HAAR MEASURES

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Abstract. This article provides a concise introduction to the theory of Haar measures on locally compact Hausdorff groups. We will in particular discuss unimodularity and coset spaces. A good reference is [KL06, Sec. 7]. Further references include [Bou04, Ch. 7] and [Kna02, Ch. VIII].

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1. Preliminaries

The natural class of groups for which to consider Haar measures is that of locally compact Hausdorff groups, due to Theorem 2.2 below.

1.1. **Locally Compact Hausdorff Groups.** After having reviewed the definitions, we show that this class is stable under taking closed subgroups and coset spaces with respect to closed subgroups.

A *topological group* is a group G with a topology such that multiplication and inversion are continuous. As a consequence, left and right multiplication by elements of G as well as inversion are homeomorphisms of G. Therefore, the neighbourhoodsystem of the identity $e \in G$ determines the topology on G. A topological space is *locally compact* if every point has a compact neighbourhood; and it is *Hausdorff* if any two distinct points have disjoint neighbourhoods in which case local compactness is equivalent to every point admitting a relatively compact open neighbourhood, i.e. an open neighbourhood with compact closure.

The class of locally compact Hausdorff groups is stable under taking closed subgroups as follows from the following Proposition. Recall that if X is a topological space and A is a subset of X, we may equip A with the *relative topology*, i.e. $U \subseteq A$ is open if and only if there is an open set $V \subseteq X$, such that $U = A \cap V$.

Proposition 1.1. Let X be a locally compact Hausdorff space and let A be a closed subset. Then A is locally compact Hausdorff.

Proof. Recalling that compact subsets of Hausdorff spaces are closed and that closed subsets of compact sets are compact, this is immediate following the definitions. □

As to coset spaces, we record the following lemma on a property of neighbourhoods that comes with the group structure.

Lemma 1.2. Let G be a topological group. Then for every $x \in G$ and every neighbourhood U of $e \in G$, there is an open neighbourhood V of x with $V^{-1}V \subseteq U$.

Proof. The map $\varphi : G \times G \to G,\ (g,h) \mapsto g^{-1}h$ is continuous. Hence there are open sets $V_1, V_2 \subseteq G$ such that $V_1^{-1}V_2 = \varphi(V_1 \times V_2) \subseteq U$. Then $V = V_1 \cap V_2$ serves. □

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If G is a topological group and H is a subgroup of G, we equip the set of cosets G/H with the *quotient topology*, i.e. $U \subseteq G/H$ is open if and only if $\pi^{-1}(U) \subseteq G$ is open where $\pi : G \to G/H,\ g \mapsto gH$. Then π is continuous and open, and left multiplication with $g \in G$ is a homeomorphism of G/H.

Proposition 1.3. Let G be a topological group and let H be a closed subgroup of G. Then G/H is Hausdorff.

Proof. Let $xH, yH \in G/H$ be distinct. Then $yHx^{-1} \subseteq G$ is closed and does not contain $e \in G$. Hence, by Lemma 1.2, there is an open neighbourhood $V \subseteq G$ of $e \in G$ such that $V^{-1}V \subseteq G - yHx^{-1}$. Then VxH and VyH are disjoint open neighbourhoods of $xH \in G/H$ and $yH \in G/H$ respectively. □

Proposition 1.4. Let G be a locally compact topological group and let H be a subgroup of G. Then G/H is locally compact.

Proof. It suffices to show that $H \in G/H$ has a compact neighbourhood. Since G is locally compact, there is a compact neighbourhood K of $e \in G$. Let V be as in Lemma 1.2. Then $\pi(V)$ is an open neighbourhood of $H \in G/H$ since π is open. We show that $\overline{\pi(V)}$ is compact. If $gH \in \overline{\pi(V)}$ then $VgH \cap VH \neq \emptyset$ and hence $gH = v_1^{-1}v_2H$ for some $v_1, v_2 \in V$. Thus $\overline{\pi(V)} \subseteq \pi(U)$ which is compact since π is continuous and hence so is $\overline{\pi(V)} \subseteq \pi(U)$. □

1.2. **Some Topological Group Theory.** We further collect several facts from topological group theory, to be used in the sequel.

First, we state a version of Urysohn's Lemma which guarantees the existence of certain compactly supported functions on locally compact Hausdorff spaces. Recall that if X is a topological space, $f \in C_c(X)$ such that $0 \leq f(x) \leq 1$ for all $x \in X$, $U \subseteq X$ open and $K \subseteq X$ compact, one writes $f \prec U$ if $\mathrm{supp}(f) \subseteq U$ and $K \prec f$ if $f(k) = 1$ for all $k \in K$.

Lemma 1.5 (Urysohn)**.** Let X be a locally compact Hausdorff space. If $K \subseteq X$ is compact and $U \subseteq X$ is open such that $K \subseteq U$, then there exists $f \in C_c(G)$ satisfying $K \prec f \prec U$.

Also, we shall need the notion of uniform continuity for functions on topological groups (which comes from giving the group the structure of a uniform space). Let G be a topological group. A function $f : G \to \mathbb{C}$ is *uniformly continuous on the left (right)* if for all $\varepsilon > 0$ there is an open neighbourhood U of $e \in G$ such that for all $x \in G$ and $g \in U$ we have $|f(gx) - f(x)| < \varepsilon$ ($|f(xg) - f(x)| < \varepsilon$).

Proposition 1.6. Let G be a locally compact Hausdorff group. Then any $f \in C_c(G)$ is uniformly continuous on the left and right.

Proof. We prove that f is uniformly continuous on the left, uniform continuity on the right being handled analogously. Let $\varepsilon > 0$. By continuity of f, there is for each $x \in \mathrm{supp}\, f$ an open neighbourhood U_x of $e \in G$ such that $|f(gx) - f(x)| < \varepsilon/2$ for all $g \in U_x$. For every U_x ($x \in G$), pick a symmetric open neighbourhood V_x of $e \in G$ such that $V_x^2 \subseteq U_x$ using Lemma 1.2. Since $\mathrm{supp}\, f$ is compact, finitely many of the sets $V_x x$ ($x \in \mathrm{supp}\, f$) cover $\mathrm{supp}\, f$, say $(V_{x_k}x_k)_{k=1}^n$. Define $V = \bigcap_{k=1}^n V_k$. Then for all $x \in \mathrm{supp}\, f$ and for all $g \in V$ we have

$$|f(gx) - f(x)| \leq |f(gx) - f(x_k)| + |f(x_k) - f(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

where $k \in \{1, \dots, n\}$ is chosen such that $x \in V_{x_k}x_k$. If $x \notin \mathrm{supp}\, f$ then for every $g \in V$ either $gx \notin \mathrm{supp}\, f$ in which case the above inequality is trivial, or $gx \in \mathrm{supp}\, f$ in which case we set $y = gx$. Then $|f(gx) - f(x)| = |f(g^{-1}y) - f(y)|$ with $y \in \mathrm{supp}\, f$ and $g^{-1} \in V$; we may then argue as before. □

Finally, the following elementary facts will be useful here and there.

Proposition 1.7. Let G be a topological group and $A, B \subseteq G$. If A and B are compact, then AB is compact. If either A or B is open, then AB is open.

Proof. If A and B are compact, then so is AB as the image of the compact set (A, B) under the continuous multiplication map from $G \times G$ to G. If either A or B is open, then AB is open as a union of open sets since $\bigcup_{a \in A} aB = AB = \bigcup_{b \in B} Ab$. $\square$

Proposition 1.8. Let G be a locally compact Hausdorff group and let H be a subgroup of G. Further, let $C \subseteq G/H$ be compact. Then there exists a compact set $K \subseteq G$ such that $\pi(K) \supseteq C$.

