

DIGITALLY-ASSISTED DISCOVERY AND PROOF

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ABSTRACT *I will argue that the mathematical community (appropriately defined) is facing a great challenge to re-evaluate the role of proof in light of the power of current computer systems, of modern mathematical computing packages and of the growing capacity to data-mine on the internet. With great challenges come great opportunities. I intend to illustrate the current challenges and opportunities for the learning and doing of mathematics. As the prospects for inductive mathematics blossom, the need to make sure that the role of proof is properly founded remains undiminished.*

INTRODUCTION

In this section I make some preliminary observations many of which have been fleshed out in (Borwein and Devlin, 2008), (Borwein and Bailey, 2008), and (Bailey et al, 2007). The core of my paper focuses on the changing nature of mathematical knowledge and in consequence asks the questions “Why do we wish to prove things?” and “How do we teach what and why to students?”

I am attracted to various notions of embodied cognition:

“the large human brain evolved over the past 1.7 million years to allow individuals to negotiate the growing complexities posed by human social living.” (Small, 2008, p. 113)

In consequence we find various modes of argument more palatable than others, and are more prone to make certain kinds of errors than others. Likewise, Steve Pinker’s observation about language as founded on

“...the ethereal notions of space, time, causation, possession, and goals that appear to make up a language of thought.” (Pinker, 2007, p. 83)

remain equally potent within mathematics.

To begin with let me briefly reprise what I mean by discovery, and by proof. The following attractive definition of discovery has the satisfactory consequence that a student can certainly discover results known to the teacher: *“In short, discovering a truth is coming to believe it in an independent, reliable, and rational way.”* (Giaquinto, 2007, p. 50)

A standard definition of proof from the Collin’s Dictionary of Mathematics follows.

PROOF, *n.* a sequence of statements, each of which is either validly derived from those preceding it or is an axiom or assumption, and the

final member of which, the *conclusion*, is the statement of which the truth is thereby established.

As a working definition of the field I offer

Mathematics, n. a group of subjects, including algebra, geometry, trigonometry and calculus, concerned with number, quantity, shape, and space, and their inter-relationships, applications, generalizations and abstractions.

We typically take for granted the distinction between induction and deduction and rarely discuss their roles with our students --- let alone the linguistically confusing fact that a proof by induction is a deduction.

Induction, n. any form of reasoning in which the conclusion, though supported by the premises, does not follow from them necessarily.

Deduction, n. a process of reasoning in which a conclusion follows necessarily from the premises presented, so the conclusion cannot be false if the premises are true.

Despite the convention identification of Mathematics with deductive reasoning Kurt Gödel in his 1951 Gibbs Lecture said

“If mathematics describes an objective world just like physics, there is no reason why inductive methods should not be applied in mathematics just the same as in physics.”

And this has been echoed or amplified by logicians as different as Quine and Chaitin.

Armed with these terms it remains to say what I connote by digital-assistance. I intend such as

- ◆ The use of Modern Mathematical Computer Packages (be they Symbolic, Numeric, Geometric, or Graphical)
- ◆ The use of More Specialist Packages or General Purpose Languages such as Fortran, C++, CPLEX, GAP, PARI, MAGMA, ...
- ◆ The use of Web Applications such as: Sloane’s Encyclopedia of Integer Sequences, the Inverse Symbolic Calculator, Fractal Explorer, Euclid in Java.¹
- ◆ The use of Web Databases including Google, MathSciNet, ArXiv, JSTOR, Wikipedia, MathWorld, Planet Math, DLMF, MacTutor, Amazon, and many more that are not always viewed as part of the palette.

All entail **data-mining** in various forms. As Franklin (Franklin 2005) argues that what Steinle has termed “exploratory experimentation” facilitated by “widening technology” as in pharmacology, astrophysics, biotechnology is

¹ All are available through the collection preserved at <http://ddrive.cs.dal.ca/~isc/portal>.

leading to a reassessment of what is viewed as a legitimate experiment in which a “local model” is not a prerequisite. Hendrik Sørensen cogently makes the case that experimental mathematics is following many of the same tracks.

“These aspects of exploratory experimentation and wide instrumentation originate from the philosophy of (natural) science and have not been much developed in the context of experimental mathematics. However, I claim that e.g. the importance of wide instrumentation for an exploratory approach to experiments that includes concept formation also pertain to mathematics.” (Sørensen, 2008)

In consequence the boundaries between mathematics and the natural sciences and between inductive and deductive reasoning are blurred and getting blurrier (Avigad 2008).

