MAXIMALITY OF SUMS OF TWO MAXIMAL MONOTONE OPERATORS IN GENERAL BANACH SPACE

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Abstract. We combine methods from convex analysis, based on a function of Simon Fitzpatrick, with a fine recent idea due to Voisei, to prove maximality of the sum of two maximal monotone operators in Banach space under various natural transversality conditions.

1. Introduction and Preliminaries

The results of this paper, especially Theorem 9, marry recent work by Voisei [12] with additional convex analysis described in [1, 2], see also [4, §5.1] or [3, §8.3], to provide an accessible short proof of the maximality of the sum of two maximal monotone operators under domain conditions such as $D(B) \cap \mathrm{core} D(A) \neq \emptyset$, while either $D(B)$ is closed and convex or $\mathrm{core}\,\mathrm{conv} D(B) \neq \emptyset$.

Recall that the *domain* of an extended-valued convex function, $\mathrm{dom}\,(f)$, is the set of points with value less than $+\infty$, and that a point s is in the *core* of a set S (denoted by $s \in \mathrm{core}\ S$) provided that s lies in S and $X = \bigcup_{\lambda>0} \lambda(S - s)$. Recall that $x^* \in X^*$ is a *subgradient* of $f : X \to (-\infty, +\infty]$ at $x \in \mathrm{dom}\, f$ provided that $f(y) - f(x) \geq \langle x^*, y - x\rangle$. The set of subgradients of f at x is the *subderivative* or *subdifferential* of f at x and is denoted $\partial f(x)$.

We shall need the *indicator* function $\iota_C(x)$ which is zero for x in C and $+\infty$ otherwise, the *Fenchel conjugate* $f^*(x^*) := \sup_x\{\langle x, x^*\rangle - f(x)\}$ and the *infimal convolution* $f \Box g(x) := \inf\{f(y) + g(z) \colon x = y + z\}$. The central examples of the *normal cone* to C at x and the *distance function* d_C, are covered by $N_C(x) = \partial\iota_C$ and $d_C = \iota_C \Box \|\cdot\|$.

We say a multifunction $T : X \mapsto 2^{X^*}$ is *monotone* provided that for any $x, y \in X$, $x^* \in T(x)$ and $y^* \in T(y)$,

$$\langle y^* - x^*, y - x\rangle \geq 0,$$

and we say that T is *maximal monotone* if its graph is not properly included in any other monotone graph. The subdifferential of a convex lower semicontinuous (lsc) function on a Banach space is a fine example of a maximal monotone multifunction (see [3, 4, 10] wherein other notation and usage may be also followed up).

2. Representative Functions

For any monotone mapping T, we associate the *Fitzpatrick function* introduced by Simon Fitzpatrick in [6] but then neglected for many years until re-popularized

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in papers by Penot [8], Buracik-Svaiter [5], and others. Some more of the related history may be found in [2]. *Fitzpatrick's function* is

$$\mathcal{F}_T(x,x^*) := \sup\{\langle x,y^*\rangle + \langle x^*,y\rangle - \langle y,y^*\rangle : y^* \in T(y), y \in \operatorname{dom} T\},$$

which is clearly lower semicontinuous and convex as an affine supremum.

Proposition 1. [6, 4] *For a maximal monotone operator* T

$$\mathcal{F}_T(x,x^*) \geq \langle x,x^*\rangle$$

with equality if and only if $x^* \in T(x)$.

Correspondingly *Penot's function* is given as the (closed) convexification

$$\mathcal{P}_T(x,x^*) := \inf\left\{\sum_{i=1}^N \lambda_i\langle x_i,x_i^*\rangle \colon \sum_i \lambda_i(x_i,x_i^*) = (x,x^*), x_i^* \in T(x_i), \sum \lambda_i = 1, \lambda_i \geq 0\right\}$$

It is easy to see that $\mathcal{P}_T$ is convex and that, with the appropriate ordering of variables x and x^* (and the conjugate restricted to $X \times X^*$) we have

$$\mathcal{P}_T^* = \mathcal{F}_T \qquad \text{while} \qquad \mathcal{F}_T^* = \mathcal{P}_T^{**} = \overline{\mathcal{P}}_T,$$

where $\overline{\mathcal{P}}_T$ is the lower-semicontinuous hull of $\mathcal{P}_T$. Note that

