regressive trajectory of u contained in U_+; in particular, $u_j \in U_+$ and $\alpha^m(u_j) \in U_+ \leq U$ for all $0 \leq m \leq j$. Therefore, $u_j \in \alpha^{-m}(U) = G_{v_{-m}}$ for all $0 \leq m \leq j$.

Part (b) follows from (a) given that $x_{i+1} = x_i u_{in} = uu_n \cdots u_{(i-1)n} u_{in}$.

For part (c), recall that we have $u \notin \alpha^{-l}(U) = G_{v_{-l}}$ by assumption and therefore $u_j \notin \alpha^{-l-j}(U) = G_{v_{-l-j}}$.

In order to prove part (d), we argue by induction: The element $x_1 = u$ satisfies $x_1 \notin G_{v_{-l}}$ by part (c). Also $x_2 v_{-l} = x_1 v_{-l}$ because $x_1^{-1} x_2 = u^{-1} u u_n = u_n$ and $u_n \in G_{\{v_m \mid -n \leq m\}}$ by part (a). Now assume the statement holds true for $i \in \mathbb{N}$ and consider $x_{i+1} = x_i u_{in}$. Then $x_{i+1} \notin G_{v_{-l-in}}$ because $u_{in} \notin G_{v_{-l-in}}$ by part (a) whereas $x_i \in G_{v_{-l-in}}$ by part (b). Also, $x_{i+1} v_{-l-jn} = x_i v_{-l-jn}$ for all $0 \leq j \leq i-1$ since $x_{i+1} = x_i u_{in}$ and $u_{in} \in G_{\{v_m \mid -in \leq m\}}$ by part (a). $\square$

By Lemma VI.9, the sequence $(x_i)_{i \in \mathbb{N}} \subseteq U_{--} \cap U_+ \subseteq U$ has the following shape, analogous to [**Möl02**, Figure 1].



Now, since U is compact, the sequence $(x_i)_{i \in \mathbb{N}} \subseteq U_{--} \cap U_+ \subseteq U$ has an accumulation point $x \in U$. However, $x \notin U_{--}$ and hence U_{--} is not closed.

For part (ii), note that $U_+ \subseteq U_{++} \cap U$ by definition. Hence, towards a contradiction, we assume that there is $u \in (U_{++} \cap U)\backslash U_+$. Since U is tidy above we may decompose $u = u_+u_-$ with $u_+ \in U_+$ and $u_- \in U_-$. Replacing u with $u_+^{-1}u \in (U_{++} \cap U)\backslash U_+$ we may hence assume $u \in U_-$.

Now, since $u \in U_{++}$, there is an α-regressive trajectory $(u_n)_{n\in\mathbb{N}}$ of u in G such that for some $N \in \mathbb{N}$ we have $u_n \in U_+$ for all $n \geq N$ and $u_{N-1} \notin U$. Consider the element $u_N \in U$. For $n \geq N$ we have $\alpha^n(u_N) = \alpha^{n-N}(u) \in U_-$. Hence $u_N \in U_{--} \cap U$. However, $u_N \notin U_-$: Indeed, $u_N \notin G_{v_{-1}} = \alpha^{-1}(U)$ because $u_{N-1} \notin U$. Therefore, by part (i), U_{--} is not closed. □

1.3. Tidiness. Finally, we combine the previous sections in order to characterize tidiness in terms of a subgraph of the graph Γ introduced above. As before, let G be a t.d.l.c. group, $\alpha \in \mathrm{End}(G)$ and $U \leq G$ compact open. Recall the definition $v_{-i} := \alpha^{-i}(U) \in \mathcal{P}(G)$ for $i \in \mathbb{N}_0$. We consider the subgraph Γ_+ of Γ defined by

$$V(\Gamma_+) := \{uv_{-i} \mid u \in U,\ i \in \mathbb{N}_0\}, \quad E(\Gamma_+) := \{(uv_{-i}, uv_{-i-1}) \mid u \in U,\ i \in \mathbb{N}_0\}.$$

Note that the action of $U \leq G$ on Γ preserves $\Gamma_+ \subseteq \Gamma$ and that $\Gamma_+ = \mathrm{desc}(v_0)$.

Lemma VI.10. Let G be a t.d.l.c. group, $\alpha \in \mathrm{End}(G)$ and $U \leq G$. If U is tidy above for α then U acts transitively on arcs of a given length issuing from $v_0 \in V(\Gamma_+)$.

Proof. Given that $\mathrm{out}_{\Gamma_+}(v_{-n+1}) = [\alpha^{-n+1}(U) : \alpha^{-n+1}(U) \cap \alpha^{-n}(U)]$ as well as $U_{\{v_{-k} \mid k \leq n-1\}} = U_{-n+1}$, this follows by induction from Proposition VI.6. □

We are now ready to characterize tidiness of U in terms of Γ_+ when the set $\{v_{-i} \mid i \in \mathbb{N}_0\}$ is infinite. Concerning the case where $\{v_{-i} \mid i \in \mathbb{N}_0\}$ is finite, Theorem VI.11 is complemented by Lemma VI.1.

Theorem VI.11. Let G be a t.d.l.c. group, $\alpha \in \mathrm{End}(G)$ and $U \leq G$ compact open. Assume $\{v_{-i} \mid i \in \mathbb{N}_0\}$ is infinite. Then U is tidy for α if and only if Γ_+ is a directed tree with constant in-valency 1, excluding v_0, as well as constant out-valency.

Proof. First, assume that U is tidy for α. Notice that for a given $i \leq 0$, the in- and out-valency is constant among the collection of vertices $\{uv_i \mid u \in U\}$ given that U acts on Γ_+ by automorphisms.

Concerning in-valencies it therefore suffices to show that each v_{-i} for $i \geq 1$ has in-valency equal to one. Suppose otherwise, that is $\mathrm{in}(v_{-i}) \geq 2$ for some $i \geq 1$. Then there is $u \in U_{v_{-i}} \subseteq \alpha^{-i}(U)$ such that $uv_{-i+1} \neq v_{-i+1}$. By Theorem VI.5 we may assume that $u \in U_+$. Now consider $u' := \alpha^i(u) \in U_{++} \cap U$. Since U is tidy below, Lemma VI.8 shows that $u' \in U_+ = U_{++} \cap U$. But $u \notin \alpha^{-i+1}(U)$ and hence $u' = \alpha^i(u) \notin \alpha(U) \supseteq U_+$, a contradiction. Thus Γ_+ is a directed tree.

Concerning out-valencies, we may also restrict our attention to $\{v_{-i} \mid i \in \mathbb{N}_0\}$. Note that $\mathrm{out}(v_0) = |U_+v_{-1}|$ by Theorem VI.5 as U is tidy above. Furthermore, $\mathrm{out}(v_{-i}) = |(U \cap \alpha^{-i}(U))v_{-i-1}| = |(U_+ \cap \alpha^{-i}(U))v_{-i-1}|$ by the same theorem. Now, since Γ_+ is a tree and U_+ fixes v_0, we obtain

$$\mathrm{out}(v_{-i}) = |(U_+ \cap U_{-i})v_{-i-1}| = |(U_{-i})_+v_{-i-1}| = |U_{-i}v_{-i-1}|$$

by Lemma VI.2. We conclude the argument by showing that

$$|U_{-i}v_{-i-1}| = |Uv_{-1}| = |U_+v_{-1}|.$$

On the one hand, we have $|U_{-i}v_{-i-1}| \leq |Uv_{-1}|$: Indeed, suppose $u \in U_{-i}$ does not fix v_{-i-1}. Then $\alpha^i(u)$ does not fix v_{-1}. If it did, we would have $\alpha^i(u) \in \alpha^{-1}(U)$ and hence $u \in \alpha^{-i-1}(U)$. On the other hand, $|U_{-i}v_{-i-1}| \geq |Uv_{-1}|$: Indeed, assume $u \in U$ does not fix v_{-1}, i.e. $u \notin \alpha^{-1}(U)$. By Theorem VI.5, we may assume $u \in U_+$. Pick an α-regressive trajectory $(u_j)_{j\in\mathbb{N}_0}$ of u in U. Then $\alpha^{i+1}(u_i) = \alpha(u) \notin U$ and hence $u_i \notin \alpha^{-i-1}(U)$, i.e. u_i does not fix v_{-i-1}.

Now assume that Γ_+ has all the stated properties. Since Γ_+ is a tree, we have $U_{--} \cap U \subseteq U_-$ while the reverse inclusion holds by definition. Hence U_{--} is closed by Lemma VI.8 and U is tidy below. Combining the constant out-valency assumption with the fact that Γ_+ is a tree we obtain the equality $|Uv_{-1}| = |U_{-i}v_{-i-1}|$. Next, $|U_{-i}v_{-i-1}| = |U_iv_{-1}|$ since $|U_iv_{-1}| \leq |Uv_{-1}|$ and due to the following observation: If $u \in U_{-i}$ is such that $uv_{-i-1} \neq v_{-i-1}$ then $\alpha^i(u) \in \alpha^i(U_{-i}) = U_i$ by Lemma IV.2 and $\alpha^n(u)v_{-1} \neq v_{-1}$. Thus $|U_iv_{-1}| \geq |U_{-i}v_{-i-1}|$. Overall, $|Uv_{-1}| = |U_iv_{-1}|$.

Finally, to see that the above implies $|Uv_{-1}| = |U_+v_{-1}|$, let $u \in U$. Then for every $i \in \mathbb{N}$ there is $u_i \in U_i$ with $uv_{-1} = u_iv_{-1}$. The sequence $(u_i)_{i\in\mathbb{N}}$ is contained in U and hence admits a convergent subsequence. Any such subsequence converges to an element $u_+ \in \bigcap_{i\geq 0} U_i = U_+$ which coincides with u on v_{-1}. Theorem VI.5 now implies that U is tidy above. □

The following Lemma is a useful test of tidiness as it relies only on calculating inverse images and indices. It is, in a sense, an algebraic way to see if Γ_+ satisfies the requirements of Theorem VI.11. We apply it multiple times in upcoming sections.

Lemma VI.12. Let G be a t.d.l.c. group, $\alpha \in \mathrm{End}(G)$ and $U \leq G$ compact open. Then U is tidy for α if and only if $[U : U\cap\alpha^{-n}(U)] = [U : U\cap\alpha^{-1}(U)]^n$ for all $n \in \mathbb{N}$.

Proof. First, assume that U is tidy for α. If $\{v_{-i} \mid i \in \mathbb{N}_0\}$ is finite, then for some $N \in \mathbb{N}_0$ we have $[U_{-N} : U_{-N-1}] = 1$ by Lemma VI.1 and Proposition VI.6 shows that $1 = [U : U \cap \alpha^{-1}(U)]$ which implies $\alpha^{-1}(U) \supseteq U$. Therefore $\alpha^{-n}(U) \supseteq U$ for all $n \in \mathbb{N}$ and the assertion follows. Now assume that $\{v_{-i} \mid i \in \mathbb{N}_0\}$ is infinite. Then Γ_+ is a rooted directed tree with constant out-valency d and we obtain

$$[U : U \cap \alpha^{-n}(U)] = [U_{v_0} : U_{v_0} \cap U_{v_{-n}}] = |Uv_{-n}| = d^n = [U : U \cap \alpha^{-1}(U)]^n$$

by the orbit-stabilizer theorem as desired.

Conversely, assume that $[U : U\cap\alpha^{-n}(U)] = [U : U\cap\alpha^{-1}(U)]^n$ for all $n \in \mathbb{N}$ and consider the graph Γ_+. We have $d := \mathrm{out}(v_0) = [U_{v_0} : U_{v_0}\cap U_{v_{-1}}] = [U : U\cap\alpha^{-1}(U)]$ as before. By definition of Γ_+, the out-valency of any other vertex is at most d. But

$$|Uv_{-n}| = [U_{v_0} : U_{v_0} \cap U_{v_{-n}}] = [U : U \cap \alpha^{-n}(U)] = [U : U \cap \alpha^{-1}(U)]^n = d^n$$

by assumption. Thus, every vertex has out-valency equal to d. Hence Γ_+ is a tree of constant in-valency 1, excluding v_0, and U is tidy for α by Theorem VI.11. □

2. A Graph-Theoretic Tidying Procedure

Let G be a totally disconnected, locally compact group and let $\alpha \in \mathrm{End}(G)$. We show that there is a compact open subgroup of G which is tidy for α.

