Fréchet-Legendre functions and reflexive Banach spaces

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Dedicated to Hedi Attouch on the occasion of his sixtieth birthday

ABSTRACT. A 2001 article by Bauschke, Borwein and Combettes [2] showed how to extend naturally the classical definitions of essential smoothness and essential strict convexity from functions on $\mathbb{R}^n$ in a compatible fashion to any Banach space. They were able, among other things, to show that substantial duality results hold for Legendre functions in reflexive spaces. That article focused on essential smoothness in the Gâteaux sense. Our goal herein is to show that similar results hold for Fréchet smoothness and to study related properties of such functions on reflexive Banach spaces.

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1 Introduction and preliminary results

We work in a real Banach space X whose closed unit ball is denoted by B_X. By a *proper function* $f : X \to (-\infty, +\infty]$ we mean a function that is somewhere finite-valued. A proper function $f : X \to (-\infty, +\infty]$ is *convex* if its domain is a convex set and

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)y \quad \text{for all } x, y \in \operatorname{dom} f,\ 0 \leq \lambda \leq 1.$$

If, additionally, the preceding inequality is strict for $0 < \lambda < 1$, then f is *strictly convex*.

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We use the notation $\partial f(x)$ for the convex *subdifferential* of f at x in the domain of f, and for $\epsilon > 0$ we denote the *ϵ-subdifferential* of f at x in the domain of f by $\partial_\epsilon f(x)$, that is,

$$\partial_\epsilon f(x) = \{\phi \in X^* : \phi(y) - \phi(x) \leq f(y) - f(x) + \epsilon,\ x \in X\};$$

when $\epsilon = 0$, this reduces to the definition of $\partial f(x)$. The *conjugate function* of $f : X \to (-\infty, +\infty]$ is defined for $x^* \in X^*$ by $f^*(x^*) = \sup_{x \in X} \langle x^*, x\rangle - f(x)$.

Relevant background material on convex analysis can be found in fine texts such as [14, 17] and in our own book [7]. Recall that a multifunction Ω is *locally bounded* at a point x if some neighbourhood U of x has a norm bounded image $\Omega(U)$. The following general definitions for essential smoothness and essential strict convexity were introduced in [2] and are also described in detail in [7, Chapter 7].

Definition 1.1. A proper lower semicontinuous convex function $f : X \to (-\infty, +\infty]$ is said to be

(a) *essentially smooth* if ∂f is both locally bounded and single-valued on its domain;

(b) *essentially strictly convex* if $(\partial f)^{-1}$ is locally bounded on its domain and f is strictly convex on every convex subset of $\operatorname{dom} \partial f$.

(c) *Legendre* if it is both essentially smooth and essentially strictly convex.

Additionally, [2, Theorem 5.11] shows these definitions are compatible on $\mathbb{R}^n$ with the classical definitions as in [14] and moreover, under these definitions, the duality results [14, Section 26] extend to reflexive Banach spaces. In particular, let us state the following capstone result:

Theorem 1.2 (Theorem 5.4 and Corollary 5.5 of [2])**.** *Suppose X is a reflexive Banach space and $f : X \to (-\infty + \infty]$ is a proper lower semicontinuous convex function. Then f is essentially smooth if and only if f^* is essentially strictly convex.*

Hence f is a Legendre function if and only if f^* is a Legendre function.

Reflexivity is necessary in the preceding theorem because one can commence with a norm $\|\cdot\|$ on ℓ_1 that is both Gâteaux differentiable and strictly convex [10], as in [2, 7]. Then $f := \frac{1}{2}\|\cdot\|^2$ is a Legendre function but its conjugate cannot be Legendre because ℓ_∞ does not admit any equivalent Gâteaux differentiable norm; see [2, Example 6.7] for further details on this example.