More generally, we say that a lower-semicontinuous convex function $\mathcal{H}_T$ *represents* a monotone operator T if

$$\mathcal{H}_T(x,x^*) \geq \langle x,x^*\rangle$$

with equality when $x^* \in T(x)$. We say a representative is *exact* if $\mathcal{H}_T(x,x^*) = \langle x,x^*\rangle$ exactly on the graph of T. Now we may check that:

Proposition 2. ([2, 8]) *Let* T *be monotone on a Banach space* X. *Then*

i.) *Penot's function* $\overline{\mathcal{P}}_T$ *represents* T.
ii.) *If* $\mathcal{H}_T$ *represents* T, *then* $\mathcal{H}_T \leq \overline{\mathcal{P}}_T$ *pointwise.*
iii.) *If* T *is maximal then* $\mathcal{F}_T \leq \mathcal{H}_T \leq \overline{\mathcal{P}}_T$.
iv.) $\mathcal{F}_T(x,x^*) \leq \langle x,x^*\rangle$ *iff* (x,x^*) *is monotonically related to the graph of* T.
v.) *Suppose* $\mathcal{F}_T$ *represents* T. *Then* $\mathcal{F}_T(x,x^*) = \langle x,x^*\rangle$ *iff* $\overline{\mathcal{P}}_T(x,x^*) = \langle x,x^*\rangle$.

Proof. (i.) is an easy computation performed in [2, 8]. (ii.) a direct consequence of $\overline{\mathcal{P}}_T = (c_T)^{**}$ and that $\mathcal{H}_T(x,x^*) \leq c_T$, where $c_T(x,x^*) := \langle x,x^*\rangle + \iota_{\mathrm{Gr}(T)}(x,x^*)$. (iii.) The lefthand inequality is established in [6, 8]. (iv.) is a direct computation. (v.) By (iv.)—as $\mathcal{F}_T$ is representative—we need only show the 'if'. We observe that if $\mathcal{F}_T(x,x^*) = \langle x,x^*\rangle$, then minorizing $\mathcal{F}_T(x+t(y-x), x^*+t(y^*-x^*))$ by $\langle x+t(y-x), x^*+t(y^*-x^*)\rangle$ we have

$$\mathcal{F}_T(y,y^*) - \mathcal{F}_T(x,x^*) \geq d^+\mathcal{F}_T((x,x^*);(y-x,y^*-x^*)) \geq \langle x,y^*-x^*\rangle + \langle y-x,x^*\rangle$$

for all y,y^*. This shows $(x^*,x) \in \partial\mathcal{F}_T(x,x^*)$. Equivalently,

$$2\langle x,x^*\rangle = \mathcal{F}_T(x,x^*) + \mathcal{F}_T^*(x,x^*) = \mathcal{F}_T(x,x^*) + \overline{\mathcal{P}}_T(x,x^*)$$

and so $\overline{\mathcal{P}}_T(x,x^*) = \langle x,x^*\rangle$. □

Note that $\mathcal{F}_T$ need not represent T if T is not maximal. The situation is however ameliorated when $T = A+B$ is the sum of maximal monotone operators satisfying

$$0 \in \operatorname{core}\{\operatorname{conv}D(A) - \operatorname{conv}D(B)\}. \tag{1}$$

We next define two partial infimal convolutions:

$$\mathcal{V}_{A,B}(x,x^*) := \inf\{\mathcal{F}_A(x,u^*) + \mathcal{F}_B(x,v^*)\colon u^*+v^*=x^*\},$$

and

$$\mathcal{W}_{A,B}(x,x^*) := \inf\{\mathcal{P}_A(x,u^*) + \mathcal{P}_B(x,v^*)\colon u^*+v^*=x^*\}.$$

The first result is very interesting in its own right it is a lovely observation first exploited by Voisei:

Theorem 3. (**Partial Convolution**, [12].) *Suppose A and B are maximal monotone and satisfy the transversality condition (1). Then $\mathcal{V}_{A,B}(x,x^*) = \mathcal{W}^*_{A,B}(x,x^*)$ is norm-weak-star lower-semicontinuous and is attained when finite.*

In consequence

$$\mathcal{V}_{A,B}(x,x^*) \geq \langle x,x^*\rangle$$

with equality if and only if $x^* \in (A+B)(x)$. In particular, $\mathcal{V}_{A,B}$ represents $A+B$ and so $\mathcal{V}_{A,B} \leq \overline{\mathcal{P}}_{A+B}$.

