A CENTURY OF TAUBERIAN THEORY

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Abstract.

A narrow path is cut through the jungle of results which started with Tauber's "corrected converse" of Abel's theorem that if $\sum_{n=0}^{\infty} a_n = \ell$, then $\sum_{n=0}^{\infty} a_n x^n \to \ell$ as $x \to 1-$.

Just over a century ago, in 1897, Tauber proved the following "corrected converse" of Abel's theorem:

Theorem T. *If* $\sum_{n=0}^{\infty} a_n x^n \to \ell$ *as* $x \to 1-$, *and*

$$(\mathrm{T}_0) \qquad na_n = o(1),$$

then $\sum_{n=0}^{\infty} a_n = \ell$.

Subsequently Hardy and Littlewood proved numerous other such converse theorems, and they coined the term *Tauberian* to describe them.

In summability language Theorem T can be expressed as:

If $\sum_{n=0}^{\infty} a_n = \ell\,(A)$, *where* A *denotes the Abel summability method, and if the Tauberian condition* (T_0) *holds, then* $\sum_{n=0}^{\infty} a_n = \ell$.

The simplest example of an Abel summable series that is not convergent is given by $a_n := (-1)^n$, for which $\sum_{n=0}^{\infty} a_n = \frac{1}{2}\,(A)$.

Tauber's innocent looking theorem was the start of a veritable Tauberian jungle of results which Korevaar, in a recent book, made a very worthwhile effort to organize and present in a coherent manner. The book's 483 pages are densely packed and there are around 800 references. Rather than attempting the impossible task of giving such a comprehensive description of the jungle in the course of a short talk, I will cut a reasonably narrow path through part of it, touching on some of the key results.

In 1914 Hardy and Littlewood proved the following generalization of Theorem T in which the strong "two-sided" Tauberian condition (T_0) is replaced by the much weaker "one-sided" condition (T_1):

Theorem H-L. *If* $\sum_{n=0}^{\infty} a_n x^n \to \ell$ *as* $x \to 1-$*, and*

(T_1) $na_n \leq C$*, a positive constant,*

then $\sum_{n=0}^{\infty} a_n = \ell$.

Note that by changing sign throughout, the Tauberian condition (T_1) could be expressed as $na_n \geq -C$.

An interesting, and non-trivial, illustration of the potency of Theorem H-L, is a proof that the series

$$\sum_{n=1}^{\infty} \frac{1}{n^z},$$

which is absolutely convergent and defines the Riemann zeta function $\zeta(z)$ when $\Re z > 1$, is not Abel summable on the line $z = 1 + it$. This amounts to observing that

$$\sum_{n=1}^{\infty} \frac{1}{n^{1+it}}$$

cannot be Abel summable, for if it were Theorem H-L (or even a weaker two-sided version of it) would imply that the series is actually convergent, which it cannot be since Hardy has shown that, for fixed $t \neq 0$,

$$\sum_{n=1}^{N} \frac{1}{n^{1+it}} = \frac{i}{tN^{it}} + \ell + o(1) \text{ as } N \to \infty,$$

where ℓ is finite and independent of N. In fact ℓ turns out to be $\zeta(1+it)$. □

Karamata simplified Hardy and Littlewood's proof of Theorem H-L in 1930, and in 1952 Wielandt elegantly modified Karamata's proof as follows:

Suppose, without loss in generality, that $\sum_{n=0}^{\infty} a_n x^n \to 0$ as $x \to 1-$. Let $\mathfrak{F}$ be the linear space of real functions f for which

$$\sum_{n=0}^{\infty} a_n f(x^n) \to 0 \text{ as } x \to 1- .$$

Then every real polynomial p with $p(0) = 0$ is in $\mathfrak{F}$. Let $g := \chi_{[1/2,1]}$, the characteristic function of $[1/2, 1]$. Given $\varepsilon > 0$, there are real polynomials p_1, p_2 with $p_1(0) = p_2(0) = 0$ and $p_1(1) = p_2(1)$ such that $p_1(x) \le g(x) \le p_2(x)$ for $0 \le x \le 1$, and