Proof. We may cover G by relatively compact open sets U_i $(i \in I)$. Since π is open and $C \subseteq G/H$ is compact, finitely many of the $\pi(U_i)$ $(i \in I)$ cover C, say $(\pi(U_k))_{k=1}^n$. Then $K = \bigcup_{k=1}^n \overline{U_k}$ serves. $\square$

1.3. **Some Measure Theory.** We now review some basic measure theory in order to give the definition of a Haar measure and some first properties.

Let X be a non-empty set. A *σ-algebra on X* is a set $\mathcal{M} \subseteq \mathcal{P}(X)$ of subsets of X, containing the empty set, which is closed under taking complements and countable unions. A pair $(X, \mathcal{M})$ where X is a set and $\mathcal{M}$ a σ-algebra on X is a *measurable space*; the sets $E \in \mathcal{M}$ are *measurable*. Given two measurable spaces $(X, \mathcal{M})$ and $(Y, \mathcal{N})$, a map $f : X \to Y$ is *measurable* if $f^{-1}(F) \in \mathcal{M}$ for all $F \in \mathcal{N}$. As a particular example, let X and Y be topological spaces equipped with their *Borel σ-algebras* $\mathcal{B}(X)$ and $\mathcal{B}(Y)$ respectively, i.e. the σ-algebra generated by the open sets. Then any continuous map from X to Y is measurable. *In the following we shall always equip topological spaces with their Borel σ-algebra.*

A *measure* on a measurable space $(X, \mathcal{M})$ is a map $\mu : \mathcal{M} \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ which is zero on the empty set and countably additive, i.e. whenever $(E_n)_{n \in \mathbb{N}}$ is a sequence of pairwise disjoint measurable sets, then $\mu(\bigcup_{n \in \mathbb{N}} E_n) = \sum_{n=1}^{\infty} \mu(E_n)$. A triple $(X, \mathcal{M}, \mu)$ where $(X, \mathcal{M})$ is a measurable space and μ is a measure on $(X, \mathcal{M})$ is a *measure space*. A set of measure zero is a *null set*. The complement of a null set is a *conull set*.

If $(X, \mathcal{M}, \mu)$ is a measure space, $(Y, \mathcal{N})$ a measurable space and $\varphi : X \to Y$ a measurable map, then $\varphi_* \mu : \mathcal{N} \to \mathbb{R}_{\geq 0} \cup \{\infty\},\ F \mapsto \mu(\varphi^{-1}(F))$ is the *push-forward measure on $(Y, \mathcal{N})$ under φ*.

The category of measure spaces is designed to allow for the following notion of an integral of certain measurable, complex-valued functions on $(X, \mathcal{M}, \mu)$.

1. If χ_E is the characteristic function of a measurable set $E \in \mathcal{M}$, define
$$\int_X \chi_E(x)\ \mu(x) = \mu(E).$$
2. If $f = \sum_{i=1}^n \lambda_i \chi_{E_i}$ is a positive real linear combination of characteristic functions of measurable sets, a *simple function*, define
$$\int_X f(x)\ \mu(x) = \sum_{i=1}^n \lambda_i \int_X \chi_{E_i}(x)\ \mu(x).$$
3. If $f : X \to \mathbb{R}$ is measurable and nonnegative, define
$$\int_X f(x)\ \mu(x) = \sup_{\varphi} \int_X \varphi(x)\ \mu(x)$$
where φ ranges over all real-valued simple functions on X with $0 \leq \varphi \leq f$.

4. If $f : X \to \mathbb{R}$ is measurable, decompose
$$f = f_+ - f_- \quad \text{where} \quad f_\pm(x) = \max(\pm f(x), 0).$$
If $\int_X |f(x)| \ \mu(x) < \infty$, define
$$\int_X f(x) \ \mu(x) = \int_X f_+(x) \ \mu(x) - \int_X f_-(x) \ \mu(x).$$
5. If $f : X \to \mathbb{C}$ is measurable and *integrable*, i.e. $\int_X |f(x)| \ \mu(x) < \infty$, define
$$\int_X f(x) \ \mu(x) = \int_X \mathrm{Re}(f(x)) \ \mu(x) + i \int_X \mathrm{Im}(f(x)) \ \mu(x).$$

The vector space of classes of measurable, integrable complex-valued functions on X modulo equality on a conull set is denoted by $L^1(X, \mu)$. The integral is a linear map from $L^1(X, \mu)$ to $\mathbb{C}$. There is the following change of variables formula.

Proposition 1.9 (Change of variables)**.** Let $(X, \mathcal{M}, \mu)$ be a measure space, $(Y, \mathcal{N})$ a measurable space and $\varphi : X \to Y$ a measurable map. For every measurable function $f : Y \to \mathbb{C}$ and every $F \in \mathcal{N}$ we have
$$\int_F f(y) \ \varphi_*\mu(y) = \int_{\varphi^{-1}(F)} f(\varphi(x)) \ \mu(x).$$
in case either of the two sides is defined.

Next, we recall Fubini's Theorem which reduces integrating over a product space to integrating over the factors. Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be measure spaces. Then so is $(X \times Y, \mathcal{M} \times \mathcal{N}, \mu \times \nu)$ where $(\mu \times \nu)$ is defined by $(\mu \times \nu)(E, F) := \mu(E)\nu(F)$ for all $(E, F) \in \mathcal{M} \times \mathcal{N}$. Also, recall that $(X, \mathcal{M}, \mu)$ is *σ-finite* if X is a countable union of sets of finite measure.

Theorem 1.10 (Fubini)**.** Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be σ-finite measure spaces. Let $f : X \times Y \to \mathbb{C}$ be measurable and suppose $\int_X \int_Y |f(x, y)| \ \nu(y) \ \mu(x) < \infty$. Then $f \in L^1(X \times Y, \mu \times \nu)$ and
$$\int_X \int_Y f(x, y) \ \nu(y) \ \mu(x) = \int_{X \times Y} f(x, y) \ (\mu \times \nu)(x, y) = \int_Y \int_X f(x, y) \ \mu(x) \ \nu(y).$$

Measures on topological spaces which appear in practice often satisfy the following additional regularity properties.

Definition 1.11 (Radon measure)**.** A *Radon measure* on a topological space X is a measure on $(X, \mathcal{B}(X))$ which additionally satisfies the following properties:

(R1) If $K \subseteq X$ is compact, then $\mu(K) < \infty$.
(R2) If $E \subseteq X$ is measurable, then $\mu(E) = \inf\{\mu(U) \mid U \supseteq E, U \text{ open}\}$.
(R3) If $U \subseteq X$ is open, then $\mu(U) = \sup\{\mu(K) \mid K \subseteq U, K \text{ compact}\}$.

The importance of Radon measures is also due to the following result of Riesz which often is employed to define a measure on a given space in the first place.

Theorem 1.12 (Riesz)**.** Let X be a locally compact Hausdorff space. Further, let $\lambda : C_c(X) \to \mathbb{C}$ be a positive, i.e. $\lambda(f) \in [0, \infty)$ whenever $f(x) \in [0, \infty)$ for all $x \in X$, linear functional. Then there exists a unique Radon measure μ on X with
$$\lambda(f) = \int_X f(x) \ \mu(x) \quad \text{for all} \quad f \in C_c(X).$$
Furthermore, μ satisfies
$$\mu(U) = \sup\{\lambda(f) \mid f \prec U\} \quad \text{and} \quad \mu(K) = \inf\{T(f) \mid K \prec f\}$$
for every open set $U \subseteq X$ and every compact set $K \subseteq X$.

2. Definition

When dealing with topological groups it is natural to look for measures which are invariant under translation. Such measures always exist for locally compact Hausdorff groups.