Proof. The argument—based on a conjugate formula of Penot [8, Prop. 13]—as in Vosei [12] and in [2, §5], or a direct Lagrangian calculation, shows $\mathcal{V}_{A,B}(x,x^*) = \mathcal{W}^*_{A,B}(x,x^*)$ and is attained when finite. The rest follows since $\mathcal{P}^*_A = \mathcal{F}_A$ and $\mathcal{P}^*_B = \mathcal{F}_B$ have the representative properties of Proposition 1.

Indeed, $\mathcal{V}_{A,B}(x,x^*) \geq \langle x,x^*\rangle$ follows directly from the definition of convolution as does $\mathcal{V}_{A,B}(x,x^*) = \langle x,x^*\rangle$ when $x^* \in (A+B)(x)$. Finally if $\mathcal{V}_{A,B}(x,x^*) = \langle x,x^*\rangle$, we let $x^* = u^*+v^*$ be the attaining values, as assured by the conjugacy formula. Then

$$0 = \mathcal{V}_{A,B}(x,x^*) - \langle x,x^*\rangle = \{\mathcal{F}_A(x,u^*) - \langle x,u^*\rangle\} + \{\mathcal{F}_B(x,v^*) - \langle x,v^*\rangle\}.$$

As the bracketed terms are non-negative we deduce that they are both zero and so $u^* \in A(x)$, $v^* \in B(x)$; and we are done. □

Let us say that a monotone operator T is *almost maximal* if

$$\mathcal{F}_T(x,x^*) \geq \langle x,x^*\rangle$$

for all $x \in X, x^* \in X^*$. This is to say that $\mathcal{F}_T$ represents T. The name is justified since Proposition 1 assures that every maximal monotone operator is almost maximal. Also, if T is maximal and $\overline{\mathrm{Gr}\, S} = \mathrm{Gr}\,(T)$ then S is almost maximal.

A nice consequence of the definition is:

Corollary 4. *Suppose T is almost maximal monotone. Then a closed convex function $\mathcal{H}$ represents T if and only if*

$$\mathcal{F}_T \leq \mathcal{H} \leq \overline{\mathcal{P}}_T.$$

Proof. By Proposition 2 we need only show that each representative function $\mathcal{H}$ is minorized by $\mathcal{F}_T$. Suppose we show that $\mathcal{H}^*$ is also a representative. Then $\mathcal{H}^* \leq \overline{\mathcal{P}}_T \Rightarrow \mathcal{H} \geq \mathcal{F}_T$ as required. To show $\mathcal{H}^*$ is a representative, since $\mathcal{H}^* \geq \mathcal{F}_T$ we need only show that $\mathcal{H}^*(x,x^*) = \langle x,x^*\rangle$ on $\mathrm{Gr}\,(T)$. This is the case by the argument of Proposition 2 v.) applied to $\mathcal{H}$. □

We offer further justification of the term in the final preliminary result.

Proposition 5. *Suppose that A and B are maximal monotone operators on a Banach space X and that the transversality condition (1) holds. Then $A+B$ is maximal as soon as it is almost maximal.*

Proof. Suppose that (x,x^*) is monotonically related to the graph of $A+B$. Then $\mathcal{F}_{A+B}(x,x^*) \le \langle x,x^*\rangle$. As $A+B$ is almost maximal we deduce that $\mathcal{F}_{A+B}(x,x^*) = \langle x,x^*\rangle$ and so, by part v.) of Proposition 2, we see that $\mathcal{P}_{A+B}(x,x^*) = \langle x,x^*\rangle$. Consequently, an appeal to Theorem 3 shows $\mathcal{V}_{A,B}(x,x^*) = \langle x,x^*\rangle$ and so that $x^* \in (A+B)(x)$, which completes the proof. □.

The next corollary shows that topologically $A+B$ is close-to-maximal. By 'bdw^*' we denote weak*-convergence for bounded nets (and hence include all weak*-convergent sequences).