With these prefatory remarks made I turn to the three mathematical examples which form the heart of my paper. I leave it to the reader to decide in each case how much credence to put in the process I describe.

THREE EXAMPLES

A. DATA MINING: What’s that number? (1995 to 2008)

In 1995 or so Andrew Granville emailed me the number

$$\alpha := 1.433127426722312\dots$$

and challenged me to identify it (our online *Inverse calculator* was new in those days). I asked *Maple* for its continued fraction? It was

$$(*) \quad [1,2,3,4,5,6,7,8,9,10,11,\dots]$$

I reached for a good book on continued fractions and found the answer

$$\alpha = \frac{I_0(2)}{I_1(2)}$$

where I_0 and I_1 are *Bessel functions* of the first kind. (Actually I knew that all arithmetic continued fractions arise in such fashion, but as we shall see one now does not need to this).

In 2008 there are at least two or three other strategies:

- Given (*), type “arithmetic progression”, “continued fraction” into Google
- Type 1,4,3,3,1,2,7,4,2 into Sloane’s Encyclopaedia of Integer Sequences

I illustrate the results of each.

On October 15, 2008, on typing “arithmetic progression”, “continued fraction” into Google, the first three hits were

Continued Fraction Constant -- from Wolfram MathWorld

- 3 visits - 14/09/07 Perron (1954-57) discusses *continued fractions* having terms even more general than the *arithmetic progression* and relates them to various special functions. ...

mathworld.wolfram.com/ContinuedFractionConstant.html - 31k

HAKMEM -- CONTINUED FRACTIONS -- DRAFT, NOT YET PROOFED

The value of a *continued fraction* with partial quotients increasing in *arithmetic progression* is $I(2/D) A/D [A+D, A+2D, A+3D, \dots]$

www.inwap.com/pdp10/hbaker/hakmem/cf.html - 25k -

On simple continued fractions with partial quotients in arithmetic ...

0. This means that the sequence of partial quotients of the *continued fractions* under investigation consists of finitely many *arithmetic progressions* (with ...

www.springerlink.com/index/C0VXH713662G1815.pdf - by P Bundschuh – 1998

Moreover the *MathWorld* entry tells us that any arithmetic continued fraction is of a ratio of Bessel functions and refers to the second hit above.

$$[A + D, A + 2D, A + 3D, \dots] = \frac{I_{A/D}\left(\frac{2}{D}\right)}{I_{1+A/D}\left(\frac{2}{D}\right)}$$

(Schroeppel 1972) for real A and $D \neq 0$.

Correspondingly – since May 2001 – Sloane’s wonderful integer sequence data base at <http://www.research.att.com/~njas/sequences/> responds splendidly: with links, code, and references as well as a definition of the requisite Bessel functions.



Greetings from [The On-Line Encyclopedia of Integer Sequences!](http://www.research.att.com/~njas/sequences/)

1,4,3,3,1,2,7,4,2

[Hints](#)

Search: 1, 4, 3, 3, 1, 2, 7, 4, 2

Displaying 1-1 of 1 results found.

page 1

Format: long | [short](#) | [internal](#) | [text](#) Sort: relevance | [references](#) | [number](#) Highlight: on | [off](#)

[A060997](#) Decimal representation of continued fraction 1, 2, 3, 4, 5, 6, 7, ... +20
6

1, 4, 3, 3, 1, 2, 7, 4, 2, 6, 7, 2, 2, 3, 1, 1, 7, 5, 8, 3, 1, 7, 1, 8, 3, 4, 5, 5, 7, 7, 5, 9, 9, 1, 8, 2, 0, 4, 3, 1, 5, 1, 2, 7, 6, 7, 9, 0, 5, 9, 8, 0, 5, 2, 3, 4, 3, 4, 4, 2, 8, 6, 3, 6, 3, 9, 4, 3, 0, 9, 1, 8, 3, 2, 5, 4, 1, 7, 2, 9, 0, 0, 1, 3, 6, 5, 0, 3, 7, 2, 6, 4, 3, 5, 7, 8, 6, 1, 1, 4, 6, 5, 9, 5, 0 ([list](#); [cons](#); [graph](#); [listen](#))

OFFSET 1,2

COMMENT The value of this continued fraction is the ratio of two Bessel functions: $BesselI(0,2)/BesselI(1,2) = \frac{A070910}{A096789}$. Or, equivalently, to the ratio of the sums: $\sum_{n=0..inf} 1/(n!n!)$ and $\sum_{n=0..inf} n/(n!n!)$. - Mark Hudson (mrmarkhudson(AT)hotmail.com), Jan 31 2003

FORMULA $1/A052119$.