Definition 2.1 (Haar measure)**.** Let G be a locally compact Hausdorff group. A *left (right) Haar measure on* G is a Radon measure μ on $(G, \mathcal{B}(G))$ which is non-zero on non-empty open sets and invariant under left-translation (right-translation):

(H1) If $U \subseteq X$ is open, then $\mu(U) \gneq 0$.
(H2) For all $E \in \mathcal{B}(G)$ and $g \in G$: $\mu(gE) = \mu(E)$ ($\mu(Eg) = \mu(E)$).

Theorem 2.2 (Haar measure)**.** Let G be a locally compact Hausdorff group. Then there exists a left (right) Haar measure on G which is unique up to strictly positive scalar multiples.

We shall not prove this theorem here. However, we make the following remark.

Remark 2.3. Whereas the uniqueness statement of Theorem 2.2 is not too hard to establish, the existence proof is more involved and not particularly fruitful. For both, see e.g. [Wei65]. However, there are three classes of locally compact Hausdorff groups for which existence may be established by classical means, see Remark 2.8.

Example 2.4. Let G be a discrete group. Then the counting measure on G, defined by $\mu : \mathcal{B}(G) = \mathcal{P}(G) \to \mathbb{R}_{\geq 0} \cup \{\infty\},\ E \mapsto |E|$, is a left and right Haar measure.

More examples are to follow in Example 2.7. For now, consider the following useful alternative description of Haar measures: Due to Riesz' Theorem 1.12, there is a one-to-one correspondence between Haar measures and *Haar functionals*, to be defined below, on a given group which is often used to obtain a Haar measure in the first place. Recall that a topological group G acts on $C_c(G)$ via the left-regular and the right-regular representation $\lambda_G(g)f(x) = f(g^{-1}x)$ and $\varrho_G(g)f(x) = f(xg)$, where $g, x \in G$ and $f \in C_c(G)$.

Definition 2.5. Let G be a locally compact Hausdorff group. A *left (right) Haar functional* on G is a non-trivial positive linear functional on $C_c(G)$ which is invariant under λ_G (ϱ_G).

Proposition 2.6. Let G be a locally compact Hausdorff group. Then there are the following mutually inverse maps.

$$\Phi : \{\text{Haar measures on } G\} \underset{\text{Riesz}}{\overset{\text{Integration}}{\rightleftarrows}} \{\text{Haar functionals on } G\} : \Psi$$

Proof. The map Φ is readily checked to range in the positive linear functionals on $C_c(G)$. For λ_G-invariance (ϱ_G-invariance), use the change of variables formula 1.9. As to non-triviality, let μ be a left (right) Haar measure on G and let K be a compact neighbourhood of some point in G. Then $\mu(K) \in (0, \infty)$ by (R1) and (H1), and by Urysohn's Lemma 1.5 there is $f \in C_c(G)$ such that $K \prec f \prec G$ and therefore $\Phi\mu(f) = \int_G f(g)\ \mu(g) \geq \mu(K) \gneq 0$.

Conversely, if λ is a left (right) Haar functional on G, its non-triviality translates to (H1) for $\mu := \Psi\lambda$ and its invariance under λ_G (ϱ_G) translates to (H2) for μ: Suppose U is a non-empty open set of measure zero with respect to μ. Then any compact set admits a finite cover by left (right) translates of U and hence has measure zero. Thus $\lambda(f) = \int_G f(g)\ \mu(g) = \int_{\operatorname{supp} f} f(g)\ \mu(g) = 0$ for all $f \in C_c(G)$, contradicting the non-triviality of λ.

As for invariance, suppose that λ is λ_G-invariant (ϱ_G-invariance being handled analogously) and let $E \in \mathcal{B}(G)$ and $g \in G$. Then by (R2),

$$\mu(gE) = \inf\{\mu(U) \mid U \supseteq gE,\ U \text{ open}\} = \inf\{\mu(gU) \mid U \supseteq E,\ U \text{ open}\}.$$

Further, by Theorem 1.12 and the λ_G-invariance of λ we have

$$\mu(gU) = \sup\{\lambda(f) \mid f \prec gU\} = \sup\{\lambda(\lambda_G(g)f) \mid f \prec U\} = \mu(U).$$

Hence μ is left invariant. The assertions $\Psi \circ \Phi = \mathrm{id}$ and $\Phi \circ \Psi = \mathrm{id}$ are immediate. $\square$

Example 2.7. Here are further examples of Haar measures.

(i) On $G = (\mathbb{R}, +)$, a left- and right Haar measure is given by the Lebesgue measure λ which can be defined as the Radon measure associated to the classical Riemann integral $\int_{\mathbb{R}} : C_c(\mathbb{R}) \to \mathbb{C}$ via Proposition 2.6.

(ii) On $G = (\mathbb{R}^n, +)$, $n \geq 1$, a left- and right Haar measure is given by the n-th power of the Lebesgue measure λ.

(iii) On $G = (\mathbb{R}^*, \cdot)$, the Lebesgue measure is not left-invariant. However, the map

$$\mu : C_c(G) \to \mathbb{C},\ f \mapsto \int_{\mathbb{R}} f(x)\ \frac{\lambda(x)}{|x|}$$

can be checked to be a left- and right Haar functional and hence defines a left- and right Haar measure on G by Proposition 2.6. Note that the above integral is always finite as the integrand has compact support; use the classical substitution rule to check left- and right-invariance.

(iv) On $G = \mathrm{GL}(n, \mathbb{R})$, $n \geq 1$, the left- and right Haar functional

$$\mu : C_c(G) \to \mathbb{C},\ f \mapsto \int_G f(X)\ \frac{\lambda(X)}{|\det X|^n}$$

defines a left- and right Haar functional on G. Here, $\lambda(X) := \prod_{i,j=1}^n \lambda(x_{ij})$ where $X = (x_{ij})_{i,j}$ is the Lebesgue measure on $\mathbb{R}^{n\cdot n}$ of which $\mathrm{GL}(n, \mathbb{R})$ is an open subset; the latter fact is key: The same construction does not work for e.g. $\mathrm{SL}(n, \mathbb{R})$ which is a submanifold of $\mathbb{R}^{n\cdot n}$ of strictly smaller dimension. Again, the integral is finite by compactness of the support of the integrand and invariance is checked by changing variables. Note that the case $G = (\mathbb{R}^*, \cdot)$ is contained via $n = 1$ in this example.

A left- and right Haar measure for $\mathrm{SL}(2, \mathbb{R})$ will be constructed in Example 4.5.

Remark 2.8. Having established the correspondence between Haar functionals and Haar measures, we now outline existence proofs of Theorem 2.2 for compact Hausdorff groups, Lie groups and totally disconnected locally compact separable Hausdorff groups.

(i) Compact Hausdorff groups. Let G be a compact Hausdorff group. Then G acts continuously on $C(G) = C_c(G)$, equipped with the supremum norm, via the left-regular representation. Therefore, G also acts on the dual space $C(G)^*$ of $C(G)$ via the adjoint representation λ_G^* of λ_G defined by

$$\langle \lambda_G^*(g)\mu, f\rangle = \langle \mu, \lambda_G(g^{-1})f\rangle.$$

for all $\mu \in C(G)^*$ and $f \in C(G)$. Since the set $P(G)$ of probability measures on G is a weak*-compact, convex and λ_G^*-invariant subset of $C(G)^*$, the compact version of the Kakutani-Markov Fixed Point Theorem (e.g. [Zim90, Thm. 2.23]) implies that it contains a λ_G^*-fixed point, i.e. a left-invariant probability measure on G, which turns out to be a left Haar measure on G.