Corollary 6. (**Graph Closedness.**) *Suppose that A and B are maximal monotone in Banach space and that the transversality condition (1) holds. Then $A+B$ has a $\|\cdot\|\times bdw^*$ closed graph and consequently has weak*-closed convex images.*

Proof. Clearly,

$$\{(x,x^*)\colon \mathcal{V}_{A,B}(x,x^*) - \langle x,x^*\rangle \le 0\} = \mathrm{Gr}(A+B)$$

is $\|\cdot\|\times bdw^*$ closed in the product space, since $\mathcal{V}_{A,B}$ is $\|\cdot\|\times bdw^*$ lower-semicontinuous while the bilinear form is $\|\cdot\|\times bdw^*$ continuous. □

Observe that the graph of a maximal monotone operator need not be $\|\dot{\|}\times w^*$ closed (or even bw^*) already for a subgradient [4, Example 5.2.31]. The w^* closure of $(A+B)(x)$ was first proven in [11].

3. Our Main Results

We first provide two useful criteria for almost maximality.

Proposition 7. *Assume that S is monotone and that either S is surjective or has full domain. Then S is almost maximal.*

Proof. Fix $x^* \in X^*$ and $x \in X$. Suppose S is surjective and write $x^* = s^* \in S(s)$. Then, by definition

$$\mathcal{F}_S(x,x^*) \ge \langle x,s^*\rangle + \langle s,x^*\rangle - \langle s,s^*\rangle = \langle x,x^*\rangle.$$

The other case is similar. □

We denote the *algebraic closure* of a set at $x \in C$ by $C^{alg}(x) := \{d\colon t_n d + (1-t_n)x \in C, \exists t_n < 1, t_n \to 1\}$. We write $C^{alg} := \bigcap_{x\in C} C^{alg}(x)$.

Proposition 8. *Assume that A and B are maximal monotone and that (1) holds. Assume also that*

$$\overline{\mathrm{conv}}\, D(A) \cap \overline{\mathrm{conv}}\, D(B) = \overline{D(A)\cap D(B)}^{alg}. \tag{2}$$

Then $A+B$ is almost maximal.

Proof. Assume that $0 \in D(A)\cap D(B)$. Let $U := \overline{\mathrm{conv}}\, D(A)$ and $V := \overline{\mathrm{conv}}\, D(B)$. Note also that (1) implies $N_U + N_V = N_{U\cap V}$. Much as in [12] we argue by maximality that $A = A + N_U$ and $B = B + N_V$. Thus, using (2) shows

$$A+B = (A+B) + N_{\overline{D(A+B)}}.$$

Now suppose that $\mathcal{F}_{A+B}(x,x^*) \le \langle x,x^*\rangle$. This implies that for every $n^* \in N_{\overline{D(A+B)}}(y)$ and $s^* \in (A+B)(y)$ we have

$$\langle s^* + tn^* - x^*, y - x\rangle \ge 0,$$

for all $t > 0$. Hence $0 \in N_{\overline{D(A+B)}}(x)$ by maximality of the normal cone.

Thus, $x \in \overline{D(A+B)}$. Now (2) implies that x lies in $\overline{D(A+B)}^{alg}$. Thus we can select $z^* \in (A+B)(\alpha\, x)$, for $0 < \alpha < 1$. Then, by convexity and by definition

$$\begin{aligned}\alpha\,\mathcal{F}_{A+B}(x,x^*) &= \alpha\,\mathcal{F}_{A+B}(x,x^*) + (1-\alpha)\mathcal{F}_{A+B}(0,0)\\ &\geq \mathcal{F}_{A+B}(\alpha\,x,\alpha\,x^*)\\ &\geq \langle \alpha\,x, \alpha\,x^*\rangle + \langle \alpha\,x, z^*\rangle - \langle \alpha\,x, z^*\rangle = \alpha^2\,\langle x, x^*\rangle.\end{aligned}$$

Thus $\mathcal{F}_{A+B}(x,x^*) \geq \alpha\,\langle x, x^*\rangle$. Letting $\alpha \uparrow 1$ completes the proof. □

Theorem 9. (**Maximality of Sums, I**.) *Suppose that A and B are maximal monotone on a Banach space. Suppose also that either*

i.) *The set* $\operatorname{int} D(A) \cap \operatorname{int} D(B)$ *is nonempty; or*
ii.) $D(A) \cap \operatorname{int} D(B) \neq \emptyset$ *while* $D(A)$ *is closed and convex; or*
iii.) *Both* $D(A), D(B)$ *are closed and convex and* $0 \in \operatorname{core}\operatorname{conv}\{D(A) - D(B)\}$.