The proof is algorithmic: Starting from an arbitrary compact open subgroup we construct a locally finite graph Γ_{++}. A certain quotient, inspired by [**Möl00**], of this graph has a connected component isomorphic to a regular rooted tree which admits an action of a subgroup of G. The stabilizer of the root in this tree is the desired tidy subgroup.

For the remainder of the section, fix $U \leq G$ compact open. Refering to Lemma VI.1, we shall assume throughout that $\{\alpha^{-i}(U) \mid i \in \mathbb{N}_0\}$ is infinite. By Theorem VI.4 we may also assume that U is tidy above for α.

2.1. The Graph Γ_{++}. Consider the graph Γ_{++} defined by

$$V(\Gamma_{++}) = \{uv_{-i} \mid u \in U_{++},\, i \in \mathbb{N}_0\}, \text{ and}$$
$$E(\Gamma_{++}) = \{(uv_{-i}, uv_{-i-1}) \mid u \in U_{++},\, i \in \mathbb{N}_0\}.$$

The following remark will be used in the proof of Theorem VI.26.

Remark VI.13. Note that Γ_{++} is a subgraph of Γ. Also, if U is tidy above for α, the graphs Γ_+ and Γ have the same out-valency by Theorem VI.5. Consequently, $\mathrm{descr}_\Gamma(v_0) = \Gamma_+ \subseteq \mathrm{descr}_{\Gamma_{++}}(v_0) \subseteq \mathrm{descr}_\Gamma(v_0) = \Gamma_+$. Hence $\mathrm{descr}_{\Gamma_{++}}(v_0) = \Gamma_+$.

The following Lemma will help to identify vertices in Γ_{++} as (un)equal. It is immediate from the assumption that $\{\alpha^{-i}(U) \mid i \in \mathbb{N}_0\}$ is infinite and the fact that left cosets of distinct subgroups are distinct.

Lemma VI.14. Retain the above notation and let $u_0 v_{-i}, u_1 v_{-j} \in V(\Gamma_{++}) \subseteq \mathcal{P}(G)$. If $u_0 v_{-i} = u_1 v_{-j}$ then $i = j$. □

Note that U_{++} acts on Γ_{++} by automorphisms. We now define an injective graph endomorphism of Γ_{++} that appears frequently. Let $uv_i \in V(\Gamma_{++})$ where $u \in U_{++}$. Since $\alpha(U_{++}) = U_{++}$, there exists $u' \in U_{++}$ such that $\alpha(u') = u$. Define $\rho(uv_i) = u'v_{i-1}$. The following proposition summarizes the properties of ρ and includes justification that ρ is a well-defined.

Proposition VI.15. Retain the above notation. The map ρ is a graph isomorphism from Γ_{++} to $\rho(\Gamma_{++})$ where

$$V(\rho(\Gamma_{++})) = \{uv_{-i} \mid u \in U_{++}, i \in \mathbb{N}\}, \text{ and}$$
$$E(\rho(\Gamma_{++})) = \{(uv_{-i}, uv_{-i-1}) \mid u \in U_{++}, i \in \mathbb{N}\}.$$

Proof. We first show ρ is well-defined. Suppose $u_0 v_{-i}, u_1 v_{-i} \in V(\Gamma_{++})$ represent the same vertex. Then $u_0^{-1}u_1 \in \alpha^{-i}(U)$. Choose $w_0, w_1 \in U_{++}$ with $\alpha(w_i) = u_i$ for $i \in \{0,1\}$. Then $\alpha(w_0^{-1}w_1) = u_0^{-1}u_1 \in \alpha^{-i}(U)$ and so $w_0^{-1}w_1 \in \alpha^{-i-1}(U)$. This implies $w_0 v_{-i-1} = w_1 v_{-i-1}$. By Lemma VI.14, this is enough to show that setting $\rho(u_0 v_{-i}) = w_0 v_{-i-1}$ is well-defined.

To see that ρ is injective suppose that $\rho(u_0 v_{-i}) = \rho(u_1 v_{-i})$. Then there are w_0 and w_1 such that $w_0 v_{-i-1} = w_1 v_{-i-1}$ and $\alpha(w_i) = u_i$ ($i \in \{0,1\}$). In particular, $w_0^{-1}w_1 \in \alpha^{-i-1}(U)$ and so $\alpha(w_0^{-1}w_1) = u_0^{-1}u_1 \in \alpha^{-i}(U)$. Thus $u_0 v_{-1} = u_1 v_{-i}$.

As to $V(\rho(\Gamma_{++}))$ we have, $V(\rho(\Gamma_{++})) \supseteq \{uv_{-i} \mid u \in U_{++},\ i \in \mathbb{N}\}$ by definition as $\alpha(U_{++}) = U_{++}$. Equality follows from Lemma VI.14.

To see that ρ preserves the edge relation, let $(uv_{-i}, uv_{-i-1}) \in E(\Gamma_{++})$. Choose $u' \in U_{++}$ with $\alpha(u') = u$. Then $(\rho(uv_{-i}), \rho(uv_{-i-1})) = (u'v_{-i-1}, u'v_{-i-2}) \in E(\Gamma_{++})$. Thus ρ is a graph morphism.

Again, we have $E(\rho(\Gamma_{++})) \supseteq \{(uv_{-i}, uv_{-i-1}) \mid u \in U_{++}, i \in \mathbb{N}\}$ by definition as $\alpha(U_{++}) = U_{++}$ and equality by Lemma VI.14. □

The following two results capture arc-transitivity of the action of U_{++} on Γ_{++}.

Lemma VI.16. Retain the above notation. Let γ_0 and γ_1 be arcs of equal length in Γ_{++} and with origin uv_0 ($u \in U_{++}$). Then there is $g \in U_{++}$ such that $g\gamma_0 = \gamma_1$.

Proof. Note that $u^{-1}\gamma_i$ ($i \in \{0,1\}$) is an arc with origin v_0 and thus is contained in $\mathrm{descr}_{\Gamma_{++}}(v_0)$. Remark VI.13 and Lemma VI.10 show that there exists $u' \in U_+$ such that $u'u^{-1}\gamma_0 = u^{-1}\gamma_1$. Then $uu'u^{-1} \in U_{++}$ and $g := uu'u^{-1}$ serves. □

In the following, we write $[v_0, v_{-k}]$ for the arc $(v_0, \ldots, v_{-k})$.

Proposition VI.17. Retain the above notation. Let γ_0 and γ_1 be arcs in Γ_{++} of equal length. Then there are $u \in U_{++}$ and $n \in \mathbb{N}_0$ with either $u\rho^n\gamma_0 = \gamma_1$ or $u\rho^n\gamma_1 = \gamma_0$. If γ_0 and γ_1 both terminate at v_{-i} ($i \in \mathbb{N}$), we may choose $n = 0$ and $u \in U_{++} \cap U_{--}$.

Proof. Suppose γ_0 originates at $u_i v_{-i_0}$ and γ_1 originates at $u_1 v_{-i_1}$. Without loss of generality assume $i_0 \geq i_1$. Then $\rho^{i_0-i_1}(\gamma_1)$ originates at $u_1' v_{-i_0} = \rho^{i_0-i_1}(u_1 v_{-i_1})$ for some $u_1' \in U_{++}$. For the first assertion it therefore suffices to show that for any two arcs γ_0 and γ_1 originating at vertices u_0v_{-i} and u_1v_{-i} ($u_0, u_1 \in U_{++}$), there exists $u \in U_{++}$ with $u\gamma_0 = \gamma_1$. Further still, by considering the image of γ_1 under multiplication by $u_0u_1^{-1}$, we can assume the $u_0 = u_1$. Now we can extend γ_j to γ_j' by concatenating on the left with the path $(u_0v_0, \ldots, u_0v_{-i})$. By Lemma VI.16, there exists $u \in U_{++}$ such that $u\gamma_0' = \gamma_1'$. We must necessarily have $u\gamma_0 = \gamma_1$.

For the second assertion, let γ be an arc terminating in v_{-k}. It suffices to show that there is $g \in U_{++} \cap U_{--}$ such that $g\gamma \subseteq [v_0, v_{-k}]$. Extending γ if necessary, we can assume without loss of generality that γ originates at some uv_0 where $u \in U_{++}$.

We now construct $g \in U_{++} \cap U_{--}$ such that $g\gamma = [v_0, v_{-k}]$. By Lemma VI.16, there exists $u' \in U_{++}$ such that $u'\gamma = [v_0, v_{-k}]$. Applying Lemma VI.16, for each $n \in \mathbb{N}_0$ there exist $w_n \in U_{++}$ such that

$$w_n(v_0, \ldots, v_{-k}, u'v_{-k-1}, \ldots, u'v_{-k-n}) = [v_0, v_{-k-n}].$$

The sequence $(w_n)_{n\in\mathbb{N}}$ is contained in U as each element fixes v_0. It hence admits a subsequence converging to some $w' \in U$. Put $g := w'u' \in U_{++}$. Since the permutation topology is coarser than the topology on G, we get $g(v_{-l}) = v_{-l}$ for all $l \geq k$. That is, $g \in U_{--}$ and $g\gamma = [v_0, v_{-k}]$. □

Remark VI.18. Restricting Proposition VI.17 to the case where γ_0 and γ_1 are single vertices we conclude that for any two vertices $u_0, u_1 \in V(\Gamma_{++})$, there are $n \in \mathbb{N}_0$ and $u \in U_{++}$ such that either $u\rho^n(u_0) = u_1$ or $u\rho^n(u_1) = u_0$.

We now show that Γ_{++} is locally finite. We will need the following Lemma which is a consequence of [**Wil15**, Proposition 4] given that $\mathcal{L}_U$, see [**Wil15**, Definition 5], is precisely $U_{++} \cap U_{--}$.

Lemma VI.19. The closure of $U_{++} \cap U_{--}$ is compact. □

The last assertion of the following proposition will be used to show that Γ_{++} admits a well-defined "depth" function.

Proposition VI.20. Retain the above notation. The graph Γ_{++}

(i) has constant out-valency,
(ii) has constant in-valency among the vertices $\{uv_{-i} \mid u \in U_{++},\ i \in \mathbb{N}\}$,
(iii) satisfies that the in-valency of uv_0 ($u \in U_{++}$) is 0,
(iv) is locally finite, and
(v) satisfies that every arc from uv_{-i} to $u'v_{-i-k}$ ($u, u' \in U_{++};\ i, k \in \mathbb{N}_0$) has length k.

Proof. If $u_0, u_1 \in V(\Gamma_{++})$, then by Remark VI.18 and swapping u_0 with u_1 if necessary, there are $g \in U_{++}$ and $n \in \mathbb{N}_0$ such that $g\rho^n(u_0) = u_1$. Proposition VI.15, shows that $|\operatorname{out}(u_1)| = |\operatorname{out}(\rho^n(u_0))|$, hence (i). Similarly, $\operatorname{in}(u) = \operatorname{in}(g\rho^n(u_0))$ if neither u_0 and u_1 are of the form uv_0 for some $u \in U_{++}$ and therefore (ii) holds.

The assertion that $|\operatorname{in}(uv_0)| = 0$ follows since for every edge $(u'v_{-i}, u'v_{-i-1})$ we have $u'v_{-i-1} \neq uv_0$ by Lemma VI.14.