(ii) Lie groups. Let G be a Lie group with Lie algebra $\mathrm{Lie}(G) \cong \Gamma(\mathrm{T}G)^G$, the space of left-invariant vector fields on G which is isomorphic to T_eG as a vector space. Further, let $X_1, \ldots, X_n$ be a basis of T_eG with associated left-invariant vector fields $X_1^G, \ldots, X_n^G \in \Gamma(\mathrm{T}G)^G$. Then for each $p \in G$, the tuple $((X_1^G)_p, \ldots, (X_n^G)_p)$ is a basis of T_pG and we may for each $i \in \{1, \ldots, n\}$ define a 1-form ω_i on G by $(\omega_i)_p((X_j)_p) = \delta_{ij}$; that is, for each $p \in G$, the tuple $((\omega_1)_p, \ldots, (\omega_n)_p)$ is the basis of $\mathrm{T}_p^* G$ dual to $((X_1^G)_p, \ldots, (X_n^G)_p)$. It is readily checked that the left-invariance of $X_1^G, \ldots, X_n^G$ implies left-invariance of the ω_i $(i \in \{1, \ldots, n\})$ in the sense that $L_g^* \omega_i = \omega_i$ for all $g \in G$ and $i\{1, \ldots, n\}$. Then so is the n-form $\omega := \omega_1 \wedge \cdots \wedge \omega_n$ since $\wedge$ commutes with pullback. Furthermore, one checks that ω is nowhere vanishing. We may then orient G such that ω is positive and hence gives rise to the left Haar functional

$$\lambda_\omega : C_c(G) \to \mathbb{C},\ f \mapsto \int_G f\ \omega$$

which in turn via Riesz' Theorem 1.12 provides a left Haar measure on G, see [Kna02, VIII.2].

(iii) Totally disconnected locally compact separable Hausdorff groups. Let G be a group of this type. By van Dantzig's theorem, G contains a compact open subgroup K. Assuming G to be non-compact, by separability and openness of K there are $g_n \in G$ $(n \in \mathbb{N})$ such that $G = \bigsqcup_{n \in \mathbb{N}} g_n K$. Using part (i), let ν be a Haar measure on K and let $\nu_n := g_{n*}\nu$ be the corresponding measure on $g_n K$. For $E \in \mathcal{B}(G)$ define

$$\mu(E) := \sum_{n \in \mathbb{N}} \nu_n(E \cap g_n K) = \sum_{n \in \mathbb{N}} \nu(g_n^{-1} E \cap K)$$

if the sum exists and infinity otherwise. Then μ is a Radon measure on G which is non-zero on non-empty open sets since ν is. Also, μ is left-invariant: Given $g \in G$, there is $\sigma \in S_{\mathbb{N}}$ such that $g g_n K = g_{\sigma(n)} K$. Then

$$\begin{aligned}\mu(g^{-1}E) &= \sum_{n \in \mathbb{N}} \nu(g_n^{-1} g^{-1} E \cap K) = \sum_{n \in \mathbb{N}} \nu(g_{\sigma(n)}^{-1} g g_n g_n^{-1} g^{-1} E \cap K) \\ &= \sum_{n \in \mathbb{N}} \nu(g_{\sigma_n^{-1}} E \cap K) = \sum_{n \in \mathbb{N}} \nu(g_n E \cap K) = \mu(E).\end{aligned}$$

where the second equality uses K-invariance of ν.

By Remark 2.8, compact Hausdorff groups have finite Haar measure. We now show that the converse holds as well.

Proposition 2.9. Let G be a locally compact Hausdorff group and let μ be a left (right) Haar measure on G. Then $\mu(G) < \infty$ if and only if G is compact.

Proof. If G is compact, then $\mu(G) < \infty$ by Definition R1. Conversely, suppose that G is not compact and let U be a relatively compact neighbourhood of $e \in G$. Then there is an infinite sequence $(g_n)_{n \in \mathbb{N}}$ of elements of G such that $g_n \notin \bigcup_{k<n} g_k U$; otherwise G would be compact as a finite union of compact sets. Let V be as in Lemma 1.2. Then the sets $g_n V$ $(n \in \mathbb{N})$ are pairwise disjoint by the fact that $VV^{-1} \subseteq U$ and the definition of $(g_n)_{n \in \mathbb{N}}$. Therefore, as V has strictly positive measure, G has infinite measure. □

3. Unimodularity

We now address and quantify the question whether left and right Haar measures on a given locally compact Hausdorff group coincide.

Definition 3.1. A locally compact Hausdorff group G is *unimodular* if every left Haar measure on G is also a right Haar measure on G and conversely.

Remark 3.2. By Theorem 2.2, it suffices in Definition 3.1 to ask for every left Haar measure on G to also be a right Haar measure.

Proposition 3.6 below will provide several classes of unimodular groups. For now, let G be a locally compact Hausdorff group and let μ be a left Haar measure on G. Then for every $g \in G$, the map $\mu_g : \mathcal{B}(G) \to \mathbb{R}_{\geq 0} \cup \{\infty\},\ E \mapsto \mu(Eg)$ is a left Haar measure on G as well. Hence, by uniqueness, there exists a strictly positive real number $\Delta_G(g)$ such that $\mu_g = \Delta_G(g)\mu$, i.e.

$$\text{(M)} \qquad \mu(Eg) = \mu_g(E) = \Delta_G(g)\mu(E) \quad \text{for all} \quad E \in \mathcal{B}(G).$$

The function $\Delta_G : G \to \mathbb{R}_{>0}$ is independent of μ and called *modular function of* G.

Let λ be the left Haar functional associated to μ by Proposition 2.6. Then by the change of variable formula 1.9 applied to $\varphi = R_{g^{-1}}$, equation (M) immediately translates to

$$\text{(M')} \qquad \lambda(\varrho_G(g^{-1})f) = \Delta_G(g)\lambda(f) \quad \text{for all} \quad f \in C_c(G).$$

Proposition 3.3. Let G be a locally compact Hausdorff group. Then the modular function $\Delta_G : G \to (\mathbb{R}_{>0}, \cdot)$ is a continuous homomorphism.

Proof. Let μ be a left Haar measure on G. The homomorphism property is immediate from (M): For all $g, h \in G$ we have

$$\Delta_G(gh)\mu = \mu_{gh} = (\mu_h)_g = \Delta_G(h)\mu_h = \Delta_G(g)\Delta_G(h)\mu.$$

Evaluating on a set of non-zero finite measure, e.g. a compact neighbourhood of some point, proves the assertion.

As to continuity, let λ be the left Haar functional associated to μ by Proposition 2.6. It suffices to check continuity at $e \in G$, since Δ_G is a homomorphism. Let K be a compact neighbourhood of $e \in G$. Using Urysohn's Lemma 1.5, we choose $\varphi \in C_c(G)$ such that $K \prec \varphi \prec G$ and $\psi \in C_c(G)$ such that $K \operatorname{supp}\varphi \prec \psi \prec G$ (see Proposition 1.7). In particular, φ is uniformly continuous on the right by Proposition 1.6: Given $\varepsilon > 0$, let $U \subseteq K$ be a symmetric open neighbourhood of $e \in G$ such that $|\varphi(xg) - \varphi(x)| < \varepsilon$ for all $x \in G$ and $g \in U$. Then by (M'),

$$|\Delta_G(g) - 1| = \frac{1}{\lambda(\varphi)}\left|\Delta_G(g)\lambda(\varphi) - \lambda(\varphi)\right| \leq \frac{1}{\lambda(\varphi)}\lambda(|\varrho_G(g^{-1})\varphi - \varphi|\psi) \leq \varepsilon\frac{\lambda(\psi)}{\lambda(\varphi)}$$

for all $g \in U$. Hence Δ_G is continuous at the identity. $\square$

Remark 3.4. We have noticed that for a locally compact Hausdorff group G with left Haar measure μ and given $g \in G$, the map $\mu_g : \mathcal{B}(G) \to \mathbb{R}_{\geq 0} \cup \{\infty\},\ E \mapsto \mu(Eg)$ is a left Haar measure on G as well. This is an instance of the following more general observation: For every continuous automorphism $\alpha \in \operatorname{Aut}(G)$, the map $\mu_\alpha : \mathcal{B}(G) \to \mathbb{R}_{\geq 0} \cup \{\infty\},\ E \mapsto \mu(\alpha(E))$ is a left Haar measure on G. In this setting, $\mu_g = \mu_{\operatorname{int}(g^{-1})}$ where $\operatorname{int}(g) : G \to G,\ x \mapsto gxg^{-1}$ denotes conjugation in G by g. One may then introduce the general *modular function* $\operatorname{mod}_G : \operatorname{Aut}(G) \to (\mathbb{R}_{>0}, \cdot)$ which remains to be a homomorphism and with the Braconnier topology on $\operatorname{Aut}(G)$, a refinement of the compact-open topology, becomes continuous, see e.g. [Pal01].