Then $A + B$ is maximal monotone.

Proof. Each of the hypotheses leads to $\overline{\operatorname{conv}}\, D(A) \cap \overline{\operatorname{conv}}\, D(B) \subset \overline{D(A+B)}^{alg}$, since $\overline{D(A)}$ is convex when $D(A)$ has nonempty interior, see [2, 9, 10]. More over the hypotheses of i.) and ii.) imply (1). Thus, Proposition 8 applies as then does Proposition 5. □

Part iii.) of Theorem 9 is the main result in [12]. In [13] corresponding results are given for compositions with closed convex domain—such also extend as above. A quite different proof of the Theorem 9 i.) follows from results in [2]:

Theorem 10. (**Maximality of Sums, II**.) *Suppose A and B are maximal monotone on a Banach space. Suppose also that* $\operatorname{core}\operatorname{conv} D(A) \cap \operatorname{core}\operatorname{conv} D(B) \neq \emptyset$.
Then $A + B$ is maximal monotone.

Proof. Suppose (x, x^*) is monotonically related to the graph of $A + B$. Let W be an arbitrary basic weak-star zero neighbourhood. Fix a finite dimensional subspace F of X containing both x and the vectors defining W. By translation we may assume that $0 \in \operatorname{core}\operatorname{conv} D(A) \cap \operatorname{core}\operatorname{conv} D(B) \neq \emptyset$. Hence, by the composition result in [2, §5], both A_F and B_F are maximal monotone; and $0 \in \operatorname{core}\operatorname{conv}\{D(A_F) - D(B_F)\}$. Thus, by the reflexive (or finite-dimensional) sum theorem

$$A_F + B_F = (A+B)_F$$

is maximal monotone. Since $x \in F$ and (x, x^*) is monotonically related to the graph of $(A + B)$ we observe that $(x, x^*|_F)$ is monotonically related to the graph of $(A + B)_F$. Hence by maximality $x^*|_F \in (A + B)_F(x)$. In consequence,

$$x^* \in (A+B)(x) + F^{\perp} \subset (A+B)(x) + W.$$

Since W is arbitrary, applying Corollary 6, we deduce that

$$x^* \in \overline{(A+B)(x)}^{*} = (A+B)(x),$$

by the Veronas' part of Corollary 6, and we are done. □

Remark 11. This argument works under the Brezis-Attouch condition (1) if we can ensure that for each finite dimensional space F there is a reflexive superspace, R, such that A_R and B_R are both maximal. This is the case, for example, if after translation $0 \in D(A) \cap \operatorname{core} D(B) \neq \emptyset$ and for each finite dimensional F there is a reflexive subspace R containing F such that $A|_R$ is maximal. Thus, any counter-example to the sum theorem has to have quite messy domains.

Another proof of this result can be obtained from Asplund's decomposition of a maximal monotone operator as the sum of cyclic and acyclic operators, as described in [2, §3].

Note also that maximality of T_Y and that of $T + \partial\iota_Y$ are equivalent for a closed subspace Y. □

Remark 12. Suppose f is lower-semicontinuous, proper and convex in Banach space. We observe—without appealing to maximality—that the representative function $(f \oplus f^*)(x, x^*) := f(x) + f^*(x^*)$ coincides with $\langle x, x^*\rangle$ exactly for x^* in $\partial f(x)$. As in Proposition 5, to prove ∂f maximal it thus suffices to show ∂f is almost maximal. Can this be done any more efficiently than directly proving maximality via an approximate mean-value theorem, as say in [4, Thm. 3.4.6]?

Remark 13. We can probably significantly improve the result in the case where one operator is a subgradient because the representative function $(f \oplus f^*)(x, x^*) := f(x) + f^*(x^*)$ is exact. We define

$$\mathcal{F}_{T,f}(x, x^*) := f(x) + \sup_{y^* \in T(y)} \{\langle x, y^*\rangle + \langle y, x^*\rangle - \langle y, y^*\rangle - f(y)\},$$

with conjugate

$$\mathcal{P}_{T,f}(x, x^*) := f(x) + \overline{\operatorname{conv}}_{y_i^* \in T(y_i)} \{\langle y_i, y_i^*\rangle - f(y_i)\},$$

and note that $\mathcal{F}_{T,0} = \mathcal{F}_T$ and $\mathcal{P}_{T,0} = \overline{\mathcal{P}}_T$, and as before $\mathcal{P}^*_{T,f} = \mathcal{F}_{T,f} \leq \mathcal{P}_{T,f}$. Likewise, $\mathcal{F}_{0,f} = f \oplus f^*$.