$$\int_0^1 \frac{p_2(t) - p_1(t)}{t(1-t)}\, dt < \frac{\varepsilon}{C}.$$

Then, by (T_1),

$$\sum_{n=0}^{\infty} a_n g(x^n) - \sum_{n=0}^{\infty} a_n p_1(x^n) \le C \sum_{n=1}^{\infty} \frac{p_2(x^n) - p_1(x^n)}{n}$$
$$= C \sum_{n=1}^{\infty} \frac{x^n(1-x^n)}{n} q(x^n) \le C\,(1-x) \sum_{n=0}^{\infty} x^n q(x^n),$$

where

$$q(x) := \frac{p_2(x) - p_1(x)}{x(1-x)} =: \sum_{k=0}^{m} b_k x^k.$$

Further, as $x \to 1-$,

$$(1-x)\sum_{n=0}^{\infty} x^n q(x^n) = \sum_{k=0}^{m} b_k \frac{1-x}{1-x^{k+1}} \to \sum_{k=0}^{m} \frac{b_k}{k+1} = \int_0^1 q(t)\,dt < \frac{\varepsilon}{C}.$$

Hence

$$\limsup_{x\to 1-} \sum_{n=0}^{\infty} a_n g(x^n) < \varepsilon,$$

and likewise

$$\liminf_{x\to 1-} \sum_{n=0}^{\infty} a_n g(x^n) > -\varepsilon.$$

It follows that $g \in \mathfrak{F}$, and therefore, for $N = \lfloor -\log 2/\log x \rfloor$,

$$\sum_{n=0}^{\infty} a_n g(x^n) = \sum_{n=0}^{N} a_n \to 0 \text{ as } x \to 1-. \qquad \square$$

Another proof of Theorem H-L is by means of Wiener's powerful Tauberian theorem involving Fourier transforms which he published in 1932:

Theorem W. *If* $K \in L(-\infty,\infty)$, $\phi \in L^\infty(-\infty,\infty)$,

$$\int_{-\infty}^{\infty} e^{-itx} K(t)\,dt \neq 0 \quad \forall x \in (-\infty,\infty), \ \textit{and}$$

$$\int_{-\infty}^{\infty} K(x-t)\phi(t)\,dt = o(1) \ \textit{as } x \to \infty,$$

then, $\forall H \in L(-\infty,\infty)$,

$$\int_{-\infty}^{\infty} H(x-t)\phi(t)\,dt = o(1) \ \textit{as } x \to \infty. \tag{1}$$

To prove Theorem H-L with $\ell = 0$ by means of Theorem W, let

$$s(x) := \sum_{n \le x} a_n, \quad \text{and} \quad F(x) := \sum_{n=0}^{\infty} a_n x^n.$$

Then, by hypothesis, $F(x) = o(1)$ as $x \to 1-$, and it follows (fairly easily) from this and (T_1) that $s(x) = O(1)$, and hence that, for $t > 0$,

$$F(e^{-t}) = \sum_{n=0}^{\infty} a_n e^{-nt} = \int_0^{\infty} e^{-tx}\, ds(x) = t \int_0^{\infty} e^{-tx} s(x)\, dx.$$

Now take $\phi(x) := s(e^x)$ and $K(x) := \exp(-x - e^{-x})$. Then

$$\int_{-\infty}^{\infty} K(x-t)\phi(t)\, dt = F\big(\exp(-e^{-x})\big) = o(1) \text{ as } x \to \infty,$$

and, $\forall x \in (-\infty, \infty)$,

$$\int_{-\infty}^{\infty} e^{-itx} K(t)\, dt = \int_0^{\infty} u^{ix} e^{-u}\, du = \Gamma(1 + ix) \neq 0.$$

Further, $\phi(x) = O(1)$, and it follows from (T_1) that, given $\delta > 0$, $\exists x_0$ such that

$$\phi(y) - \phi(x) \le 2\delta \text{ for } x_0 \le x \le y \le x + \delta.$$

Taking $H := \delta^{-1}\chi_{[0,\delta]}$ and then $H := \delta^{-1}\chi_{[-\delta,0]}$ in (1), we obtain respectively

$$\limsup_{x\to\infty} \phi(x) \le 2\delta \quad \text{and} \quad \liminf_{x\to\infty} \phi(x) \ge -2\delta,$$

from which it follows that $\phi(x) \to 0$ and hence that $s(x) \to 0$ as $x \to \infty$. $\square$

Wiener's theorem yields Tauberian theorems for many standard summability methods.