We obtain the following useful criterion for unimodularity.

Corollary 3.5. Let G be a locally compact Hausdorff group. Then G is unimodular if and only if $\Delta_G \equiv 1$.

Proof. If $\Delta_G \equiv 1$, then G is unimodular by (M) and Remark 3.2. Conversely, if G is unimodular, let μ be a Haar measure on G and let E be a compact neighbourhood of some point in G. Then $\mu(E) \in (0, \infty)$ and hence $\Delta_G \equiv 1$ by (M). $\square$

Corollary 3.5 provides us with the following list of classes of unimodular groups. Yet another class will be given in Proposition 4.12.

Proposition 3.6. Let G be a locally compact Hausdorff group. Then G is unimodular if, in addition, it satisfies one of the following properties: being abelian, compact, topologically simple, topologically perfect, discrete, connected semisimple Lie or connected nilpotent Lie.

Proof. Let G be a locally compact Hausdorff abelian group with left Haar measure μ. Since $Eg = gE$ for every subset $E \subseteq G$ and all $g \in G$, the left-invariance of μ implies right-invariance.

If G is compact Hausdorff and μ is a left Haar measure on G, then $\mu(G) \in (0, \infty)$ and hence $\Delta_G \equiv 1$ by (M).

If G is topologically simple, then $\overline{[G,G]}$, which is a closed normal subgroup of G, either equals $\{e\}$ or G. In the former case, G is abelian and hence unimodular; in the latter case, continuity of Δ_G implies:

$$\Delta_G(G) = \Delta_G(\overline{[G,G]}) \subseteq \overline{\Delta_G([G,G])} = \{1\}$$

whence G is unimodular. When G is topologically perfect, i.e. $G = \overline{[G,G]}$, the same argument applies.

For a discrete group, the left Haar measures are the strictly positive scalar multiples of the counting measure which certainly is right-invariant.

Suppose now, that G is a connected semisimple Lie group. Note that in this case the modular function $\Delta_G : G \to (\mathbb{R} \setminus \{0\}, \cdot)$ is a continuous and hence smooth ([War83, Thm. 3.39]) homomorphism of Lie groups. Thus $D_e\Delta_G : \mathrm{Lie}(G) \to \mathbb{R}$ is a morphism of Lie algebras. Since $\mathrm{Lie}(G)$ is semisimple and $\mathbb{R}$ is abelian we have

$$D_e\Delta_G(\mathrm{Lie}(G)) = D_e\Delta_G([\mathrm{Lie}(G), \mathrm{Lie}(G)]) = [D_e\Delta_G(\mathrm{Lie}(G)), D_e\Delta_G(\mathrm{Lie}(G))] = \{0\}$$

and hence $\Delta_G \equiv 1$ by the Lie correspondence, passing to the universal cover of G.

For the case of a connected nilpotent Lie group, we appeal to the fact that for any Lie group G we have $\Delta_G(g) = |\det \mathrm{Ad}(g)|$, where $\mathrm{Ad} : G \to \mathrm{Aut}(\mathrm{Lie}(G))$ is the adjoint representation of G, see e.g. [Kna02, Prop. 8.27] (this follows in the setting of Remark 2.8). If, in addition, G is connected and nilpotent, then the exponential map $\exp : \mathrm{Lie}(G) \to G$ is surjective ([Kna02, Thm. 1.127]) and hence for every $g \in G$ there is some $X \in \mathrm{Lie}(G)$ such that $g = \exp(X)$ and

$$\Delta_G(g) = |\det \mathrm{Ad}(g)| = |\det e^{\mathrm{ad}X}| = e^{\mathrm{tr}\,\mathrm{ad}X} = 1$$

where the last equality follows from $\mathrm{Lie}(G)$ and hence $\mathrm{ad}X$ being nilpotent. □

The following proposition provides a class of totally disconnected locally compact Hausdorff groups that are unimodular. Recall that if T is a locally finite tree then $\mathrm{Aut}(T)$ is a totally disconnected locally compact separable Hausdorff group with the permutation topology. We adopt Serre's graph theory conventions, see [Ser80].

Proposition 3.7. Let $T = (V, E)$ be a locally finite connected graph. If $G \leq \mathrm{Aut}(T)$ is closed and locally transitive then G is unimodular.

Proof. Let μ be a left Haar measure on G, see Remark 2.8. Since G is locally transitive there is for every triple (x, e_0, e) of a vertex $x \in V$ and edges $e_0, e \in E(x)$ an element $g_e \in G_x$ such that $g_e e_0 = e$. Then $G_x = \bigsqcup_{e \in E(x)} g_e G_{e_0})$. In particular, $\mu(G_x) = |E(x)|\mu(G_{e_0})$ for every $e_0 \in E(x)$. Since $G_e = G_{\overline{e}}$ for all $e \in E$ we further conclude that $\mu(G_e) = \mu(G_{e'})$ for all $e, e' \in E$. Given $g \in G$ we therefore have

$$\mu(G_e) = \mu(G_{ge}) = \mu(gG_e g^{-1}) = \mu(G_e g^{-1}) = \Delta_G(g^{-1})\mu(G_e)$$

and hence G is unimodular. □

Example 3.8. We now provide two related examples of non-unimodular groups.

(i) Consider the group
$$P := \left\{ \begin{pmatrix} x & y \\ & x^{-1} \end{pmatrix} \middle| \, x \in \mathbb{R}\backslash\{0\}, \ y \in \mathbb{R} \right\} \leq \mathrm{SL}(2, \mathbb{R}).$$
Then the functionals $\mu, \nu : C_c(P) \to \mathbb{C}$, given by
$$\mu : f \mapsto \int_{\mathbb{R}^2} f(X) \ \frac{\lambda(x)\lambda(y)}{x^2} \quad \text{and} \quad \nu : f \mapsto \int_{\mathbb{R}^2} f(X) \ \lambda(x)\lambda(y)$$
are left- and right Haar functionals respectively as can be checked by changing variables. However, P is a closed subgroup of $\mathrm{SL}(2,\mathbb{R})$ which is unimodular as a connected simple Lie group by Proposition 3.6. We shall shed some light on the origin of this example in Remark 4.6.

(ii) Let $T_d = (V, E)$ be the d-regular tree and let $\omega \in \partial T_d$ be a boundary point of T_d. Set $G := \mathrm{Aut}(T_d)_\omega$, the stabilizer of ω in $\mathrm{Aut}(T_d)$. Then G is not unimodular: Indeed, let $t \in G$ be a translation of length 1 towards ω and let $x \in V$ be on the translation axis of t. Then
$$\begin{aligned} \Delta(t) = \frac{\mu(G_x)}{\mu(G_{tx})} &= \frac{\mu(G_x)}{\mu(G_{x,tx})} \frac{\mu(G_{x,tx})}{\mu(G_{tx})} \\ &= \frac{[G_x : G_{x,tx}]}{[G_{tx} : G_{x,tx}]} = \frac{|G_x(tx)|}{|G_{tx}x|} = \frac{1}{d-1}. \end{aligned}$$
See Remark 4.6 for how this relates to part (i).