Our goal is to show that duality results similar to Theorem 1.2 hold for an appropriately defined Fréchet-Legendre function. For this, recall that a function f on X is *Fréchet differentiable* at x_0 with Fréchet derivative $f'(x_0) \in X^*$ if the limit

$$\lim_{t \to 0} \frac{f(x_0 + th) - f(x_0)}{t} = \langle f'(x_0), h \rangle$$

exist uniformly for $h \in B_X$; when the limit exists pointwise, this defines Gâteaux differentiability. We will say a proper lower semicontinuous convex function is *essentially Fréchet smooth* if it is essentially smooth and additionally is Fréchet differentiable at each point in the interior of its domain—equivalently ∂f is locally bounded, single-valued, and norm-to-norm continuous throughout its domain.

Because Fréchet and Gâteaux differentiability are equivalent for convex functions on $\mathbb{R}^n$, this definition is again compatible with the classical definition of essential smoothness. It follows from Theorem 1.2 that a proper lower semicontinuous convex function f on a reflexive space is Legendre if and if f and f^* are essentially smooth (this again can fail outside of reflexive spaces). Thus, it is also compatible with the classical definitions to say a proper lower semicontinuous convex function f on a reflexive space is *Fréchet-Legendre* if (and only if) both f and f^* are essentially Fréchet smooth.

A compelling reason for examining Fréchet-Legendre functions is the duality of Fréchet differentiability with strong minimizations principles. For this, recall that a function f attains its *strong minimum* at $x_0 \in \operatorname{dom} f$ if $\|x_n - x_0\| \to 0$ whenever $f(x_n) \to f(x_0)$ and $f(x_0) = \inf_X f$. If $f(x_0) < f(x)$ for all $x \in X \setminus \{x_0\}$ then f is said to attain its *strict minimum* at x_0. The following result is taken from [13, Proposition 4] and has its origins in [1].

Theorem 1.3. *Let X be a Banach space and $f : X \to (-\infty, +\infty]$ be proper lower semicontinuousand let y be a point in* $\operatorname{int} \operatorname{dom} f^*$ *then f^* is Fréchet differentiable at y if and only if $f - y$ attains its strong minimum.*

This result leads neatly to Stegall's variation principle; see [13] and also [8, 7] for a slightly different approach.

Remark 1 An overarching theme of both [8, 13] is that Fréchet differentiability is connected dually to various powerful minimization principles. This reinforces the argument that it is well worth the effort to investigate the theory Fréchet-Legendre functions. Moreover, continuous convex functions are generically differentiable on reflexive spaces [7, 10], and many natural globally Fréchet differentiable functions exist on those spaces because they have an abundance of Fréchet differentiable norms [10]. A less well known but important supply of Fréchet differentiable convex functions arises with *spectral functions* of matrices and of Hilbert-Schmidt or Schatten p-class operators [7, §6.5] and [9, §7.3]. $\square$

Lets us recall that if $\phi \in X^*$, and $f - \phi$ attains its strong minimum at x_0, then the functional $(\phi, -1)$ strongly exposes the epigraph of f at $(x_0, f(x_0))$. Likewise if $f - \phi$ attains its strict minimum at x_0, then $(\phi, -1)$ exposes the epigraph of f at $(x_0, f(x_0))$ [7, 10]. Hence in these cases we will say ϕ *strongly exposes* (resp. *exposes*) f at x_0. In this language, much of the natural duality theory developed for norms translates seamlessly to continuous convex functions.

Based on the corresponding standard definition for norms, it is natural to say a convex function $f : X \to (-\infty, +\infty]$ is *locally uniformly convex* if whenever a (bounded) sequence $x_0, x_1, x_2, \ldots \subset \operatorname{dom} f$ satisfies

$$\frac{f(x_n) + f(x_0)}{2} - f\left(\frac{x_n + x_0}{2}\right) \to 0,$$

then $\|x_n - x_0\| \to 0$. (It is reasonably easy to show that one may drop the requirement that the sequence be bounded [7].)

The following is perhaps the most useful tool for testing differentiability. Proofs for it can be found in [17, Theorem 3.3.2] and[6, Fact 2.3].

Theorem 1.4 (Šmulian)**.** *Suppose the convex function f is continuous at x_0. Then f is Fréchet (resp. Gâteaux) differentiable at x_0 if and only if $\phi_n \to \phi$ (resp. $\phi_n \to_{w^*} \phi$) whenever $\phi_n \in \partial_{\epsilon_n} f(x_0)$, $\phi \in \partial f(x_0)$ and $\epsilon_n \to 0^+$, and necessarily ϕ is the Fréchet (resp. Gâteaux) derivative at f at x_0.*

Using Šmulian's theorem one can capture the typical duality between exposedness and smoothness as in the following whose proof can be found in, for example, [7, Proposition 5.2.4].