For local finiteness it now suffices to show that both $\operatorname{out}(v_0)$ and $\operatorname{in}(v_{-1})$ are finite. Note that by Remark VI.13 we have

$$|\operatorname{out}(v_0)| = |Uv_{-1}| = [U : U \cap \alpha^{-1}(U)]$$

which is finite by compactness of U and continuity of α. To see that $\operatorname{in}(v_{-1})$ is finite, note that by Proposition VI.17 each vertex of $\operatorname{in}(v_{-1})$ can be written as uv_0 where $u \in U_{++} \cap U_{--} \cap \alpha^{-1}(U)$. Conversely, any such u yields a vertex in $\operatorname{in}(v_0)$. Thus

$$|\operatorname{in}(v_{-1})| = [U_{++} \cap U_{--} \cap \alpha^{-1}(U) : U_{++} \cap U_{--} \cap \alpha^{-1}(U) \cap U].$$

If $u_0, u_1 \in U_{++} \cap U_{--} \cap \alpha^{-1}(U)$ with $u_0 u_1^{-1} \notin U$ then $u_0, u_1 \in \overline{U_{++} \cap U_{--}} \cap \alpha^{-1}(U)$ a fortiori and $u_0 u_1^{-1} \notin U$. Thus

$$|\operatorname{in}(v_{-1})| \leq [\overline{U_{++} \cap U_{--}} \cap \alpha^{-1}(U) : \overline{U_{++} \cap U_{--}} \cap \alpha^{-1}(U) \cap U].$$

Applying Lemma VI.19 and noting that $\alpha^{-1}(U)$ is closed, $\overline{U_{++} \cap U_{--}} \cap \alpha^{-1}(U)$ is compact. Furthermore, since U is open, we derive that $\overline{U_{++} \cap U_{--}} \cap \alpha^{-1}(U) \cap U$ is open in $\overline{U_{++} \cap U_{--}} \cap \alpha^{-1}(U)$. Thus $\operatorname{in}(v_{-1})$ is finite.

For part (v), let γ be an arc from uv_{-i} to uv_{-i-k}. Note that by Proposition VI.17, there is $g \in U_{++}$ with $g\gamma \subseteq (v_0, v_{-1}, \ldots)$. By Lemma VI.14, $guv_{-i} = v_{-i}$ and $gu'v_{-i-k} = v_{-i-k}$. Thus $g\gamma = (v_{-i}, \ldots, v_{-k})$ has length k and so does γ because U_{++} acts by automorphisms. □

2.2. The quotient T. The tidying procedure relies on identifying a certain quotient T of Γ_{++} as a forest of regular rooted trees. To define this quotient, we first introduce a "depth" function $\psi : V(\Gamma_{++}) \to \mathbb{N}$ on Γ_{++} as follows: For $v \in V(\Gamma_{++})$, choose an arc γ originating from some uv_0 ($u \in U_{++}$) and terminating at v. Set $\psi(v)$ to be the length of γ. The following is immediate from Proposition VI.20.

Lemma VI.21. Retain the above notation. The map ψ is well-defined and $\psi(uv_{-i}) = i$ for all $u \in U_{++}$ and $i \in \mathbb{N}_0$. □

By virtue of Lemma VI.21 we may define the level sets $V_k := \psi^{-1}(k) \subseteq V(\Gamma_{++})$ for $k \geq 0$ and the edge sets $E_k := \{(w, w') \in E(\Gamma_{++}) \mid \psi(w') = k\}$ for $k \geq 1$. It is a consequence of Lemma VI.21 and Lemma VI.14 that $(w, w') \in E_k$ if and only if there is $u \in U_{++}$ such that $(w, w') = (uv_{-k+1}, uv_{-k})$. On V_k ($k \geq 1$) we introduce an equivalence relation by $w \sim w'$ if w and w' belong to the same connected component of $\Gamma_{++} \backslash E_k$. Similarly, for $w, w' \in V_0$ we put $w \sim w'$ if they belong to the same connected component of Γ_{++}. Write $[w]$ for the collection of vertices w' with $w \sim w'$. Note that for every $g \in U_{++}$ and $k \in \mathbb{N}_0$ we have $gV_k = V_k$ and $gE_k = E_k$. Since the action of U_{++} on Γ_{++} preserves connected components we see that $w \sim w'$ if and only if $gw \sim gw'$. The following Lemma extends this to ρ.

Lemma VI.22. Retain the above notation and let $k \in \mathbb{N}_0$. Then $\rho(V_k) = V_{k+1}$ and $\rho(E_k) = E_{k+1}$. Hence, for $w, w' \in V(\Gamma_{++})$ we have $w \sim w'$ if and only if $\rho(w) \sim \rho(w')$.

Proof. The assertions $\rho(V_k) = V_{k+1}$ and $\rho(E_k) = E_{k+1}$ are immediate from the definitions. Suppose now that $w, w' \in V_k$ are in the same connected component of $\Gamma_{++} \setminus E_k$. By Proposition VI.15, this can occur if and only if $\rho(w), \rho(w') \in V_{k+1}$ are in the same connected component of $\rho(\Gamma_{++}) \setminus E_{k+1}$. By Proposition VI.20 and the definition of E_{k+1}, the embedding $\rho(\Gamma_{++}) \to \Gamma_{++}$ maps connected components of $\rho(\Gamma_{++}) \setminus E_{k+1}$ to connected components of $\Gamma_{++} \setminus E_{k+1}$ and is surjective on V_{k+1}. □

Lemma VI.23. Retain the above notation. There is $N \in \mathbb{N}$ such that for every $v \in \operatorname{desc}_{\Gamma_{++}}(v_0)$ with $\psi(v) \geq N$ we have $\operatorname{in}(v) \subseteq \operatorname{desc}_{\Gamma_{++}}(v_0)$.

Proof. By Proposition VI.20, we can choose $u_0, \ldots, u_k \in U_{++} \cap \alpha^{-1}(U)$ such that $\operatorname{in}(v_{-1}) = \{u_0 v_0, \ldots, u_k v_0\}$. Since $u_i \in U_{++}$ for all $i \in \{0, \ldots, k\}$, we may pick α-regressive trajectories $(w_j^i)_{j \in \mathbb{N}_0}$ and $N_i \in \mathbb{N}$ such that $w_0^i = u_i$ and $w_n^i \in U$ for all $n \geq N_i$. Set $N = \max\{N_i \mid i \in \{0, \ldots, k\}\} + 1$.

Suppose $n \geq N$. To see that $\operatorname{in}(v_{-n}) \subseteq \operatorname{desc}_{\Gamma_{++}}(v_0)$ note that by Proposition VI.20 we have $\operatorname{in}(v_{-n}) = \rho^{n-1}(\operatorname{in}(v_{-1})) = \{w_{n-1}^i v_{-N+1} \mid i \in \{0, \ldots, k\}\}$. Since $n - 1 \geq N_i$ for all $i \in \{0, \ldots, k\}$, the path $(w_{n-1}^i v_0, \ldots, w_{n-1}^i v_{-n+1})$ is contained in $\operatorname{desc}_{\Gamma_{++}}(v_0)$. This shows $\operatorname{in}(v_{-n}) \subseteq \operatorname{desc}_{\Gamma_{++}}(v_0)$.

In general, let $v \in \operatorname{desc}_{\Gamma_{++}}(v_0)$ with $\psi(v) = n \geq N$. Applying Proposition VI.17 to the arc $(v_0, \ldots, v_{-n})$ and any arc connecting v_0 to v, there is $u \in U \cap U_{++}$

such that $uv_{-n} = v$. Furthermore, $u\,\mathrm{desc}_{\Gamma_{++}}(v_0) = \mathrm{desc}_{\Gamma_{++}}(v_0)$ as $uv_0 = v_0$ and it follows that $\mathrm{in}(v) = u\,\mathrm{in}(v_{-n}) \subseteq \mathrm{desc}_{\Gamma_{++}}(v_0)$. □

Lemma VI.24. Retain the above notation. Then the equivalence classes on Γ_{++} induced by $\sim$ have finite constant size.

Proof. By Proposition VI.17 and Lemma VI.22, it suffices to show that a single equivalence class is finite. Using Lemma VI.23, choose $N \in \mathbb{N}$ such that for every $v \in \mathrm{desc}_{\Gamma_{++}}(v_0)$ with $\psi(v) \geq N$ we have $\mathrm{in}(v) \subset \mathrm{desc}(v_0)$. We show that $[v_{-N}] \subseteq \mathrm{desc}_{\Gamma_{++}}(v_0)$. Since $\mathrm{desc}(v_0) \cap V_k$ is finite for all $k \in \mathbb{N}$, this assertion will follow.

Suppose $v \in [v_{-N}]$. Then v_{-N} and v are in the same connected component of $\Gamma_{++} \backslash E_N$. Hence there is a path from v_{-N} to v contained in $\Gamma_{++} \backslash E_N$. Choosing arcs within this path and extending them to V_N if necessary, we see that there are vertices $u_0, \ldots, u_n \in V_N$ with $u_0 = v_{-N}$, $u_n = v$ and $\mathrm{desc}_{\Gamma_{++}}(u_i) \cap \mathrm{desc}_{\Gamma_{++}}(u_{i+1}) \neq \emptyset$. We use induction to show that $u_i \in \mathrm{desc}_{\Gamma_{++}}(v_0)$. Clearly, $u_0 = v_{-N} \in \mathrm{desc}_{\Gamma_{++}}(v_0)$. Suppose $u_k \in \mathrm{desc}_{\Gamma_{++}}(v_0)$ and let $(w_0, \ldots, w_l)$ be an arc such that $w_0 = u_{k+1}$ and $w_l \in \mathrm{desc}_{\Gamma_{++}}(u_k) \cap \mathrm{desc}_{\Gamma_{++}}(u_{k+1})$. Then $w_l \in \mathrm{desc}_{\Gamma_{++}}(v_0)$ and $\psi(w_{-l}) = N + l > N$. This implies $w_{l-1} \in \mathrm{in}(w_l) \subseteq \mathrm{desc}_{\Gamma_{++}}(v_0)$ by the choice of N. Repeating this process until we have $u_{k+1} = w_0 \in \mathrm{in}(w_1) \subseteq \mathrm{desc}_{\Gamma_{++}}(v_0)$ completes the induction. □

Now define a directed graph T as the quotient of Γ_{++} by the vertex equivalence relation introduced above. In particular, $([w], [w'])$ is an edge in T if and only if there are representatives $w \in [w]$ and $w' \in [w']$ such that (w, w') is an edge in Γ_{++}. The following result collects properties of T. For the statement, we let $d_+ = |\,\mathrm{out}_{\Gamma_{++}}(v_0)|$ and $d_- = |\,\mathrm{in}_{\Gamma_{++}}(v_{-1})|$. We let $\varphi : \Gamma_{++} \to T$ denote the quotient map.

Lemma VI.25. Retain the above notation. The quotient T is a forest of regular rooted trees of degree d_+/d_-. The map ρ and the action of U_{++} on Γ_{++} descend to T. Furthermore, we have the following.

(i) The map ρ is a graph morphism from T onto $\rho(T)$ where

$$V(\rho(T)) = \{[uv_{-i}] \mid u \in U_{++},\ i \in \mathbb{N}\}, \text{ and}$$

$$E(\rho(T)) = \{([uv_{-i}], [uv_{-i-1}]) \mid u \in U_{++},\ i \in \mathbb{N}\}.$$

(ii) For every $v \in V(T)$, the stabilizer $(U_{++})_v$ acts transitively on $\mathrm{out}_T(v)$.

Proof. It is clear that if $v \in V(\Gamma_{++}) \cap V_0$, then $|\,\mathrm{in}_T([v])| = 0$ since $|\,\mathrm{in}_{\Gamma_{++}}(u)| = 0$ for all $u \in V_0$. We now show that if $v \in \Gamma_{++} \backslash V_0$, then $|\,\mathrm{in}_T([v])| = 1$. Since $|\,\mathrm{in}_{\Gamma_{++}}(v)| \geq 1$, we have $|\,\mathrm{in}_T([v])| \geq 1$. Suppose now that $(u_0, [v])$ and $(u_1, [v])$ are edges in T. Then there are representatives $u_i', w_i' \in V(\Gamma_{++})$ such that $u_i' \in [u_i]$, $w_i \in [v]$ and $(u_i', w_i') \in E(\Gamma_{++})$ for $i \in \{0, 1\}$. In particular, w_0 is in the same connected component of $\Gamma_{++} \setminus E_{\psi(w_0)}$ as w_1. Consequently, u_0' is in the same connected component of $E_{\psi(w_0)-1}$ as u_1'. As $\psi(u_0') = \psi(w_0) - 1 = \psi(w_1) - 1 = \psi(u_1')$, this shows that $u_0 = [u_0'] = [u_1'] = u_1$ and so $(u_0, [v]) = (u_1, [v])$. Hence $|\,\mathrm{in}([v])| = 1$.