Uilizing the modular function, we can turn left Haar measures into right Haar measures as in the following Proposition. Let $i : G \to G$ denote the inversion on G.

Proposition 3.9. Let G be a locally compact Hausdorff group with left Haar measure μ. Then $\overline{\mu} = i_*\mu : \mathcal{B}(G) \to \mathbb{R}_{\geq 0} \cup \{\infty\}, \ E \mapsto \mu(E^{-1})$ is a right Haar measure on G with associated right Haar functional $\varrho : C_c(G) \to \mathbb{C}, \ f \mapsto \int_G f(x)\Delta_G(x^{-1}) \ \mu(x)$. If G is unimodular, then $\overline{\mu} = \mu$.

Proof. The map $\overline{\mu}$ is readily checked to be a right Haar measure on G. The map ϱ is clearly positive and linear. Its non-triviality follows as in the proof of Proposition 2.6 using $\Delta_G(g) \gneq 0$ for all $g \in G$. As to ϱ_G-invariance, changing variables via Proposition 1.9 using $R_{g*}\mu = \mu_{g^{-1}}$ yields
$$\begin{aligned} \varrho(\varrho_G(g)f) &= \int_G f(xg)\Delta_G(x^{-1}) \ \mu(x) = \int_G f(x)\Delta_G(gx^{-1}) \ \mu_{g^{-1}}(x) = \\ &= \int_G f(x)\Delta_G(g)\Delta_G(x^{-1})\Delta_G(g^{-1}) \ \mu(x) = \int_G f(x)\Delta_G(x^{-1}) \ \mu(x) = \varrho(f). \end{aligned}$$
for every $f \in C_c(G)$ and $g \in G$. Overall, ϱ is a right Haar functional on G.

Now, let $\Phi\overline{\mu}$ denote the right Haar functional associated to $\overline{\mu}$ as in Proposition 2.6. Then there is a strictly positive real number c such that $\Phi\overline{\mu} = c\varrho$. Applying the change of variables formula 1.9, we obtain for all $f \in C_c(G)$:
$$\begin{aligned} \int_G f(x) \ \overline{\mu}(x) &= c\int_G f(x)\Delta_G(x^{-1}) \ \mu(x) = c\int_G f(x^{-1})\Delta_G(x) \ \overline{\mu}(x) \\ &= c^2 \int_G f(x^{-1})\Delta_G(x)\Delta_G(x^{-1}) \ \mu(x) = c^2 \int_G f(x) \ \overline{\mu}(x). \end{aligned}$$
Let K be a compact symmetric neighbourhood of some point in G and $f \in C_c(G)$ such that $K \prec f \prec G$. Then $\int_G f(x^{-1}) \ \mu(x) \in (0, \infty)$ and hence $c = 1$. In particular, unimodularity of G implies $\mu = \overline{\mu}$. $\square$

4. Coset spaces

Let G be a locally compact Hausdorff group and let H be a closed subgroup of G. If H is normal in G, there exists a left (right) Haar measure on G/H by Theorem 2.2. We now address the question under which circumstances there exists a G-invariant Radon measure on G/H which is non-zero on non-empty open sets if H is not normal in G, and we shall refer to such a measure as a *Haar measure on* G/H by abuse of notation. The following example shows that a Haar measure on G/H may or may not exist.

Example 4.1. Let $G = \mathrm{SL}(2, \mathbb{R})$.

(i) Consider the natural action of G on $X = \mathbb{R}^2 \setminus \{0\}$. Then
$$H := \mathrm{stab}_G((1,0)^T) = \left\{ \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \middle| \, x \in \mathbb{R} \right\}$$
and hence $G/H \cong X$ has a Haar measure, namely the restricted two-dimensional Lebesgue measure.

(ii) On the other hand, G acts on $X = \mathbb{P}^1\,\mathbb{R} = \{V \leq \mathbb{R}^2 \mid \dim V = 1\}$. Here,
$$H := \mathrm{stab}_G(\langle e_1 \rangle) = \left\{ \begin{pmatrix} x & y \\ & x^{-1} \end{pmatrix} \middle| \, x \in \mathbb{R}\setminus\{0\}, y \in \mathbb{R} \right\}$$
which is the non-unimodular group of Example 3.8. The space $G/H \cong X$ does not admit a Haar measure: For instance, consider the compact subsets $E_1 := \{\langle (1,t)^T \rangle \mid t \in [0,1]\}$ and $E_2 := \{\langle (t,1)^T \rangle \mid t \in [0,1]\}$ of $\mathbb{P}^1\,\mathbb{R}$. Then
$$\begin{pmatrix} 1 & -1 \\ & 1 \end{pmatrix} E_1 = E_1 \cup E_2 \quad \text{and} \quad \begin{pmatrix} 1 & -1 \\ 1 & \end{pmatrix} E_1 = E_2.$$
A Haar measure on G/H would assign finite non-zero measure to the compact sets E_1 and E_2 which combined with G-invariance contradicts the above two equalities.

Theorem 4.2. Let G be a locally compact Hausdorff group with left Haar measure μ and let H be a closed subgroup of G with left Haar measure ν. Then there exists a Haar measure ξ on G/H if and only if $\Delta_G|_H \equiv \Delta_H$. In this case, ξ is unique up to strictly positive scalar multiples and suitably normalized satisfies for all $f \in C_c(G)$:
$$\text{(W)} \qquad \int_G f(g)\ \mu(g) = \int_{G/H} \int_H f(gh)\ \nu(h)\ \xi(gH).$$

In the context of Theorem 4.2, formula (W) can be extended to hold for $f \in L^1(G)$, see [KL06, Theorem 7.12] and the surrounding explanations.

Proof. (Theorem 4.2, "$\Rightarrow$"). If ξ exists as above, then the map
$$\lambda : C_c(G) \to \mathbb{C},\ f \mapsto \int_{G/H} \int_H f(gh)\ \nu(h)\ \xi(gH)$$
is a left Haar functional on G and thus defines a left Haar measure μ on G. In particular, $\lambda(\varrho_G(t^{-1})f) = \Delta_G(t)\lambda(f)$ for all $t \in G$ and $f \in C_c(G)$ by (M'). On the other hand, we have for all $t \in H$ and $f \in C_c(G)$:
$$\begin{aligned}\lambda(\varrho_G(t^{-1})f) &= \int_{G/H} \int_H (\varrho_G(t^{-1})f)(gh)\ \nu(h)\ \xi(gH) = \\ &= \int_{G/H} \int_H \Delta_H(t) f(gh)\ \nu(h)\ \xi(gH) = \Delta_H(t)\lambda(f).\end{aligned}$$
If, by Urysohn's Lemma 1.5, we choose $f \in C_c(G)$ to satisfy $K \prec f \prec G$ where K is a compact neighbourhood of some point in G, then $\int_G f(g)\ \mu(g) = \lambda(f) \in (0, \infty)$ and hence $\Delta_G|_H \equiv \Delta_H$. □

The proof of the converse assertion of Theorem 4.2 relies on the following description of compactly supported functions on G/H. Once more, Riesz' Theorem 1.12 will be used to produce a measure.