Proposition 1.5. *Let $f : X \to (-\infty, +\infty]$ be a lower semicontinuous convex function that is continuous at x_0.*

(a) Then f^* is strongly exposed by $x \in X$ at $\phi \in X^*$ if and only if f is Fréchet differentiable at x with $f'(x) = \phi$.

(b) Then f^* is exposed by $x \in X$ at $\phi \in X^*$ if and only if f is Gâteaux differentiable at x with $f'(x) = \phi$.

2 Legendre functions on reflexive Banach spaces

Let X be a reflexive Banach space. We will say a continuous convex function $f : X \to \mathbb{R}$ is *strongly convex* it is strictly convex and $\|x_n - x_0\| \to 0$ whenever $f(x_n) \to f(x_0)$ and $x_n \to_w x_0$, moreover, it is consistent with the eponymous definition for norms from [4, Definition 6.4]. This is also consistent with—though less general than—the definition of strongly rotund functions which play a central role in the theory of maximum entropy reconstruction [5, 7] while strongly rotund norms are key to the study of non-convex best approximation problems [4].

We provide five theorems describing the ambit of Fréchet-Legendre functions and their relatives. We begin with a natural duality theorem in reflexive spaces.

Theorem 2.1 (Duality in reflexive space)**.** *Suppose X is a reflexive Banach space and $f : X \to \mathbb{R}$ is a continuous cofinite convex function. Then:*

(a) f is Gâteaux differentiable if and only if f^* is strictly convex.

(b) The following are equivalent:

(i) f is Fréchet differentiable;

(ii) f^* is strongly exposed at each $x^* \in X^*$ by each subgradient in $\partial f(x^*)$;

(iii) f^* is strongly exposed at each $x^* \in X^*$.

(iv) f^* is strongly convex.

Proof. The reader may find a proof of (a) and of (b) (ii) $\Rightarrow$ (i) $\Rightarrow$ (iii) that relies on Proposition 1.5, for example, in [7, Theorem 5.3.7]. For the remainder we argue as follows.

(iii) $\Rightarrow$ (iv): Condition (iii) implies f^* is strictly convex so we assume $f^*(x_n) \to f^*(x)$ and $x_n \to_w x$. Let $\phi \in X$ strongly expose f^* at x. It is immediate that $(f^* - \phi)(x_n) \to (f^* - \phi)(x)$. Because ϕ strongly exposes f^* at x, $\|x_n - x\| \to 0$ and so f^* is strongly convex.

(iv) $\Rightarrow$ (ii): Conversely suppose f^* is strongly convex. Let $\phi \in \partial f^*(x)$. Then $(f^* - \phi)$ attains its strict minimum at x because f^* is strictly convex. Suppose by way of contradiction that ϕ does not strongly expose f^* at x. Then there is a sequence (x_n) such that $(f^* - \phi)(x_n) \to (f^* - \phi)(x)$, but $\|x_n - x\| \geq \epsilon > 0$ for all n. Because $f^* - \phi$ is convex and attains its minimum at x, we may assume (x_n) is bounded. The Eberlein-Šmulian theorem implies there is a subsequence $(x_k) \to_w \bar{x}$ for some $\bar{x} \in X^*$. Now

$\phi(x_k) \to \phi(\bar{x})$ and it follows $f^*(x_k) \to f^*(\bar{x})$. Consequently, $(f^* - \phi)(\bar{x}) = (f^* - \phi)(x)$. Thus $\bar{x} = x$ since $f^* - \phi$ attains its strict minimum at x. Thus we arrive at the contradiction $\|x_k - x\| \to 0$. □