The map ρ and the action of U_{++} on Γ_{++} descend to T by Lemma VI.22 and the preceding paragraph. The assertions concerning ρ and $\rho(T)$ are immediate from Proposition VI.15. The same Proposition implies $u\rho^n(\mathrm{in}_T(v)) = \mathrm{in}_T(u\rho^n(v))$. Proposition VI.17 shows that an analogue of Remark VI.18 also holds for T. Hence T is a forest of regular rooted trees and has constant out-valency.

Let d denote the out-valency of T. As in [**Möl00**, Lemma 5], we argue that $d = d^+/d^-$. By Lemma VI.24, equivalence classes of vertices in Γ_{++} have constant finite order $k \in \mathbb{N}$. Given $v \in V(T)$, let $A := \varphi^{-1}(v)$. The d edges issuing from v end in vertices $w_1, \ldots, w_d \in V(T)$. Put $B := \varphi^{-1}(\{w_1, \ldots, w_d\})$. Then all edges in Γ_{++} ending in B originate in A because T has in-valency 1. The number of edges issuing from A, which is kd^+, and the number of edges terminating in B, which is kdd^-, are thus equal. Hence $d = d^+/d^-$.

For (ii), let $v \in V(T)$ and $u_0, u_1 \in \mathrm{out}_T(v)$. Pick representatives w_0, w_0', w_1, w_1' in $V(\Gamma_{++})$ such that $([w_i], [w_i']) = (v, u_i)$ for $i \in \{0, 1\}$ and choose $g \in U_{++}$ such that $g(w_0, w_0') = (w_1, w_1')$ by Proposition VI.17. Then $gv = v$ and $gu_0 = u_1$. □

Theorem VI.26. Let G be a t.d.l.c. group and $\alpha \in \mathrm{End}(G)$. Then there exists a compact open subgroup $V \leq G$ which is tidy for α.

Proof. By Lemma VI.1 we may assume that $\{v_{-i} \mid i \in \mathbb{N}_0\}$ is infinite. Furthermore, by Theorem VI.4, we may assume that U is tidy above for α.

For $i \in \mathbb{N}_0$, let $v_i' := \varphi(v_i) \in V(T)$. In view of the fact that $\Gamma_{++} \subseteq \Gamma$, consider $V := G_{\{X_0\}}$ where $X_0 := [v_0] \subseteq V(\Gamma_{++})$ is the equivalence class of v_0 in Γ_{++}. Then V is open in the permutation topology coming from Γ as $G_{X_0} \leq V = G_{\{X_0\}}$ and hence also open (and closed) in G. Since X_0 is finite by Lemma VI.24 we conclude that V is compact as it contains the compact group U as a finite index subgroup.

We have $\mathrm{desc}_{\Gamma_{++}}(X_0) = \mathrm{desc}_{\Gamma}(X_0)$ by Remark VI.13. Since the group V preserves $\mathrm{desc}_{\Gamma}(X_0)$ it acts on $\mathrm{desc}_{\Gamma_{++}}(X_0)$ by automorphisms.

It is clear that V preserves V_k, E_k and connected components. So the action of V descends to T and V stabilizes $v_0' \in V(T)$. Note that $(U_{++})_{v_0'} \leq V$ and so iterated application of Lemma VI.25 shows that V acts transitively on vertices of fixed depth in T. Also, $V_{v_i'} = V \cap \alpha^{-i}(V)$: Suppose $g \in V$ and $gv_i = uv_i$, where $u \in U_{++}$. Then $g^{-1}u \in \alpha^{-i}(U)$. Thus $\alpha^i(g^{-1}u) \in U$ and so $\alpha^i(g)v_0 = \alpha^i(u)v_0$. Applying Lemma VI.22, we see that $gv_i \sim v_i$ if and only if $\alpha^i(g)v_0 \sim v_0$. Finally, applying the orbit-stabilizer theorem and Lemma VI.25 we have

$$[V : V \cap \alpha^{-n}(V)] = |Vv_{-n}'| = (d_+/d_-)^n = |Vv_{-1}'|^n = [V : V \cap \alpha^{-1}(V)]^n.$$

for all $n \in \mathbb{N}$. Hence V is tidy for α by Lemma VI.12 □

Remark VI.27. Retain the above notation and assume that U is tidy. We argue that in this case Γ_{++} and T coincide: It suffices to show that $|\mathrm{in}(v)| = 1$ for some $v = uv_{-i}$ with $i > 0$ as Proposition VI.20 shows that the relation $\sim$ on Γ_{++} is trivial. By Remark VI.13 and Theorem VI.11, the graph $\mathrm{desc}_{\Gamma_{++}}(v_0) = \Gamma_+$ is already a tree. Lemma VI.23 shows that there exists a vertex v with $\mathrm{in}(v) \subset \Gamma_+$. Thus $|\mathrm{in}(v)| = 1$.

The following lemma will be used in Section 4.

Lemma VI.28. Suppose U is tidy for α. Then $U_{++} \cap U_{--} \leq U_+ \cap U_- \leq U$.

Proof. Since U is tidy for α, the graph Γ_{++} is a forest of rooted trees by Remark VI.27 . Note that for each $u \in U_{++} \cap U_{--}$, there exists $i \in \mathbb{N}_0$ such that $uv_{-i} = v_{-i}$. Hence $U_{++} \cap U_{--}$ preserves $\mathrm{desc}_{\Gamma_{++}}(v_0)$. Since this is a tree with root v_0, $U_{++} \cap U_{--}$ is contained within $\mathrm{stab}_G(v_0) = U$. The claim now follows from Lemma VI.8. □

3. The Scale Function and Tidy Subgroups

In this section we link the concept of tidy subgroups to the scale function and thereby recover results of [**Wil15**] in a geometric manner. First, we make a preliminary investigation into the intersection of tidy subgroups. Let G be a t.d.l.c. group, $\alpha \in \mathrm{End}(G)$ and $U^{(1)}, U^{(2)} \leq G$ compact open as well as tidy for α.

Proposition VI.29. Retain the above notation. Then

$$[U^{(1)} : U^{(1)} \cap \alpha^{-1}(U^{(1)})] = [U^{(2)} : U^{(2)} \cap \alpha^{-1}(U^{(2)})]$$

To prove Proposition VI.29, we need some preparatory lemmas concerning inverse images of $U^{(1)}$ and $U^{(2)}$. The first one complements Lemma VI.1.

Lemma VI.30. Let G be a t.d.l.c. group, $\alpha \in \mathrm{End}(G)$ and $U \leq G$ compact open and tidy above for α. If $\{\alpha^{-n}(U) \mid n \in \mathbb{N}_0\}$ is finite then $\alpha(U) = U = \alpha^{-1}(U)$.

Proof. By assumption, the intersection $\bigcap_{k=0}^{\infty} \alpha^{-k}(U)$ has only finitely many terms and hence stabilizes eventually. For sufficiently large $n \in \mathbb{N}_0$ we therefore have $[U_{-n} : U_{-n-1}] = 1$. By Proposition VI.6, we get for all $m \in \mathbb{N}_0$ that

$$1 = [U_{-n} : U_{-n-1}] = [U : U_{-1}] = [U_{-m} : U_{-m-1}] = [\alpha^{-m}(U) : \alpha^{-m}(U) \cap \alpha^{-m-1}(U)].$$

For $m = 1$, we obtain $[U : U_{-1}] = [U : U \cap \alpha^{-1}(U)] = 1 = [\alpha^{-1}(U) : U \cap \alpha^{-1}(U)]$. That is, $\alpha^{-1}(U) \supseteq U$ and $U \supseteq \alpha^{-1}(U)$ which yields the assertion. □

The next lemma settles Proposition VI.29 when both $\{\alpha^{-n}(U^{(1)}) \mid n \in \mathbb{N}_0\}$ and $\{\alpha^{-n}(U^{(2)}) \mid n \in \mathbb{N}_0\}$ are finite.

Lemma VI.31. Retain the above notation. If $\{\alpha^{-n}(U^{(i)}) \mid n \in \mathbb{N}_0\}$ is finite for both $i \in \{1, 2\}$ then $[U^{(1)} : U^{(1)} \cap \alpha^{-1}(U^{(1)})] = [U^{(2)} : U^{(2)} \cap \alpha^{-1}(U^{(2)})]$ and $U^{(1)} \cap U^{(2)}$ is tidy for α.

Proof. The first assertion follows from Lemma VI.30. By the same Lemma we have $\alpha^{-1}(U^{(1)} \cap U^{(2)}) = \alpha^{-1}(U^{(1)}) \cap \alpha^{-1}(U^{(2)}) = U^{(1)} \cap U^{(2)}$. Lemma VI.1 now entails that $(U^{(1)} \cap U^{(2)})_- = U^{(1)} \cap U^{(2)}$ is tidy for α. □

Retain the above notation and set $V := U^{(1)} \cap U^{(2)}$. Consider the graph Γ_+ associated to V.

Lemma VI.32. Retain the above notation. Then either Γ_+ is a directed infinite tree, rooted at v_0, with constant in-valency 1 excluding the root, or there exists $n \in \mathbb{N}_0$ such that $\alpha^{-n}(V) = \alpha^{-n-k}(V)$ for all $k \in \mathbb{N}_0$.

Proof. Note that if $\alpha^{-n}(V) = \alpha^{-n-1}(V)$ then $\alpha^{-n}(V) = \alpha^{-n-k}(V)$ for all $k \in \mathbb{N}_0$. Suppose instead that $\alpha^{-n}(V) \neq \alpha^{-n-1}(V)$ for all $n \in \mathbb{N}_0$. By Lemma VI.31 we may assume, without loss of generality, that $\{\alpha^{-n}(U^{(1)}) \mid n \in \mathbb{N}_0\}$ is infinite. In particular, we may consider the graph $\Gamma_+^{(1)}$ associated to $U^{(1)}$ which is an infinite rooted tree by Theorem VI.11.

We have to show that Γ_+ does not contain a cycle, the in-valency of $v_0 \in V(\Gamma_+)$ is 0 and the in-valency of every other vertex in Γ_+ is precisely 1. Note that every vertex excluding v_0 has in-valency at least 1: By assumption, $v_{-i} \neq v_{-i-1}$ for all $i \in \mathbb{N}$. In particular $v_{-i} \in \mathrm{in}(v_{-i-1})$ for all $i \in \mathbb{N}$.

Now, suppose there is a cycle $(u_0 v_{-i}, \ldots, u_n v_{-i-n} = u_0 v_{-i})$ in Γ_+, where $u_j \in V$ for all $j \in \{0, \ldots, n\}$. Then $\alpha^{-i}(V) = \alpha^{-i-n}(V)$ and so $(v_{-i}, \ldots, v_{-i-n})$ is a non-trivial cycle. We aim to show that v_{-i} has in-valency at least 2 in this case. We can choose $u \in \alpha^{-i-1}(V) \setminus \alpha^{-i}(V)$: If $\alpha^{-i-1}(V) \subseteq \alpha^{-i}(V)$ then iterated applications of α^{-1} show $\alpha^{-i}(V) \supseteq \alpha^{-i-1}(V) \supseteq \alpha^{-i-n}(V) = \alpha^{-i}(V)$, in contradiction to the assumption. Since $\alpha^{-i-1}(V) = \alpha^{-1}\alpha^{-i}(V) = \alpha^{-1}\alpha^{-i-n}(V)$, we also obtain $u \in \alpha^{-i-n-1}(V) \setminus \alpha^{-n-i}(V)$. This implies that (uv_{-i-n}, v_{-i-n-1}) is an edge in Γ_+ which is distinct from (v_{-n-i}, v_{-n-i-1}).

Noting that if v_0 has non-zero in-valency then we have a cycle, it remains to show that no vertex has in-valency at least 2. We split into two cases: First, consider the case where $\{\alpha^{-n}(U^{(2)}) \mid n \in \mathbb{N}_0\}$ is finite. Then $\alpha^{-n}(U^{(2)}) = U^{(2)}$ for all $n \in \mathbb{N}_0$ by Lemma VI.30 and

$$\begin{aligned} |\,\mathrm{in}_{\Gamma_+}(v_i)| &= [\alpha^{-i}(V) : \alpha^{-i} \cap \alpha^{-i+1}(V)] \\ &= [\alpha^{-i}(U^{(1)}) \cap U^{(2)} : \alpha^{-i}(U^{(1)}) \cap \alpha^{-i+1}(U^{(1)}) \cap U^{(2)}] \\ &\leq [\alpha^{-i}(U^{(1)}) : \alpha^{-i}(U^{(1)}) \cap \alpha^{-i+1}(U^{(1)})] = |\,\mathrm{in}_{\Gamma_+^{(1)}}(v_{-i}^{(1)})| = 1 \end{aligned}$$

for all $i \in \mathbb{N}$ which suffices.