Lemma 4.3. Let G be a locally compact Hausdorff group and H a closed subgroup of G with left Haar measure ν. Then the following map is surjective:

$$C_c(G) \to C_c(G/H),\ f \mapsto \left(f_H : gH \mapsto \int_H f(gh)\ \nu(h) \right).$$

Proof. Several things need to be checked. First of all, for all $f \in C_c(G)$ and for all $gH \in G/H$, the integral $\int_H f(gh)\ \nu(h)$ is independent of the representative of gH and finite. Next, for all $f \in C_c(G)$, the function $f_H f$ is continuous as a parametrized integral as in the proof of the continuity of the modular function. Clearly, $\operatorname{supp} f_H \subseteq p(\operatorname{supp}(f))$ and hence $f_H \in C_c(G/H)$. It remains to prove surjectivity. To this end, let $F \in C_c(G/H)$. Pick $K \subseteq G$ such that $\pi(K) \supseteq \operatorname{supp} F$ (Proposition 1.8) and let $\eta \in C_c(G)$ satisfying $K \prec \eta$ (Urysohn's Lemma). Now define $f \in C_c(G)$ by

$$f : G \to \mathbb{C},\ g \mapsto \begin{cases} \frac{F(gH)\eta(g)}{\eta_H(gH)} & \eta_H(gH) \neq 0 \\ 0 & \eta_H(gH) = 0 \end{cases}$$

Again, we need to show that this function is continuous and has compact support. As for compact support, clearly $\operatorname{supp} f \subseteq \operatorname{supp} \eta$. In fact, if G was compact, we could choose $\eta \equiv 1$. To show that f is continuous, we show that it is continuous at every point of two open sets $U_1 \subseteq G$ and $U_2 \subseteq G$ satisfying $U_1 \cup U_2 = G$. On the set $U_1 := \{g \in G \mid \eta_H(gH) \neq 0\}$ it is continuous as a quotient of continuous functions; and on the set $U_2 := G \backslash KH$ it is continuous as it vanishes there. Further, if $g \notin U_1$, then $0 = \eta_H(gH) = \int_H \eta(gh)\ \nu(h)$. Since η is a non-negative continuous function, this implies $\eta(gh) = 0$ for all $h \in H$, hence $g \notin KH$, i.e. $g \in U_2$. Continuity and compact support being established, it remains to show that $f_H \equiv F$. Compute

$$f_H(gH) = \int_H \frac{F(ghH)\eta(gh)}{\eta_H(ghH}\ \nu(h) = F(gH)\frac{\int_H \eta(gh)\ \nu(h)}{\eta_H(gH)} = F(gH).$$

Hence the map $(-)_H : C_c(G) \to C_c(G/H)$ is surjective. $\square$

Proof. (Theorem 4.2, "$\Leftarrow$"). Let $\sigma : C_c(G/H) \to C_c(G)$ be a right-inverse for the map $C_c(G) \to C_c(G/H),\ f \mapsto f_H$ of Lemma 4.3 and consider the map

$$\lambda : C_c(G/H) \to \mathbb{C},\ f \mapsto \int_G (\sigma f)(g)\ \mu(g).$$

Once λ is independent of σ, it is a positive linear functional. To prove that it is independent of σ, it suffices to show that $\int_G f(g)\ \mu(g) = 0$ whenever $f_H \equiv 0$. By Lemma 4.3 and Urysohn's Lemma 1.5 there exists a function $\eta \in C_c(G)$ such that $(\operatorname{supp} f)H \prec \eta_H \prec G/H$. Then by Proposition 3.9 we have

$$\begin{aligned} \int_G f(g)\ \mu(g) &= \int_G \eta_H(gH) f(g)\ \mu(g) = \int_G \int_H \eta(gh) f(g)\ \nu(h)\ \mu(g) \\ &= \int_G \int_H \eta(gh^{-1}) f(g) \Delta_H(h^{-1})\ \nu(h)\ \mu(g). \end{aligned}$$

We may as well integrate over the compact and hence σ-finite spaces $\operatorname{supp} f \subseteq G$ and $(\operatorname{supp} \eta)^{-1} \operatorname{supp} f \cap H \subseteq H$ (see Proposition 1.7). Therefore, Fubini's Theorem

1.10 allows us to continue the above computation by

$$= \int_H \int_G \eta(gh^{-1}) f(g) \Delta_H(h^{-1})\ \mu(g)\ \nu(h)$$
$$= \int_H \int_G \eta(g) f(gh) \Delta_H(h^{-1}) \Delta_G(h)\ \mu(g)\ \nu(h).$$

Applying Fubini's Theorem 1.10 again, we deduce using $\Delta_G|_H \equiv \Delta_H$ and $f_H \equiv 0$:

$$= \int_G \eta(g) \int_H f(gh)\ \nu(h)\ \mu(g) = \int_G \eta(g) f_H(gH) = 0$$

which completes the proof that λ is a positive linear functional. Hence, by Riesz' Theorem 1.12, there exists a unique Radon measure ξ on G/H such that

$$\int_G (\sigma f)(g)\ \mu(g) = \lambda(f) = \int_{G/H} f(gH)\ \xi(gH) =$$
$$= \int_{G/H} (\sigma f)_H(gH)\ \xi(gH) = \int_{G/H} \int_H (\sigma f)(gh)\ \nu(h)\ \xi(gH).$$

for all $f \in C_c(G/H)$. The measure ξ is checked to be non-zero on non-empty open sets and G-invariant, i.e. ξ is a Haar measure on G/H. Since the above equation is independent of σ, we may as well start with a function $f \in C_c(G)$; we have thus proven the existence of a unique Haar measure ξ on G/H satisfying (W). To complete the proof, we need to show that any Haar measure on G/H (not necessarily satisfying (W)) is a strictly positive scalar multiple of ξ: Let ξ_1, ξ_2 be Haar measures on G/H. Then there are left Haar measures μ_1, μ_2 on G satisfying (W) for ξ_1 and ξ_2 respectively (see the converse direction of the proof). By uniqueness, $\mu_2 = c\mu_1$ for some strictly positive real number c. Then ξ_2 and $c\xi_1$ both satisfy (W) for μ_2. From the uniqueness proven above we conclude $\xi_2 = c\xi_1$. □

Remark 4.4. Retain the notation of Theorem 4.2. If G is compact, then the function η in the proof of Lemma 4.3 can be chosen to identically equal one. The constructed left Haar functional on G/H is then given by

$$\lambda : C_c(G/H) \to \mathbb{C},\ f \mapsto \int_G \frac{f(gH)}{1_H(gH)}\ \mu(g) = \frac{1}{\nu(H)} \int_G f(gH)\ \mu(g).$$

Notice that $\nu(H)$ is finite by Proposition 2.9 since H is compact as a closed subset of a compact space. Now, it is a fact (see [KL06, Thm. 7.12]) that the Haar measure ξ on G/H associated to λ can be computed by evaluating λ on characteristic functions. Thus, if $E \subseteq G/H$ is measurable, we have

$$\xi(E) = \frac{\mu(\pi^{-1}(E))}{\nu(H)}, \quad \text{in particular} \quad \xi(G/H) = \frac{\mu(G)}{\nu(H)}.$$

The reader is encouraged to think about how the auxiliary function η mends the issues that arise in the case where G is not compact.