I conjecture that

$$\mathcal{F}_{T,f}(x, x^*) \leq \langle x, x^*\rangle \tag{3}$$

for (x, x^*) monotonically related to $\operatorname{Gr}(T + \partial f)$ as holds in the extreme cases. □

Theorem 14. (**Maximality of Sums, III**.) *Suppose T is maximal monotone and f is convex and closed. Suppose that (3) holds and that*

$$\operatorname{dom} f \cap \operatorname{core} \operatorname{conv} D(T) \tag{4}$$

is nonempty. Then $T + \partial f$ is maximal.

Proof. *We define*

$$\mathcal{V}_{T,f}(x, x^*) := f(x) + (\mathcal{F}_{T,f}(x, \cdot)\Box f^*)(x^*).$$

Then

$$\mathcal{V}_{T,f}(x, x^*) \leq \mathcal{P}_{T,f}(x, x^*).$$

By assumption $\mathcal{F}_{T,f}(x, x^*) \leq \langle x, x^*\rangle$ for (x, x^*) monotonically related to Gr($T+\partial f$) while, as in Proposition 3, $\mathcal{V}_{T,f}(x, x^*)$ is an exact representative for $T + \partial f$. Now *the weakened constraint qualification (4) still ensures* $\mathcal{F}_{T,f}$ *represents* $T + \partial f$ *as it implies that* $\operatorname{dom} f \cap \overline{\operatorname{conv}} D(T) \subseteq \overline{\operatorname{dom} f \cap \operatorname{conv} D(T)}^{alg}$. *Much as before*

$$\mathcal{F}_{T,f}(x, x^*) = \langle x, x^* \rangle \Rightarrow \mathcal{P}_{T,f}(x, x^*) = \langle x, x^* \rangle \Rightarrow \mathcal{V}_{T,f}(x, x^*) = \langle x, x^* \rangle,$$

and we are done. □

We finish with two especially nice consequences of part ii.) of Theorem 9.

Corollary 15. (**Normal Cones**.) *Suppose in an arbitrary Banach space that* T *is maximal monotone and* C *is closed and convex while* $C \cap \operatorname{int} D(T) \neq \emptyset$. *Then* $T + N_C$ *is maximal monotone.*

Proof. The maximality of $T + N_C$ is an immediate consequence of part ii.) of Theorem 9. □

Recall that a maximal monotone mapping T is maximal monotone locally [11], or *type (FPV)*, if for every open convex set V in X with $V \cap D(T) \neq \emptyset$ the following holds for every $x \in V$: $\langle y^* - x^*, y - x \rangle \geq 0$ for all $y^* \in T(y)$, and all $y \in V$ implies that $x^* \in T(x)$.

Corollary 16. *Suppose in an arbitrary Banach space that* T *is maximal monotone and* $D(T)$ *has nonempty interior, or is closed and convex, then* T *is of type (FPV).*

Proof. We argue as follows. Fix x, V and x^* as in the definition of (FPV). We may select a closed convex set C such that $x \in \operatorname{int} C \subset V$ and $\operatorname{int} D(T) \cap C \neq \emptyset$. It follows from Corollary 15 that $T + N_C$ is maximal. Let $y^* \in T(y), n^* \in N_C(y), y \in Y$ be given. Then $\langle y^* + n^* - x^*, y - x \rangle = \langle y^* - x^*, y - x \rangle + \langle n^*, y - x \rangle \geq 0$ since $x \in C$. By maximality $x^* \in T(x) + N_C(x) = T(x)$ since $x \in \operatorname{int} C$.

The same argument shows that every maximal monotone mapping with closed convex domain is type (FPV). □

The case $D(T) = X$ of Corollary 16 was first established in [7].

4. Final Remarks

As distinct from the reflexive case, we note that our arguments make no use of the duality map. Indeed, outside of reflexive space, the Rockafellar-Minty approach via surjectivity of $T + J$ is of little use, see [2, §6] and [4, §5.1]. That said, Theorem 9 certainly suggests that $A + B$ may well be maximal given only (1) and no auxiliary conditions.

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