Karamata proved various Tauberian theorems, the most famous being the following one about Laplace transforms which he proved in 1931:

Theorem K. *Let A be a non-decreasing, unbounded function on $[0,\infty)$ with $A(0) \geq 0$, and let L be a slowly varying function (i.e., $\forall t > 0$, $L(xt)/L(x) \to 1$ as $x \to \infty$). Then, for $\sigma \geq 0$,*

$$B(x) := \int_0^\infty e^{-t/x} dA(t) \sim x^\sigma L(x) \text{ as } x \to \infty$$

(i.e., B is regularly varying with index σ) if and only if

$$A(x) \sim \frac{x^\sigma L(x)}{\Gamma(1+\sigma)} \text{ as } x \to \infty.$$

From this theorem Karamata derived:

Theorem $\mathbf{K_1}$. *Let A be a non-decreasing, unbounded and regularly varying function on $[0,\infty)$ with $A(0) \geq 0$, and let the function s be continuous and bounded below on $[0,\infty)$. If*

$$\int_0^\infty e^{-yt} s(t)\, dA(t) \sim \ell \int_0^\infty e^{-yt}\, dA(t) \text{ as } y \to 0+\,, \tag{2}$$

then

$$\frac{1}{A(x)} \int_0^x s(t)\, dA(t) \to \ell \text{ as } x \to \infty. \tag{3}$$

This is also a Tauberian theorem since (3) $\Rightarrow$ (2) without the one-sided boundedness condition on s. It follows from a theorem established by Korenblum in 1955 that the condition in Theorem K_1 that A be regularly varying can be replaced by the weaker condition

$$\frac{A(y)}{A(x)} \to 1 \text{ when } \frac{y}{x} \to 1,\ y > x \to \infty, \tag{4}$$

(i.e., $\log A(x)$ is slowly oscillating). From this extension of Theorem K_1, I was able to prove:

Theorem DB. *Let A be a non-decreasing, unbounded and function on $[0,\infty)$ with $A(0) \geq 0$, and let the function s be continuous $[0,\infty)$. If (2) and (4) are satisfied, and in addition*

$$(5) \qquad \liminf\{s(y) - s(x)\} \geq 0 \text{ when } \frac{y}{x} \to 1,\ y > x \to \infty,$$

then $s(x) \to \ell$ as $x \to \infty$.

The proof uses a variant of a method developed by Vijayaraghavan in 1926 to first deduce that $s(x)$ is bounded.

Theorem DB can be specialized by taking

$$A(x) := n \text{ for } n \le x < n+1,\ n = 0, 1, \dots ,\ \text{and}$$

$$s(n) := s_n := \sum_{k=0}^{n} a_k,$$

to obtain as a corollary the following result which Schmidt established in 1925:

Corollary. *If*

$$(6) \qquad \sum_{n=0}^{\infty} s_n e^{-ny} \sim \ell \sum_{n=0}^{\infty} e^{-ny} = \frac{\ell}{1-e^{-y}} \ \textit{as } y \to 0+$$

and

$$(7) \qquad \liminf (s_m - s_n) \ge 0 \ \textit{when } \frac{m}{n} \to 1,\ m > n \to \infty$$

(i.e., s_n is slowly decreasing), then $s_n \to \ell$.

Note that (6) is equivalent to

$$(1-x) \sum_{n=0}^{\infty} s_n x^n = \sum_{n=0}^{\infty} a_n x^n \to \ell \text{ as } x \to 1-,$$

and that $na_n > -C \Rightarrow$ (7), so that the Corollary generalizes Theorem H-L.