In the case where $\{\alpha^{-n}(U^{(2)}) \mid n \in \mathbb{N}_0\}$ is infinite, suppose for the sake of a contradiction that $uv_{-n} \in V(\Gamma_+)$ $(n \in \mathbb{N})$ has in-valency at least 2. Choose vertices $wv_{-n+1}, zv_{-n+1} \in V(\Gamma_+)$ such that (wv_{-n+1}, uv_{-n}) and (zv_{-n+1}, vv_{-n}) are distinct edges in Γ_+. Let $\varphi_i : \Gamma_+ \to \Gamma_+^{(i)}$ $(i \in \{1,2\})$ be the graph morphism given by $\varphi_i(uv_{-j}) = uv_{-j}^{(i)}$ for all $j \in \mathbb{N}_0$ and $u \in V \subseteq U^{(i)}$. Since each vertex excluding the root in $\Gamma_+^{(i)}$ has in-valency 1, we have $\varphi_i(wv_{-n+1}) = \varphi_i(zv_{-n+1})$. This implies $w^{-1}z \in \alpha^{-n+1}(U^{(1)}) \cap \alpha^{-n+1}(U^{(2)}) = \alpha^{-n+1}(V)$. Thus $wv_{-n+1} = zv_{-n+1}$ in contradiction to the assumption. □

Set $k_i = [U^{(i)} : V]$ and $d_i = [U^{(i)} : U^{(i)} \cap \alpha^{-1}(U^{(i)})]$.

Lemma VI.33. Retain the above notation. We have $k_i d_i^n \geq |Vv_{-n}| \geq d_i^n / k_i$. Also, if $\{\alpha^{-i}(V) \mid i \in \mathbb{N}_0\}$ is finite then $d_1 = 1 = d_2$.

Proof. Since $U^{(i)}$ is tidy, either the graph $\Gamma_+^{(i)}$ is a tree with out-valency d_i by Theorem VI.11, or $\{\alpha^{-i}(U^{(i)}) \mid i \in \mathbb{N}_0\}$ is finite and $\alpha(U^{(i)}) = U^{(i)} = \alpha^{-1}(U^{(i)})$ by Lemma VI.30, whence $d_i = 1$. In both cases, $k_i d_i^n = k_i |U^{(i)} v_{-n}^{(i)}|$, as the following arguments show: In the former case this follows from Lemma VI.10, in the latter we have $v_{-n}^{(i)} = v_0^{(i)}$ whence $|U^{(i)} v_{-n}^{(i)}| = 1$. Next, we have

$$k_i |U^{(i)} v_{-n}^{(i)}| = [U^{(i)} : V][U^{(i)} : U^{(i)} \cap \alpha^{-n}(U^{(i)})].$$

Since $[\alpha^{-n}(U^{(i)}) : \alpha^{-n}(V)] \leq [U^{(i)} : V]$ we obtain

$$\begin{aligned} k_i |U^{(i)} v_{-n}^{(i)}| &\geq [U^{(i)} : U^{(i)} \cap \alpha^{-n}(U^{(i)})][\alpha^{-n}(U^{(i)}) : \alpha^{-n}(V)] \\ &\geq [U^{(i)} : U^{(i)} \cap \alpha^{-n}(U^{(i)})][\alpha^{-n}(U^{(i)}) \cap U^{(i)} : U^{(i)} \cap \alpha^{-n}(V)] \\ &= [U^{(i)} : U^{(i)} \cap \alpha^{-n}(V)] \\ &= |U^{(i)} v_{-n}| \end{aligned}$$

where $U^{(i)} v_{-n}$ is the orbit of v_{-n} under the action $U^{(i)}$ in $\mathcal{P}(G)$. Since $V \leq U^{(i)}$, we have $k_i d_i^n \geq |U^{(i)} v_{-n}| \geq |Vv_{-n}|$ which is the first inequality.

Since $\alpha^{-n}(V) = \alpha^{-n}(U^{(1)}) \cap \alpha^{-n}(U^{(2)}) \leq \alpha^{-n}(U^{(i)})$, we have $|Vv_{-n}| \geq |Vv_{-n}^{(i)}|$ when considered as orbits in $\mathcal{P}(G)$. The orbit-stabilizer theorem now implies

$$\begin{aligned} |Vv_{-n}^{(i)}| &= \frac{[U^{(i)} : V][V : \operatorname{stab}_V(v_{-n}^{(i)})]}{[U^{(i)} : V]} = \frac{[U^{(i)} : \operatorname{stab}_V(v_{-n}^{(i)})]}{k_i} \\ &\geq \frac{[U^{(i)} : \operatorname{stab}_{U^{(i)}}(v_{-n}^{(i)})]}{k_i} = \frac{|Uv_{-n}^{(i)}|}{k_i} = \frac{d_i^n}{k_i}, \end{aligned}$$

as required. Finally, if $\{\alpha^{-i}(V) \mid i \in \mathbb{N}_0\}$ is finite, then $\alpha^{-n}(V) = \alpha^{-n-k}(V)$ for n sufficiently large and $k \in \mathbb{N}_0$ by Lemma VI.32. Thus $(|Vv_{-n}|)_{n \in \mathbb{N}_0}$ eventually stabilizes. This implies $d_i = 1$. □

Proof. (Proposition VI.29). By Lemma VI.33, we may assume that $\{\alpha^{-i}(V) \mid i \in \mathbb{N}_0\}$ is infinite. In this case, Lemma VI.32 shows that Γ_+ is a rooted tree with root v_0. Let $t_n = |\operatorname{out}_{\Gamma_+}(v_{-n})|$ for $n \in \mathbb{N}_0$. Since Γ_+ is a rooted tree, $t_n = [V_{-n} : V_{-n-1}]$.

The sequence $(t_n)_{n \in \mathbb{N}_0}$ is non-increasing: Indeed, we have

$$t_{n-1} = [V_{-n+1} : V_{-n}] \geq [V_{-n} : V_{-n-1}] = t_n$$

for all $n \in \mathbb{N}$ by the following argument: If $u, u' \in V_{-n}$ with $uV_{-n-1} \neq u'V_{-n-1}$, then $\alpha(u) \in \alpha(V_{-n}) \leq V_{-n+1}$ by Lemma IV.2. Similarly $\alpha(u') \in V_{-n+1}$. However since $u^{-1}u' \notin \alpha^{-n-1}(U)$, $\alpha(u^{-1})\alpha(u') \notin V_{-n}$.

Since the sequence $(t_n)_{n \in \mathbb{N}_0}$ is non-negative, non-increasing and takes integer values it is eventually constant equal to some integer t. Since Γ_+ is a tree, we have $|Vv_{-n}| = \prod_{i=1}^{n-1} t_i$. Given that $t_i = t$ for almost all $i \in \mathbb{N}_0$ there is a constant $l \in \mathbb{Q}$

such that $|Vv_{-n}| = lt^n$ for sufficiently large n. Then

$$k_i d_i^n \geq |Vv_{-n}| = lt^n \geq \frac{d_i^n}{k_i}$$

for large enough $n \in \mathbb{N}$ and $i \in \{1, 2\}$ by the first claim. As a consequence, we have $t = d_i$ for $i \in \{1, 2\}$ which implies the overall assertion. □

The following theorem links the concept of being tidy to the scale function.

Theorem VI.34. Let G be a t.d.l.c. group, $\alpha \in \mathrm{End}(G)$ and $U \leq G$ compact open. Then U is tidy for α if and only if U is minimizing for α. In this case, we have $s(\alpha) = |\operatorname{out}_{\Gamma_+}(v_0)|$.

Proof. Suppose that U is minimizing for α. If $\{\alpha^{-k}(U) \mid k \in \mathbb{N}_0\}$ is finite then $s(\alpha) = 1$ by Lemma VI.1. Consequently, $\alpha(U) \leq U$. Therefore, we have $U = U_-$ and $U_{--} \geq U_- = U$ is open and hence closed.

Assume now that $\{\alpha^{-k}(U) \mid k \in \mathbb{N}\}$ is infinite. First, we show that U is tidy above for α. Suppose otherwise. Then by Theorem VI.4 and Lemma VI.3 there is $n \in \mathbb{N}$ such that with $v_{-1} \in V(\Gamma)$ we have $|U_n v_{-1}| = |U_+ v_{-1}| \lneq |Uv_{-1}|$ and so that U_{-n} is tidy above for α. Then

$$\begin{aligned}[][\alpha(U_{-n}) : \alpha(U_{-n}) \cap U_{-n}] &= [U_{-n} : U_{-n} \cap \alpha^{-1}(U_{-n})] = [U_n : U_n \cap \alpha^{-1}(U)] \\ &= |U_n v_{-1}| \lneq |Uv_{-1}| = [U : U \cap \alpha^{-1}(U)] = [\alpha(U) : \alpha(U) \cap U].\end{aligned}$$

where the equalities follow by applying the appropriate power of α to the respective quotient, using Lemma IV.2. This contradicts the assumption that U is minimizing.

Now consider the graph Γ_{++} associated to U with out-valency d^+, and in-valency d^-, excluding all $v \in V(\Gamma_{++})$ with $\psi(v) = 0$. Since U is tidy above, Theorem VI.5 and Remark VI.13 imply that

$$d^+ = |Uv_{-1}| = [U : U \cap \alpha^{-1}(U)] = [\alpha(U) : \alpha(U) \cap U].$$

Let V denote the tidy subgroup constructed from the graph Γ_{++} associated to U by Theorem VI.11. Then the quotient T of Γ_{++} has out-valency

$$d = [V : V \cap \alpha^{-1}(V)] = [\alpha(V) : \alpha(V) \cap V].$$

Furthermore, $d = d^+/d^-$ by Lemma VI.25. The fact that U is minimizing now implies $d^- = 1$. It follows that Γ_+ is already a tree and U is tidy by Theorem VI.26.

Conversely, assume that U is tidy for α. Let $V \leq G$ be a compact open subgroup which is minimizing. Then V is tidy by the above and Proposition VI.29 implies

$$s(\alpha) = [\alpha(V) : \alpha(V) \cap V] = [V : V \cap \alpha^{-1}(V)] = [U : U \cap \alpha^{-1}(U)] = [\alpha(U) : \alpha(U) \cap U].$$

That is, U is minimizing. □

Corollary VI.35. Let G be a t.d.l.c. group and $\alpha \in \mathrm{End}(G)$. Then $s(\alpha^n) = s(\alpha)^n$.

Proof. By Theorem VI.26 there is a compact open subgroup $U \leq G$ which is tidy for α. Following Theorem VI.34 the group U is minimizing and therefore

$$s(\alpha) = [\alpha(U) : \alpha(U) \cap U] = [U : U \cap \alpha^{-1}(U)].$$

Since U is also tidy for α^n by Lemma VI.12 we conclude, using the same lemma, that

$$s(\alpha^n) = [\alpha^n(U) : \alpha^n(U) \cap U] = [U : U \cap \alpha^{-n}(U)] = [U : U \cap \alpha^{-1}(U)]^n = s(\alpha)^n. \quad \square$$

Möller's spectral radius formula [**Möl02**, Theorem 7.7] for the scale may be proven as in [**Wil15**, Proposition 18] but with reference to Theorem VI.26 for the existence of tidy subgroups.

Theorem VI.36. Let G be a t.d.l.c. group, $\alpha \in \mathrm{End}(G)$ and $U \leq G$ compact open. Then $s(\alpha) = \lim_{n \to \infty} [\alpha^n(U) : \alpha^n(U) \cap U]^{1/n}$. □

4. The Tree-Representation Theorem

In this section, we prove an analogue of the following tree representation theorem for automorphisms due to Baumgartner and Willis [**BW04**], see also [**Hor15**].