EXAMPLE C=1.433127426722311758317183455775 ...

MATHEMATICA RealDigits[FromContinuedFraction[Range[44]], 10, 110] [[1]]
 (* Or *) RealDigits[BesselI[0, 2] / BesselI[1, 2], 10, 110] [[1]]
 (* Or *) RealDigits[Sum[1/(n!n!), {n, 0, Infinity}] / Sum[n/(n!n!), {n, 0, Infinity}], 10, 110] [[1]]

CROSSREFS Cf. [A052119](#), [A001053](#).

Adjacent sequences: [A060994](#) [A060995](#) [A060996](#) this_sequence [A060998](#)
[A060999](#) [A061000](#)

Sequence in context: [A016699](#) [A060373](#) [A090280](#) this_sequence [A129624](#)
[A019975](#) [A073871](#)

KEYWORD [cons](#), easy, nonn

AUTHOR Robert G. Wilson v (rgwv(AT)rgwv.com), May 14 2001

Since the mid-nineties the [Inverse Symbolic Calculator](#) at <http://ddrive.cs.dal.ca/~isc> has returned

Best guess: **BesI(0,2)/BesI(1,2)**

Most of the functionality of ISC has been built into the “identify” function in *Maple* in versions since 9.5.

B. INSTRUMENTAL COMPUTING: Pi and 22/7 (Year do through 2008)

The following integral was made popular through a 1971 *Eureka* (a Cambridge undergraduate journal) article

$$0 < \int_0^1 \frac{(1-x)^4 x^4}{1+x^2} dx = \frac{22}{7} - \pi$$

Set on a 1960 Sydney honours final, and as a Monthly Problem after that, it perhaps originated in 1941 with Dalziel (author of the 1971 article who did not reference himself)!

Why should we trust the evaluation? Well both *Maple* and *Mathematica* can ‘do it’.

A better approach is to ask *Maple* to evaluate the indefinite integral

$$\int_0^1 \frac{(1-x)^4 x^4}{1+x^2} dx$$

It will return

$$\int_0^t \frac{x^4 (1-x)^4}{1+x^2} dx = \frac{1}{7}t^7 - \frac{2}{3}t^6 + t^5 - \frac{4}{3}t^3 + 4t - 4 \arctan(t)$$

and now differentiation – either by hand or computer – and the *Fundamental theorem of calculus* proves the result. More details are given in Chapter three of Borwein and Bailey 2008. This is perhaps not a conventional proof but it is a totally rigorous one: and provides an ‘instrumental use’ of the computer. In general, students should be encouraged to see if a computer algebra system can evaluate indefinite sums and integrals whenever it has successfully evaluate a definite version.

C. CONCRETIZATION: Some matrices conquered

In the course of proving conjectures about *multiple zeta values* (Borwein and Bailey 2008) I needed to obtain the closed form partial fraction decomposition for

$$\frac{1}{x^s (1-x)^t} = \sum_{j \geq 0} \frac{a_j^{s,t}}{x^j} + \sum_{j \geq 0} \frac{b_j^{s,t}}{(1-x)^j}$$

The closed form for a is

$$a_j^{s,t} = \binom{s+t-j-1}{s-j}$$

(with a symmetric form for b) as was known to Euler, but is easily discovered by looking at the first few cases in *Maple* and, if the pattern is still not clear, by asking Sloane’s Encyclopedia. Once discovered a conventional proof by induction is easy. We needed also to show that $\mathbf{M}=\mathbf{A}+\mathbf{B}\cdot\mathbf{C}$ was invertible where the n by n matrices A, B, C respectively had entries

$$(-1)^{k+1} \binom{2n-j}{2n-k}, \quad (-1)^{k+1} \binom{2n-j}{k-1}, \quad (-1)^{k+1} \binom{j-1}{k-1}$$

Thus, A and C are triangular and B is full. For example in 6 dimensions

$$M(6) = \begin{bmatrix} 1 & -22 & 110 & -330 & 660 & -924 \\ 0 & -10 & 55 & -165 & 330 & -462 \\ 0 & -7 & 36 & -93 & 162 & -210 \\ 0 & -5 & 25 & -56 & 78 & -84 \\ 0 & -3 & 15 & -31 & 35 & -28 \\ 0 & -1 & 5 & -10 & 10 & -6 \end{bmatrix}$$

After messing around with lots of cases it occurred to me to ask *Maple* for the *minimal polynomial* of M.