Example 4.5. To illustrate the usefulness of Theorem 4.2, we now provide a Haar functional for $G := \mathrm{SL}(2, \mathbb{R})$. Recall that G acts transitively on the upper half plane $\mathbb{H} := \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ via fractional linear transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z := \frac{az+b}{cz+d} \quad \text{and} \quad \begin{pmatrix} \sqrt{y} & x\sqrt{y}^{-1} \\ & \sqrt{y}^{-1} \end{pmatrix} i = x + iy$$

for $x \in \mathbb{R}$ and $y \in \mathbb{R}_{>0}$. Also, one readily verifies that $H := \mathrm{stab}_G(i) = \mathrm{SO}(2, \mathbb{R})$; therefore the maps

$$G/H \to \mathbb{H},\ gH \mapsto gi \quad \text{and} \quad \mathbb{H} \to G/H,\ x + iy \mapsto \begin{pmatrix} \sqrt{y} & x\sqrt{y}^{-1} \\ & \sqrt{y}^{-1} \end{pmatrix}$$

are mutually inverse G-isomorphisms. In fact they are homeomorphisms. Since G is unimodular as a connected semisimple Lie group and H is unimodular as a compact group by Proposition 3.6, we by Theorem 4.2 conclude the existence of a Haar measure ξ on $G/H \cong \mathbb{H}$. Let ν be the left Haar measure on H. Then the map

$$C_c(G) \to \mathbb{C},\ f \mapsto \int_{G/H} \int_H f(gH)\ \nu(h)\ \xi(gH)$$

is a left Haar functional on G. To make this computable, we use the homeomorphisms $H \cong S^1$ and $G/H \cong \mathbb{H}$ to change variables with Proposition 1.9, and the fact that the hyperbolic geometry on $\mathbb{H}$ provides an $\mathrm{SL}(2,\mathbb{R})$-invariant Radon measure on $\mathbb{H}$. All together, the Haar functional on $G = \mathrm{SL}(2,\mathbb{R})$ then reads

$$f \mapsto \int_{-\infty}^{\infty} \int_0^{\infty} \int_0^{2\pi} f\left(\begin{pmatrix} \sqrt{y} & x\sqrt{y}^{-1} \\ & \sqrt{y}^{-1} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\right)\ d\theta\ \frac{d\lambda(y)\ d\lambda(x)}{y^2}.$$

Remark 4.6. In the setting of Example 4.5 (i), the group P of Example 3.8 is the stabilizer in $\mathrm{SL}(2,\mathbb{R})$ of the boundary point of $\mathbb{H}$ associated to the (unit-speed) geodesic $\gamma : [0,\infty) \to \mathbb{H},\ t \mapsto i + ie^{it}$. Basically, P translates γ to asymptotic geodesics. More generally, if M is a symmetric space of non-compact type, such as $\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$, let $G := \mathrm{Iso}(M)^\circ$, $p \in M$ and $x \in \partial M$ be a boundary point. Then there is the following dichotomy of stabilizers, see e.g. [Ebe96, Sec. 2.17].

$\mathrm{stab}_G(p)$	$\mathrm{stab}_G(x)$
compact	non-compact
connected	not in general connected
not transitive on M	transitive on M
one conjugacy class	in general several conjugacy classes
unimodular	not in general unimodular

4.1. **Discrete Subgroups.** If, in the above discussion, $H = \Gamma$ is a discrete subgroup of G and G is second-countable, then integration over G/Γ can be realized by integrating over a *fundamental domain* for G/Γ in G, to be explained below. We shall always pick the counting measure ν as Haar measure on Γ.

Definition 4.7. Let G be a locally compact Hausdorff group and let Γ be a discrete subgroup of G. A *strict fundamental domain* for G/Γ in G is a set $F \in \mathcal{B}(G)$ such that $\pi : F \to G/\Gamma$ is a bijection. A *fundamental domain* for G/Γ in G is a set $F \in \mathcal{B}(G)$ which differs from a strict fundamental domain by a set of measure zero with respect to any left Haar measure on G.

Proposition 4.8. Let G be a locally compact Hausdorff, second-countable group with a discrete subgroup Γ. Then there exists a fundamental domain for G/Γ in G.

Remark 4.9. Retain the notation of Proposition 4.8. Note that second-countability of G in particular implies that Γ is countable.

Proof. (Proposition 4.8). The canonical projection $\pi : G \to G/\Gamma$ is a local homeomorphism. Combined with second-countability, this implies the existence of an open cover $(U_n)_{n\in\mathbb{N}}$ of G such that $\pi : U_n \to \pi(U_n)$ is a homeomorphism for every $n \in \mathbb{N}$. Let $F_1 = U_1$ and define inductively $F_n = U_n \backslash (n \cap \pi^{-1}\pi(\bigcup_{k<n} U_k))$. Then $F := \bigcup_{n\in\mathbb{N}} F_n$ is a fundamental domain for G/Γ in G. □

Integration over G/Γ now reduces to integration over G as follows.

Proposition 4.10. Let G be a locally compact Hausdorff, second-countable group with left Haar measure μ and let Γ be a discrete subgroup of G. Assume that $\Delta_G|_\Gamma \equiv \Delta_\Gamma$. Further, let F be a fundamental domain for G/Γ in G. Then a Haar measure ξ on G/Γ satisfying (W) exists and is associated to the following functional: $\lambda : C_c(G/\Gamma) \to \mathbb{C},\ f \mapsto \int_F f(g\Gamma)\ \mu(g)$, i.e.

$$\int_{G/\Gamma} f(g\Gamma)\ \xi(g\Gamma) = \int_F f(g\Gamma)\ \mu(g) \quad \text{for all} \quad f \in C_c(G/\Gamma).$$

Proof. The functional λ is positive and linear; the associated Radon measure ξ on G/Γ is checked to be non-zero on non-empty open sets and G-invariant. Hence ξ is a Haar measure on G/Γ. To prove that it satisfies (W), note that changing F by a set of measure zero, we may assume that F is a strict fundamental domain. Then G is a countable disjoint union $G = \bigsqcup_{\gamma\in\Gamma} F\gamma$ and hence we have for all $f \in C_c(G)$:

$$\begin{aligned}\int_G f(g)\ \mu(g) &= \sum_{\gamma\in\Gamma} \int_{F\gamma} f(g)\ \mu(g) = \sum_{\gamma\in\Gamma} \int_F f(g\gamma)\ \mu(g) = \int_\Gamma \int_F f(g\gamma)\ \mu(g)\ \nu(\gamma)\\ &= \int_F \int_\Gamma f(g\gamma)\ \nu(\gamma)\ \mu(g) = \int_F f_\Gamma(g\Gamma)\ \mu(g) = \int_{G/\Gamma} f_\Gamma(g\Gamma)\ \xi(g\Gamma)\\ &= \int_{G/\Gamma} \int_\Gamma f(g\gamma)\ \nu(\gamma)\ \xi(g\Gamma).\end{aligned}$$

where the second equality follows from the assumption $\Delta_G|_\Gamma \equiv \Delta_\Gamma \equiv 1$, and the the application of Fubini's Theorem 1.10 is valid since G is σ-finite as a locally compact, second-countable space and Γ is σ-finite as it is countable. □

Remark 4.11. Retain the notation of Proposition 4.10. The assumption $\Delta_G|_\Gamma \equiv \Delta_\Gamma$ is not automatic. For instance, the subgroup

$$\Gamma := \left\{ \begin{pmatrix} e^t & \\ & e^{-t} \end{pmatrix} \middle|\, t \in \mathbb{Z} \right\}$$

of the group P of Example 3.8 is isomorphic to $\mathbb{Z}$ and discrete in P. However, for $\gamma = \operatorname{diag}(e^t, e^{-t}) \in \Gamma\backslash\{\mathrm{Id}\}$ we have $\Delta_P(\gamma) = e^{-2t} \neq 1 \equiv \Delta_\Gamma$ by Example 3.8.

We end this section with the following result about groups containing lattices: Recall that if G is a locally compact Hausdorff group and Γ is a discrete subgroup of G then Γ is a *lattice in* G if G/Γ supports a finite Haar measure.

Proposition 4.12. Let G be a locally compact Hausdorff group. If G contains a lattice, then G is unimodular.

Proof. Let Γ be a lattice in G. Since G/Γ supports a finite Haar measure ξ, Theorem 4.2 implies that $\Delta_G|_\Gamma \equiv \Delta_\Gamma \equiv 1$ and hence $\ker \Delta_G \supseteq \Gamma$. Therefore, Δ_G factors through $G \to G/\Gamma$ via $\widetilde{\Delta}_G : G/\Gamma \to (\mathbb{R}^*_{\geq 0}, \cdot)$. Then $(\widetilde{\Delta}_G)_*\xi$ is a non-zero, finite measure on $\mathbb{R}^*_{\geq 0}$ which is invariant under the image of Δ_G. This forces $\Delta_G \equiv 1$. □

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