Theorem VI.37 ([**BW04**, Theorem 4.1]). Let G be a t.d.l.c. group, $\alpha \in \mathrm{Aut}(G)$ of infinite order and $U \leq G$ compact open as well as tidy for α. Then there is a regular tree T of degree $s(\alpha)+1$ and a homomorphism $\varphi : U_{++} \rtimes \langle\alpha\rangle \to \mathrm{Aut}(T)$ such that

(i) $\varphi(U_{++} \rtimes \langle\alpha\rangle)$ fixes an end $\omega \in \partial T$ and is transitive on $\partial T \setminus \{\omega\}$,
(ii) the stabilizer of each end in $\partial T \setminus \{\omega\}$ is conjugate to $(U_+ \cap U_-) \rtimes \langle\alpha\rangle$,
(iii) $\ker(\varphi)$ is the largest compact normal subgroup $N \trianglelefteq U_{++}$ with $\alpha(N) = N$,
(iv) $\varphi(U_{++})$ is the set of elliptic elements in $\varphi(U_{++} \rtimes \langle\alpha\rangle)$.

To prove an analogous statement for endomorphisms, we let $\alpha \in \mathrm{End}(G)$ have infinite order and $U \leq G$ compact open as well as tidy for α. Let $S := U_{++} \rtimes \langle\alpha\rangle$ be the topological semidirect product semigroup of the (semi)group $U_{++} \leq G$ and the semigroup $\langle\alpha\rangle \leq \mathrm{End}(G)$, where $\mathrm{End}(G)$ is equipped with the compact-open topology and $\langle\alpha\rangle$ acts continuously on U_{++} by endomorphisms as $\alpha(U_{++}) = U_{++}$, see [**CHK83**, Theorem 2.9, Theorem 2.10]. In particular:

(1) Elements of S have the form (u, α^k) for some $u \in U_{++}$ and $k \in \mathbb{N}_0$. We identify (U_{++}, id) with U_{++}, and $(\mathrm{id}, \langle\alpha\rangle)$ with $\langle\alpha\rangle$.
(2) Composition in S is given by $(u_0, \alpha^{k_0})(u_1, \alpha^{k_1}) = (u_0\alpha^{k_0}(u_1), \alpha^{k_0+k_1})$.
(3) The topology on S is the product topology on the set $U_{++} \times \langle\alpha\rangle$.
(4) The subsemigroup of S generated by (id, α) is isomorphic to $(\mathbb{N}, +)$ because $\alpha \in \mathrm{End}(G)$ has infinite order.

We split the construction of the desired tree into the cases $s(\alpha) = 1$ and $s(\alpha) > 1$. First, assume $s(\alpha) > 1$. Recall that $v_{-i} := \alpha^{-i}(U) \in \mathcal{P}(G)$ for $i \geq 0$. We extend this definition to positive indices by setting $v_i := \alpha^i(U) \in \mathcal{P}(G)$ for all $i \in \mathbb{Z}$. The following lemma shows that these vertices are all distinct.

Lemma VI.38. Retain the above notation. In particular, assume $s(\alpha) > 1$. Suppose $\alpha^m(U) = \alpha^n(U)$ for some $n, m \in \mathbb{Z}$. Then $m = n$.

Proof. For $m, n \leq 0$, an equality $\alpha^{-m}(U) = \alpha^{-n}(U)$ with $m \neq n$ implies that the set $\{\alpha^{-k}(U) \mid k \in \mathbb{N}_0\}$ is finite and hence $s(\alpha) = 1$ by Lemma VI.1.

Now, let $0 \leq m < n$. Then Lemma VI.7, Lemma VI.12 and Corollary VI.35 show that

$$\begin{aligned} s(\alpha)^n &= [\alpha^n(U_+) : U_+] \\ &= [\alpha^n(U_+) : \alpha^m(U_+)][\alpha^m(U_+) : U_+] \\ &= [\alpha^n(U_+) : \alpha^m(U_+)]s(\alpha)^m. \end{aligned}$$

Since $m < n$ and $s(\alpha) > 1$, we get $[\alpha^n(U_+) : \alpha^m(U_+)] \neq 1$. Hence there exists $u \in \alpha^n(U_+) \backslash \alpha^m(U_+) \subseteq \alpha^n(U)$. For the sake of a contradiction, suppose $u \in \alpha^m(U)$. Since U is tidy above, there exists $u_\pm \in U_\pm$ such that $u = \alpha^m(u_+)\alpha^m(u_-)$. It follows that $\alpha^m(u_+)^{-1}u \in \alpha^n(U_+) \leq U_{++}$ since $\alpha^m(U_+) \leq \alpha^n(U_+)$. Also, we have $\alpha^m(u_-) \in \alpha^m(U_-) \leq U_- \leq U_{--}$, and so applying Lemma VI.28,

$$\alpha^m(u_+)^{-1}u \in U_{++} \cap U_{--} \leq U_+ \cap U_- \leq \alpha^m(U_+).$$

It follows that $u \in \alpha^m(U_+)$, a contradiction. Thus $u \notin \alpha^m(U)$ and $\alpha^n(U) \neq \alpha^m(U)$.

Finally, suppose $m < 0 < n$ and $\alpha^m(U) = \alpha^n(U)$. Then $\alpha^m(U)$ is a compact open subgroup which is stabilized by α^{n-m}. This shows $s(\alpha^{n-m}) = 1$ which implies $s(\alpha) = 1$ by Corollary VI.35. This contradicts the assumption $s(\alpha) > 1$. $\square$

We define a directed graph $\overline{\Gamma}_{++}$ by setting

$$V(\overline{\Gamma}_{++}) = \{uv_i \mid i \in \mathbb{Z}, u \in U_{++}\} \text{ and } E(\overline{\Gamma}_{++}) = \{(uv_i, uv_{i-1} \mid i \in \mathbb{Z}, u \in U_{++}\}.$$

Note that Γ_{++} is a subgraph of $\overline{\Gamma}_{++}$ and that U_{++} acts on $\overline{\Gamma}_{++}$ by automorphisms. We will show that the map ρ, defined in the paragraph preceding Proposition VI.15, extends to an automorphism of $\overline{\Gamma}_{++}$. To do so, consider the following subgroups associated to α:

$$\mathrm{par}^-(\alpha) := \{x \in G \mid \text{there exists a bounded } \alpha\text{-regressive trajectory for } x\},$$

$$\mathrm{bik}(\alpha) := \overline{\{x \in \mathrm{par}^-(\alpha) \mid \alpha^n(x) = e \text{ for some } n \in \mathbb{N}\}}.$$

It follows from [**Wil15**, Proposition 20], [**Wil15**, Definition 12] and Theorem VI.34 that $\mathrm{bik}(\alpha) \leq U$. The same proposition implies that for $u_1, u_2 \in U_{++} \leq \mathrm{par}^-(\alpha)$ with $\alpha(u_1) = \alpha(u_2)$ we have $u_1^{-1}u_2 \in \mathrm{bik}(\alpha) \leq U$.

Now define $\rho : \overline{\Gamma}_{++} \to \overline{\Gamma}_{++}$ as follows: Given $uv_i \in V(\overline{\Gamma}_{++})$, choose $u' \in U_{++}$ such that $\alpha(u') = u$ and set $\rho(uv_i) = u'v_{i-1}$.

Proposition VI.39. Retain the above notation. Then ρ is an automorphism of $\overline{\Gamma}_{++}$.

Proof. We first show that ρ is well-defined: By Lemma VI.38, it suffices to suppose $u_0, u_1, u_0', u_1' \in U_{++}$ and $i \in \mathbb{Z}$ are such that $u_0v_i = u_1v_i$, $\alpha(u_0') = u_0$ and $\alpha(u_1') = u_1$. Then $u_0^{-1}u_1 \in \alpha^i(U)$ and $(u_0')^{-1}u_1' \in \alpha^{-1}(\alpha^i(U)) \cap U_{++}$. For any $u_3 \in \alpha^{i-1}(U)$ with $\alpha(u_3) = u_0^{-1}u_1$ we get $((u_0')^{-1}u_1)^{-1}u_3 \in \mathrm{bik}(\alpha) \leq \alpha^{i-1}(U)$ as $\mathrm{bik}(\alpha) \leq U$ and $\alpha(\mathrm{bik}(\alpha)) = \mathrm{bik}(\alpha)$. Hence $(u_0')^{-1}u_1 \in \alpha^{i-1}(U)$. This shows $u_0'v_{i-1} = u_1'v_{i-1}$, hence ρ is well-defined. To see that ρ is a bijection on $V(\overline{\Gamma}_{++})$ note $\rho(\alpha(u)v_{i+1}) = uv_i$ and that ρ^{-1} defined by $uv_i \mapsto \alpha(u)v_{i+1}$ is well-defined by the following argument: If $uv_i = u'v_i$, then $u^{-1}u' \in \alpha^i(U)$ and $\alpha(u)^{-1}\alpha(u') \in \alpha^{i+1}(U)$. Thus $\alpha(u)v_{i+1} = \alpha(u')v_{i+1}$. □

Note that $\overline{\Gamma}_{++}$ contains Γ_{++} as a subgraph and Γ_{++} is a forest of rooted regular trees by Remark VI.27. For $v \in V(\overline{\Gamma}_{++})$, there is $n \in \mathbb{N}_0$ such that $\rho^n(v) \in V(\Gamma_{++})$. This shows that the in-valency of v is 1. We find that $\overline{\Gamma}_{++}$ is a regular tree with constant out-valency $s(\alpha)$ by Theorem VI.11 and Remark VI.13. Since ρ is a translation in $\mathrm{Aut}(\overline{\Gamma}_{++})$ we see that the subsemigroup generated by ρ^{-1} is isomorphic to $(\mathbb{N}, +)$.

Define $\varphi : U_{++} \sqcup \langle\alpha\rangle \to \mathrm{Aut}(\overline{\Gamma}_{++})$ by $\varphi(u)(u'v_i) = uu'v_i$ for all $u, u' \in U_{++}$ and $\varphi(\alpha^k) = \rho^{-k}$ for all $k \in \mathbb{N}_0$.

Lemma VI.40. Retain the above notation. The map φ extends to a continuous semigroup homomorphism $\varphi : S \to \mathrm{Aut}(\overline{\Gamma}_{++})$.

Proof. Note that φ extends separately both to a semigroup homomorphism of U_{++}, and the semigroup generated by α. To show that it extends to a semigroup homomorphism of S it suffices to show that $\varphi(\alpha)\varphi(u) = \varphi(\alpha(u))\varphi(\alpha)$. Then $\varphi(u, \alpha^n) := \varphi(u)\varphi(\alpha^n)$ is well-defined for all $u \in U_{++}$ and $n \in \mathbb{N}_0$. Given a vertex $u'v_i \in V(\overline{\Gamma}_{++})$, we obtain as required:

$$\varphi(\alpha)\varphi(u)u'v_i = \rho^{-1}(uu'v_i) = \alpha(uu')v_{i+1} = \alpha(u)\rho^{-1}(u'v_i) = \varphi(\alpha(u))\varphi(\alpha)uv_i.$$

To see that φ is continuous it suffices to show that $\{x \in S \mid \varphi(x)w = w'\}$ is open in S for all $w, w' \in V(\overline{\Gamma}_{++})$. This follows from the fact that the stabilizer V of w' in U_{++} is an open subgroup of U_{++}, so x is contained in the open subset $(V, \mathrm{id})x \subseteq S$ and $\varphi((V, \mathrm{id})x)w = w'$. □

We are now in a position to prove an analogue of Theorem VI.37 for endomorphisms.

Theorem VI.41. Let G be a t.d.l.c. group, $\alpha \in \mathrm{End}(G)$ of infinite order, $U \le G$ compact open as well as tidy for α, and $S := U_{++} \rtimes \langle \alpha \rangle$. Then there is a tree T and a continuous semigroup homomorphism $\varphi : S \to \mathrm{Aut}(T)$ such that

(i) T has constant valency $s(\alpha) + 1$,
(ii) $\varphi(S)$ fixes an end $\omega \in \partial T$ and is transitive on $\partial T \setminus \{\omega\}$,
(iii) $\ker(\varphi)$ is the largest compact normal subgroup $N \trianglelefteq U_{++}$ with $\alpha(N) = N$,
(iv) $\varphi(U_{++})$ is the set of elliptic elements of $\varphi(S)$.