>linalg[minpoly](M(12),t); returns $-2+t+t^2$

Embodened I tried

> linalg[minpoly](B(20),t);

> linalg[minpoly](A(20),t);

> linalg[minpoly](C(20),t);

and was rewarded with $-1+t^3, -1+t^2, -1+t^2$. Since a typical matrix has a full degree minimal polynomial, we are assured that A, B, C really are roots of unity. Armed with this discovery we are lead to try to prove

$$A^2 = I, \quad BC = A, \quad C^2 = I, \quad CA = B^2$$

which is a nice combinatorial exercise (by hand or computer). Clearly then we obtain also

$$B^3 = B \cdot B^2 = B(CA) = (BC)A = A^2 = I$$

and the requisite formula

$$M^{-1} = \frac{M + I}{2}$$

follows with a little algebraic manipulation of the prior identities.. Characteristic or minimal polynomials (rather abstract for me when I was a student) now become members of a rapidly growing box of concrete symbolic tools, as do many matrix decompositions, Groebner bases, etc ...

CONCLUSION

The students of **2010** live in an information-rich, judgement-poor world in which the explosion of information, and of tools, is not going to diminish. So we

have to teach judgement (not just obsessive concern with plagiarism) when it comes to using what is already possible digitally. This means mastering the sorts of tools I have illustrated.

Additionally, it seems to me critical that we mesh our software design -- and our teaching style more generally -- with our growing understanding of our cognitive strengths and limitations as a species (as touched upon in the introduction) . Moreover, there is some body of evidence from Cliff Nass's *CHIME lab* at Stanford that cognitive styles are changing for the "born digital" as illustrated by measurement of the *Stroop effect* (which measures cognitive interference) for proficient multi-taskers.²

We also have to acknowledge that most of our classes will contain a very broad variety of skills and interests (and relatively few future mathematicians). Properly balanced, discovery and proof, assisted by good software, can live side-by-side and allow for the mediocre and the talented to flourish in their own fashion.

Impediments to the assimilation of the tools I have illustrated are myriad as I am only too aware from recent my own teaching experiences. These impediments include our own inertia and organizational and technical bottlenecks (this is often from poor IT design - not so much from too few dollars). The impediments certainly include under-prepared or mis-prepared colleagues and the dearth of good material from which to teach a modern syllabus.

Finally, it will never be the case that quasi-inductive mathematics supplants proof. We need to find a new equilibrium. Consider the following empirically-discovered identity

$$\sum_{n=-\infty}^{\infty} \operatorname{sinc}(n/2)\operatorname{sinc}(n/3)\operatorname{sinc}(n/5)\cdots\operatorname{sinc}(n/23)\operatorname{sinc}(n/29) \\ = \int_{-\infty}^{\infty} \operatorname{sinc}(x/2)\operatorname{sinc}(x/3)\operatorname{sinc}(x/5)\cdots\operatorname{sinc}(x/23)\operatorname{sinc}(x/29) dx$$

where the denominators range over the primes. Provably, the following is true. The analogous 'sum equals integral' identity remains valid for more than the first 10^{176} primes but stops holding after some larger prime, and thereafter the error is positive but less than one part in a googolplex. It is hard to imagine that inductive mathematics alone will ever be able to handle such behaviour (Baillie et al, 2008).

That said, we are only beginning to scratch the surface of a very exciting set of tools for the enrichment of mathematics, not to mention the growing power of

² See www.snre.umich.edu/eplab/demos/st0/stroop_program/stroopgraphicnonshockwave.gif

formal proof engines. I conclude with one of my favourite quotes from George Polya and Jacques Hadamard

*“This "quasi-experimental" approach to proof can help to de-emphasise a focus on rigor and formality for its own sake, and to instead support the view expressed by Hadamard when he stated "The object of mathematical rigor is to sanction and legitimize the conquests of intuition, and there was never any other object for it.”*³

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³ From E. Borel, *Lecons sur la theorie des fonctions*, 3rd ed. 1928, quoted in Polya (1981), (p. 2/127).