Another classical Tauberian result concerns the Cesàro method C_α, $\alpha > -1$, and the Borel method B defined by:

$$\sum_{n=0}^{\infty} a_n = \ell\,(C_\alpha),$$

$$\text{if } \frac{1}{\binom{n+\alpha}{n}} \sum_{k=0}^{n} \binom{k+\alpha}{k} a_{n-k} \to \ell \text{ as } n \to \infty;$$

$$\sum_{n=0}^{\infty} a_n = \ell\,(B), \text{ or } s_n \to \ell\,(B),$$

$$\text{if } e^{-x} \sum_{n=0}^{\infty} \frac{s_n x^n}{n!} \to \ell \text{ as } x \to \infty, \qquad \text{where } s_n := \sum_{k=0}^{n} a_k.$$

Theorem B. *If* $\sum_{n=0}^{\infty} a_n = \ell\,(B)$, *and*

(T$_2$) $\qquad \frac{\sqrt{n}a_n}{n^r} \leq C,\ r \geq 0,$

then $\sum_{n=0}^{\infty} a_n = \ell(C_{2r})$.

In 1960 Rajagopal proved a version of this result with a weaker Tauberian condition than (T$_2$). The case $r = 0$ of the result with (T$_2$) replaced by the stronger two-sided condition $\sqrt{n}a_n = O(1)$ was proved by Hardy and Littlewood in 1916. In 1925 Schmidt showed in the case $r = 0$ that (T$_2$) can be relaxed to

$$\liminf(s_m - s_n) \geq 0 \ \textit{when}\ 0 < \sqrt{m} - \sqrt{n} \to 0,\ n \to \infty.$$

Recently Kratz and I established a quantatitive version of Vijayaraghavan's classical 1926 result and used it to give a short proof of the case $r = 0$ of Theorem B. It is worth noting that though summability C_0 (i.e., convergence) implies summability B, summability C_α with $\alpha > 0$ does not in general imply summability B.

Various Tauberian theorems have been used in assorted proofs of the prime number theorem. A particularly interesting one is the following one proved in 1931 by Ikehara, a student and colleague of Wiener's:

Theorem I-W. *Suppose that the function F has the following properties:*

(i) *For $\Re z > 1$, $F(z) = \int_0^\infty e^{-zt} A(t)\, dt$, where A is a non-decreasing function with $A(0) \geq 0$.*

(ii) *For $\Re z > 1$, $z \neq 1$, $F(z) = G(z) + \dfrac{1}{z-1}$, where $G(z)$ is continuous on the half-plane $\Re z \geq 1$.*

Then $e^{-t} A(t) \to 1$ as $t \to \infty$.

The prime number theorem can be proved with the aid of Theorem I-W as follows: Let

$$A(t) := \psi(e^t), \text{ where } \psi(x) := \sum_{p^n \leq x} \log p.$$

The $p's$ in the sum defining the Chebyshev function ψ are the odd primes, and it is known that the prime number theorem, viz.,

$$\pi(x) := \sum_{p \leq x} \sim \frac{x}{\log x} \text{ as } x \to \infty,$$

is equivalent to $\psi(x) \sim x$ as $x \to \infty$.

For $\Re z > 1$, we have that

$$F(z) = \int_0^\infty e^{-zt} A(t)\, dt = \int_1^\infty u^{-z-1} \psi(u)\, du = -\frac{\zeta'(z)}{z\zeta(z)} = G(z) + \frac{1}{z-1},$$

the function G satisfying the requirements of Theorem I-W since the Riemann zeta function $\zeta(z)$ has no zeros in the half plane $\Re z \geq 1$ and is holomorphic in the whole plane, except for a simple pole at $z = 1$ with residue 1. Hence, by Theorem I-W, $e^{-t}\psi(e^t) \to 1$ as $t \to \infty$ and so $\psi(x) \sim x$ as $x \to \infty$. $\square$

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