Proof. First, assume $s(\alpha) > 1$. Let T be the undirected graph underlying $\overline{\Gamma}_{++}$, i.e. the graph with vertex set $V(\overline{\Gamma}_{++})$ and edge-relation the symmetric closure of $E(\overline{\Gamma}_{++}) \subseteq V(\overline{\Gamma}_{++}) \times V(\overline{\Gamma}_{++})$. The continuous semigroup homomorphism φ from S to $\mathrm{Aut}(\overline{\Gamma}_{++})$ defined above induces a continuous semigroup homomorphism $S \to \mathrm{Aut}(T)$ for which we use the same letter.

Part (i) is now immediate from the fact that every vertex in $\overline{\Gamma}_{++}$ has out-valency $s(\alpha)$ and in-valency 1.

For part (ii), let $\omega \in \partial T$ be the end associated to the sequence $(v_i)_{i \in \mathbb{N}_0}$. Then $\rho(\omega) = \omega$. If $u \in U_{++}$, then there exists an α-regressive trajectory for u eventually contained in U. That is $u \in \alpha^n(U)$ for all sufficiently large $n \in \mathbb{N}$ whence $uv_n = v_n$ for sufficiently large n. This shows that $u\omega = \omega$. Overall, we conclude $\varphi(S)\omega = \omega$.

Now consider the end $-\omega \in \partial T$ associated to the sequence $(v_{-i})_{i \in \mathbb{N}_0}$. Given another end $\omega' \in \partial T$ defined by $(u_{k-i} v_{k-i})_{i \in \mathbb{N}_0}$ for $k \in \mathbb{Z}$ and a sequence $(u_{k-i})_{i \in \mathbb{N}_0}$ in U_{++}, the sequence $u_k^{-1} \rho^k \omega'$ represents an end $\omega'' \in \partial T$ originating from v_0 and it suffices to show that there is an element $u \in U_{++}$ which maps the sequence of $-\omega$ to that of ω''. This is a consequence of Lemma VI.16 by picking a convergent subsequence inside the compact set $U \cap U_{++}$.

As to (iii), the kernel of φ consists of those elements $s \in S$ such that $\varphi(s)$ fixes every vertex of T. That is,

$$\ker(\varphi) = U_{++} \cap \bigcap_{i \in \mathbb{Z}} \bigcap_{u \in U_{++}} u\alpha^i(U).$$

In particular, $\ker(\varphi)$ is compact and satisfies $\alpha(\ker(\varphi)) = \ker(\varphi)$ as $\alpha(U_{++}) = U_{++}$.

Now, let N be any compact normal subgroup of U_{++} with $\alpha(N) = N$. Then $\varphi(N) \le \mathrm{Aut}(\overline{\Gamma}_{++})_v$ for some $v \in V(\overline{\Gamma}_{++})$ because $\varphi(N)$ is compact by Lemma VI.40. Since N is normal in U_{++}, we conclude that

$$\varphi(N) = \varphi(u)\varphi(N)\varphi(u)^{-1} \le \varphi(N) \cap \mathrm{Aut}(\overline{\Gamma}_{++})_{\varphi(u)v} \le \mathrm{Aut}(\overline{\Gamma}_{++})_{v,\varphi(u)v}$$

for all $u \in U_{++}$. Similarly, given that $\alpha(N) = N$ we have

$$\begin{aligned}\varphi(N) = \varphi(\alpha(N))\varphi(\alpha)\varphi(\alpha)^{-1} &= \varphi(\alpha(N) \circ \alpha)\varphi(\alpha)^{-1} \\ &= \varphi(\alpha \circ N)\varphi(\alpha)^{-1} = \rho^{-1}\varphi(N)\rho \le \mathrm{Aut}(\overline{\Gamma}_{++})_{v,\rho^{-1}(v)}.\end{aligned}$$

as well as

$$\begin{aligned}\varphi(N) = \varphi(\alpha)^{-1}\varphi(\alpha)\varphi(N) &= \varphi(\alpha)^{-1}\varphi(\alpha \circ N) \\ &= \varphi(\alpha)^{-1}\varphi(\alpha(N))\varphi(\alpha) = \rho\varphi(N)\rho^{-1} \le \mathrm{Aut}(\overline{\Gamma}_{++})_{v,\rho(v)}.\end{aligned}$$

As a consequence, $\varphi(N)$ fixes every vertex in the orbit of v under the action of the group generated by $\varphi(S)$. This group acts vertex-transitively as it contains $\varphi(U_{++})$ and both ρ and ρ^{-1}. This shows that $\varphi(N)$ fixes T, i.e. $\varphi(N) \le \ker \varphi$.

For part (iv), write $s = (u, \alpha^k)$ $(u \in U_{++},\ k \in \mathbb{N})$ for elements of S. Given that $\varphi(\alpha) = \rho^{-1}$, we necessarily have $k = 0$ in order for $\varphi(s)$ to fix a vertex, so $s \in U_{++}$. Conversely, every element $u \in U_{++}$ is contained in $\alpha^n(U)$ for all sufficiently large $n \in \mathbb{N}$, so $\varphi(u)$ fixes v_n for the same values of n.

Now, assume $s(\alpha) = 1$. Then $\alpha(U_+) = U_+$ by Lemma VI.7. This shows that $U_{++} = U_+$ is a compact subgroup with $\alpha(U_{++}) = U_{++}$. Let T be the (undirected) tree with vertex set $\mathbb{Z}$ and $i, j \in V(T)$ connected by an edge whenever $|i - j| = 1$. Define $\varphi : S \to \mathrm{Aut}(T)$ by setting $\varphi(\alpha)$ to be the translation of length 1 in the direction of $\omega := (i)_{i\in\mathbb{N}_0} \in \partial T$, and $\varphi(u)$ to be the identity automorphism of T for all $u \in U_{++}$. Then φ satisfies all the conclusions of Theorem VI.41. $\square$

Remark VI.42. The action in Theorem VI.41 relates to Theorem VI.37 in the following manner: Results from [**Wil15**, Section 9] show that if U is tidy for α, then $\mathrm{bik}(\alpha) \trianglelefteq U_{++}$ and the endomorphism $\overline{\alpha}$ of $U_{++}/\mathrm{bik}(\alpha)$ induced by $\alpha|_{U_{++}}$ is an automorphism. Let $q : U_{++} \to U_{++}/\mathrm{bik}(\alpha)$ be the quotient map. Then $q(U_+)$ is tidy for $\overline{\alpha}$, $(q(U_+))_{++} = q(U_{++})$ and $s(\overline{\alpha}) = s(\alpha)$. Extend q to a semigroup homomorphism from S to $q(U_{++}) \rtimes \langle\overline{\alpha}\rangle$ by setting $q(\alpha) = \overline{\alpha}$. Also, let $\varphi : S \to \mathrm{Aut}(T)$ be as in Theorem VI.41 and $\varphi' : q(U_{++}) \rtimes \langle\overline{\alpha}\rangle \to T'$ as in Theorem VI.37. Then there exists a graph isomorphism $\psi : T' \to T$ such that the diagram

$$\begin{array}{ccc} S & \xrightarrow{\ \varphi\ } & \mathrm{Aut}(T) \\ {\scriptstyle q}\downarrow & & \uparrow{\scriptstyle \widetilde{\psi}} \\ q(U_{++}) \rtimes \langle\overline{\alpha}\rangle & \xrightarrow[\ \varphi'\ ]{} & \mathrm{Aut}(T'), \end{array}$$

where $\widetilde{\psi}$ is conjugation by ψ, commutes.

5. New Endomorphisms From Old

We conclude with a construction that produces new endomorphisms of totally disconnected, locally compact groups from old, inspired by [**Wil15**, Example 5].

Let G_1 and G_2 be totally disconnected compact groups. Assume that there are isomorphisms $\varphi_i : G_i \to H_i \cong G_i \leq G_i$ $(i \in \{1, 2\})$ of G_i onto compact open subgroups $H_i \leq G_i$. Consider the HNN-extension G of $G_1 \times G_2$ which makes the isomorphic subgroups $H_1 \times G_2 \cong G_1 \times G_2 \cong G_1 \times H_2$ conjugate:

$$G := \langle G_1 \times G_2, t \mid \{t^{-1}(h_1, g_2)t = (\varphi_1^{-1}(h_1), \varphi_2(g_2)) \mid (h_1, g_2) \in H_1 \times G_2\}\rangle.$$

Set $U := G_1 \times G_2 \leq G$. Given that G commensurates U, it admits a unique group topology which makes the inclusion of U into G continuous and open, see [**Bou98**, Chapter III, §1.2, Proposition 1]. Then G is a non-compact t.d.l.c. group which contains $U := G_1 \times G_2$ as a compact open subgroup. Define $\beta \in \mathrm{End}(G)$ by setting $\beta(t) = t$ and $\beta(g_1, g_2) = (\varphi_1(g_1), g_2)$ for all $(g_1, g_2) \in G_1 \times G_2$. Then

$$\beta(t^{-1}(h_1, g_2)t) = t^{-1}(\varphi_1(h_1), g_2)t = (h_1, g_2) = \beta(\varphi_1^{-1}(h_1), g_2).$$

for all $(h_1, g_2) \in H_1 \times G_2$ and hence β indeed extends to G. Note that β is continuous: Let $V \leq G$ be open. Then so is $V \cap (H_1 \cap G_2)$ and

$$\beta^{-1}(V) \supseteq \beta^{-1}(V \cap (H_1 \cap G_2)) \cap U$$

which is open in U and therefore in G since φ_1 is continuous. Observe that $s(\beta) = 1$ as $\beta(U) \leq U$. Let $\alpha := c_t \circ \beta \in \mathrm{End}(G)$ where $c_t : G \to G,\ g \mapsto tgt^{-1}$ is conjugation by t. For $(g_1, h_2) \in G_1 \times H_2$ we have

$$\text{(E)} \qquad \alpha(g_1, h_2) = t\beta(g_1, h_2)t^{-1} = t(\varphi_1(g_1), h_2)t^{-1} = (\varphi_1^2(g_1), \varphi_2^{-1}(h_2))$$

We proceed to show that U is tidy for α and compute $s(\alpha)$.

Lemma VI.43. Retain the above notation. Then U is tidy for α and $s(\alpha) = [G_2 : H_2]$.

Proof. We proceed via Lemma VI.12. First, we show that $\alpha^{-n}(U) \cap U = G_1 \times \varphi_2^n(G_2)$. The inclusion $G_1 \times \varphi_2^n(G_2) \leq \alpha^{-n}(U) \cap U$ follows from equation (E). Suppose $g \notin G_1 \times \varphi_2^n(G_2)$. We will show $g \notin \alpha^{-n}(U) \cap U$. If $g \notin U$, then we are done and so we may write $g = (g_1, g_2) \in G_1 \times (G_2 \setminus \varphi_2^n(H_2))$. By equation (E), there exists $0 \leq m < n$ such that $\alpha^m(g_1, g_2) \in G_1 \times (G_2 \setminus H_2)$. We therefore show that $\alpha^l(g_1', g_2') \notin U$ for all $l \in \mathbb{N}$ whenever $(g_1', g_2') \in G_1 \times (G_2 \backslash H_2)$. Indeed, $\alpha^l(g_1, g_2) = t^l(\varphi_1^l(g_1), g_2)t^{-l}$ is not contained in U: If $t^l(\varphi_1^l(g_1), g_2)t^{-l} = (g_1', g_2') \in U$ then

$$t \cdots t(\varphi_1^l(g_1), g_2)t^{-1} \cdots t^{-1}(g_1'^{-1}, g_2'^{-1}) = 1,$$

contradicting Britton's Lemma on words in HNN-extensions, see [**Bri63**, Lemma 4] or [**LS15**, Theorem 2.1].

We have shown that $\alpha^{-n}(U) \cap U = G_1 \times \varphi_2^n(G_2)$. Since $\varphi_2^n(G_2)$ is a nested series of subgroups for $n \in \mathbb{N}$, we have

$$\begin{aligned}[U : U \cap \alpha^{-n}(U)] &= [G_1 \times G_2 : G_1 \times \varphi_2^n(G_2)] = [G_2 : \varphi_2^n(G_2)] \\ &= \prod_{i=0}^{n-1} [\varphi_2^i(G_2) : \varphi^{i+1}(G_2)] = [G_2 : H_2]^n.\end{aligned}$$

Lemma VI.12 shows that U is tidy. By Lemma VI.43, we have

$$s(\alpha) = [U : U \cap \alpha^{-1}(U)] = [G_1 \times G_2 : G_1 \times H_2] = [G_2 : H_2]. \qquad \square$$

Bibliography

[AT08] A. Arhangelskii and M. Tkachenko, *Topological Groups and Related Structures, An Introduction to Topological Algebra*, vol. 1, Springer, 2008.

[BCGM12] U. Bader, P.-E. Caprace, T. Gelander, and S. Mozes, *Simple groups without lattices*, Bulletin of the London Mathematical Society **44** (2012), no. 1, 55–67.

[BEW15] C. Banks, M. Elder, and G. A. Willis, *Simple groups of automorphisms of trees determined by their actions on finite subtrees*, Journal of Group Theory **18** (2015), no. 2, 235–261.

[BGT16] T. P. Bywaters, H. Glöckner, and S. Tornier, *Endomorphisms of totally disconnected locally compact groups*, arXiv preprint 1612.06958 (2016).

[BM00a] M. Burger and S. Mozes, *Groups acting on trees: from local to global structure*, Publications Mathématiques de l'IHÉS **92** (2000), no. 1, 113–150.

[BM00b] ———, *Lattices in products of trees*, Publications Mathématiques de l'IHÉS **92** (2000), no. 1, 151–194.

[BMW12] U. Baumgartner, R. Möller, and G. A. Willis, *Hyperbolic groups have flat-rank at most 1*, Israel Journal of Mathematics **190** (2012), no. 1, 365–388.

[Bou98] N. Bourbaki, *General topology. Chapters 1–4*, Springer **6** (1998), 1–3.

[Bou16] A. Le Boudec, *Groups acting on trees with almost prescribed local action*, Commentarii Mathematici Helvetici **91** (2016), no. 2, 253–293.

[Bri63] J. L. Britton, *The word problem*, Annals of Mathematics (1963), 16–32.

[BT17] T. P. Bywaters and S. Tornier, *Willis Theory via Graphs*, arXiv preprint 1711.09610 (2017).

[BV16] A. G. Bruno and S. Virili, *Topological entropy in totally disconnected locally compact groups*, Ergodic Theory and Dynamical Systems (2016), 1–24.

[BW04] U. Baumgartner and G. A. Willis, *Contraction groups and scales of automorphisms of totally disconnected locally compact groups*, Israel Journal of Mathematics **142** (2004), no. 1, 221–248.

[BW06] ———, *The direction of an automorphism of a totally disconnected locally compact group*, Mathematische Zeitschrift **252** (2006), no. 2, 393–428.

[CH09] Z. Chatzidakis and E. Hrushovski, *An invariant for difference field extensions*, arXiv preprint 0902.0844 (2009).

[CHK83] J. H. Carruth, J. A. Hildebrant, and R. J. Koch, *The theory of topological semigroups*, Pure and Applied Mathematics, M. Dekker, 1983.

[CL89] M. Conder and P. Lorimer, *Automorphism groups of symmetric graphs of valency 3*, Journal of Combinatorial Theory, Series B **47** (1989), no. 1, 60–72.

[CM11] P.-E. Caprace and N. Monod, *Decomposing locally compact groups into simple pieces*, Mathematical Proceedings of the Cambridge Philosophical Society, vol. 150, Cambridge University Press, 2011, pp. 97–128.

[CRW17] P.-E. Caprace, C. Reid, and G. A. Willis, *Locally normal subgroups of totally disconnected groups. Part I: General theory*, Forum of Mathematics, Sigma, vol. 5, Cambridge University Press, 2017.

[Day60] D. E. Daykin, *On the Rank of the Matrix f(A) and the Enumeration of Certain Matrices over a Finite Field*, Journal of the London Mathematical Society **1** (1960), no. 1, 36–42.

[DM80] D. Ž. Djoković and G. L. Miller, *Regular groups of automorphisms of cubic graphs*, Journal of Combinatorial Theory, Series B **29** (1980), no. 2, 195–230.

[DM96] J. D. Dixon and B. Mortimer, *Permutation groups*, vol. 163, Springer, 1996.

[DS91] S. Dani and R. Shah, *Contraction subgroups and semistable measures on p-adic lie groups*, Mathematical Proceedings of the Cambridge Philosophical Society, vol. 110, Cambridge University Press, 1991, pp. 299–306.

[DSW06] S. Dani, N. Shah, and G. Willis, *Locally compact groups with dense orbits under $\mathbb{Z}^d$-actions by automorphisms*, Ergodic Theory and Dynamical Systems **26** (2006), no. 5, 1443–1465.

[GAP17] The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.8.7*, 2017.

[GGT16] A. Garrido, Y. Glasner, and S. Tornier, *Automorphism Groups of Trees: Generalities and Prescribed Local Actions*, arXiv preprint 1602.03575 (2016).

[Gle52] A. Gleason, *Groups without small subgroups*, Annals of mathematics (1952), 193–212.

[GM14] M. Giudici and L. Morgan, *A class of semiprimitive groups that are graph-restrictive*, Bulletin of the London Mathematical Society **46** (2014), no. 6, 1226–1236.

[GM16] ———, *A theory of semiprimitive groups*, arXiv preprint 1607.03798 (2016).

[Hor15] D. Horadam, *Tree actions associated with the scale of an automorphism of a totally disconnected locally compact group*, The University of Newcastle, 2015.

[HR12] E. Hewitt and K. A. Ross, *Harmonic Analysis I*, vol. 115, Springer, 2012.

[JRW96] W. Jaworski, J. Rosenblatt, and G. A. Willis, *Concentration functions in locally compact groups*, Mathematische Annalen **305** (1996), no. 1, 673–691.

[Led17] W. Lederle, *Coloured Neretin groups*, arXiv preprint 1701.03027 (2017).

[LPS88] M. W. Liebeck, C. E. Praeger, and J. Saxl, *On the O'Nan-Scott theorem for finite primitive permutation groups*, Journal of the Australian Mathematical Society (Series A) **44** (1988), no. 03, 389–396.

[LS91] M. W. Liebeck and J. Saxl, *On point stabilizers in primitive permutation groups*, Communications in Algebra **19** (1991), no. 10, 2777–2786.

[LS15] R. C. Lyndon and P. E. Schupp, *Combinatorial group theory*, Springer, 2015.

[Möl00] R. G. Möller, *Descendants in highly arc transitive digraphs*, Discrete Mathematics **247** (2000), 147–157.

[Möl02] ———, *Structure theory of totally disconnected locally compact groups via graphs and permutations*, Canadian Journal of Mathematics **54** (2002), no. 4, 795–827.

[Möl10] ———, *Graphs, permutations and topological groups*, arXiv preprint 1008.3062 (2010).

[Mon01] N. Monod, *Continuous bounded cohomology of locally compact groups*, no. 1758, Springer, 2001.

[MŚ59] A. M. Macbeath and S. Świerczkowski, *On the set of generators of a subgroup*, Indagationes Mathematicae, vol. 62, Elsevier, 1959, pp. 280–281.

[MSV14] J. Morris, P. Spiga, and G. Verret, *Semiregular automorphisms of cubic vertex-transitive graphs*, arXiv preprint 1401.2940 (2014).

[MV12] R. G. Möller and J. Vonk, *Normal subgroups of groups acting on trees and automorphism groups of graphs*, Journal of Group Theory **15** (2012), no. 6, 831–850.

[MZ52] D. Montgomery and L. Zippin, *Small subgroups of finite-dimensional groups*, Proceedings of the National Academy of Sciences **38** (1952), no. 5, 440–442.

[Ner03] Y. A. Neretin, *Groups of hierarchomorphisms of trees and related hilbert spaces*, Journal of Functional Analysis **200** (2003), no. 2, 505–535.

[Pra96] C. Praeger, *Finite quasiprimitive graphs*, University of Western Australia. Department of Mathematics, 1996.

[PSV12] P. Potočnik, P. Spiga, and G. Verret, *On graph-restrictive permutation groups*, Journal of Combinatorial Theory, Series B **102** (2012), no. 3, 820–831.

[PW03] W. Previts and T. Wu, *Dense orbits and compactness of groups*, Bulletin of the Australian Mathematical Society **68** (2003), no. 1, 155–159.

[Rat04] D. Rattaggi, *Computations in groups acting on a product of trees: Normal subgroup structures and quaternion lattices*, Ph.D. thesis, ETH Zurich, 2004.

[Rei13] C. Reid, *Local Sylow theory of totally disconnected, locally compact groups*, Journal of Group Theory **16** (2013), no. 4, 535–555.

[Rob96] D. Robinson, *A Course in the Theory of Groups*, Springer, 1996.

[RW15] C. Reid and P. Wesolek, *The essentially chief series of a compactly generated locally compact group*, arXiv preprint 1509.06593 (2015).

[Ser03] J.-P. Serre, *Trees*, Springer, 2003.

[Slo] N.J.A. Sloane, *The On-Line Encyclopedia of Integer Sequences*, published electronically at https://oeis.org.

[SW13] Y. Shalom and G. A. Willis, *Commensurated subgroups of arithmetic groups, totally disconnected groups and adelic rigidity*, Geometric and Functional Analysis **23** (2013), no. 5, 1631–1683.

[Tho70] J. G. Thompson, *Bounds for orders of maximal subgroups*, Journal of Algebra **14** (1970), no. 2, 135–138.

[Tit70] J. Tits, *Sur le groupe des automorphismes d'un arbre*, Essays on topology and related topics, Springer, 1970, pp. 188–211.

[Tor17] S. Tornier, *Prime localizations of Burger–Mozes-type groups*, Journal of Group Theory (2017).

[Tut47] W. T. Tutte, *A family of cubical graphs*, Mathematical Proceedings of the Cambridge Philosophical Society, vol. 43, Cambridge University Press, 1947, pp. 459–474.

[Tut59] ———, *On the symmetry of cubic graphs*, Canadian Journal of Mathematics **11** (1959), 621–624.

[vD31] D. van Dantzig, *Studien over topologische algebra*, University of Groningen, 1931.

[Wei78] R. Weiss, *s-Transitive graphs*, Algebraic methods in graph theory **25** (1978), 827–847.

[Wes15] P. Wesolek, *Elementary totally disconnected locally compact groups*, Proceedings of the London Mathematical Society (2015), pdv013.

[Wie71] H. Wielandt, *Subnormal subgroups and permutation groups*, Ohio State University, 1971.

[Wil94] G. A. Willis, *The structure of totally disconnected locally compact groups*, Mathematische Annalen **300** (1994), no. 1, 341–363.

[Wil01] ———, *Further properties of the scale function on a totally disconnected group*, Journal of Algebra **237** (2001), no. 1, 142–164.

[Wil04] ———, *Tidy subgroups for commuting automorphisms of totally disconnected groups: An analogue of simultaneous triangularisation of matrices*, New York J. Math **10** (2004), 1–35.

[Wil07] ———, *Compact open subgroups in simple totally disconnected groups*, Journal of Algebra **312** (2007), no. 1, 405–417.

[Wil15] ———, *The scale and tidy subgroups for endomorphisms of totally disconnected locally compact groups*, Mathematische Annalen **361** (2015), 403–442.

[Yam53] H. Yamabe, *A generalization of a theorem of Gleason*, Annals of Mathematics (1953), 351–365.

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Curriculum Vitae

Stephan Tornier

Personal data

Birth	Born April 5, 1988 in Gifhorn, Germany
Citizenship	Germany

Education

03/2018	Ph.D. in Mathematics, ETH Zurich, Switzerland
	Examiner: Marc Burger. Co-Examiner: George Willis
	Various research stays at The University of Newcastle, Australia.
05/2013	MSc ETH in Mathematics, ETH Zurich, Switzerland.
11/2012	BSc ETH in Mathematics, ETH Zurich, Switzerland.
	Exchange semester at ANU, Canberra, Australia.
09/2008	Civil Service, Munich, Germany.
07/2007	Abitur, Gymnasium Hankensbüttel, Germany.

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