Remark 2 If one removes the reflexivity assumption, the previous theorem may fail. Indeed, consider Talagrand's construction of a Fréchet differentiable norm $\|\cdot\|$ on $C[0, \omega_1]$ whose dual is not strictly convex [15]; then let $f := \frac{1}{2}\|\cdot\|^2$. It may even fail on $\mathbb{R}$ without the cofinite assumption. Indeed, $f :=$ exp is locally uniformly convex, but its conjugate, $x \mapsto x \log x - x$,is not differentiable everywhere on its domain. Further, in the reflexive case, one can not strengthen (ii) or (iii) to say that f^* is locally uniformly convex. For this, let $f := \frac{1}{2}\|\cdot\|^2$ where $\|\cdot\|$ is a Fréchet differentiable norm whose dual is not locally uniformly convex as was constructed by Yost [16]. □

Recalling that a function is *cofinite* if its conjugate is everywhere finite, the existence of cofinite Fréchet-Legendre functions actually characterizes reflexive Banach spaces.

Theorem 2.2 (Reflexive characterization of Fréchet differentiability)**.** *Suppose X is a Banach space. Then X is reflexive if and only if it admits a continuous cofinite and convex function f such that f and f^* are both Fréchet differentiable.*

Proof. If X is reflexive consider $f := \frac{1}{2}\|\cdot\|^2$ where $\|\cdot\|$ is an equivalent locally uniformly convex and Fréchet differentiable norm on X [10]. For the converse, use the Moreau-Rockafellar theorem (see [7, Theorem 4.4.10]) to deduce that f^* is coercive. From such an f^* one can deduce in a standard fashion that X^* has an equivalent dual Fréchet differentiable norm (see the proof of [6, Theorem 3.5(a)]), and as a consequence X is reflexive. See [7, Exercise 5.1.28] for further details. □

This result underlines how tightly the existence of conjugate convex Fréchet differentiable functions is coupled to reflexivity.

Let us say that a proper function f is *supercoercive* if

$$\lim_{\|x\|\to\infty} \frac{f(x)}{\|x\|} = \infty.$$

Recall that f is *coercive* if merely $\lim_{\|x\|\to\infty} f(x)/\|x\| > 0$. Thus $|\cdot|$ is coercive but not supercoercive while $x \mapsto x \log x$ is supercoercive. (Note that for convex functions coercivity is equivalent to $f(x) \to \infty$ when$\|x\| \to \infty$ [7].)

Theorem 2.3 (Fréchet-differentiable and locally uniformly convex functions)**.** *Let X be a reflexive Banach space and assume that $f : X \to (-\infty, +\infty]$ is a lower semicontinuous proper function.*

(a) Suppose f is continuous, cofinite and f and f^* are both Fréchet differentiable. Then f and f^* are both locally uniformly convex.

(b) A function f is supercoercive, bounded on bounded sets, locally uniformly convex, and Fréchet differentiable if and only if f^* is.

Proof. (a) We will show that f is locally uniformly convex. For this, suppose that $(x_n)_{n=1}^{\infty}$ is bounded and

$$\frac{f(x) + f(x_n)}{2} - f\left(\frac{x + x_n}{2}\right) \to 0. \tag{2.1}$$

Let $\phi \in \partial f(x)$ and $\phi_n \in \partial f(\frac{x+x_n}{2})$. Then

$$\begin{aligned} \frac{1}{2}f(x) + \frac{1}{2}f(x_n) - \left\{\phi_n\left(\frac{x+x_n}{2}\right) - f^*(\phi_n)\right\} &\to 0 \text{ and so} \\ \frac{1}{2}\{f(x) - \phi_n(x)\} + \frac{1}{2}\{f(x_n) - \phi_n(x_n)\} + f^*(\phi_n) &\to 0. \end{aligned}$$

Now both $f(x) - \phi_n(x) \geq -f^*(\phi_n)$ and $f(x_n) - \phi_n(x_n) \geq -f^*(\phi_n)$, and so we may conclude that

$$\phi_n(x) - f(x) - f^*(\phi_n) \to 0 \ \text{ and } \ \phi_n(x_n) - f(x_n) - f^*(\phi_n) \to 0. \tag{2.2}$$

Thus, $f(x) + f^*(\phi_n) \leq \phi_n(x) + \epsilon_n$ where $\epsilon_n \to 0^+$, and hence one can check that $\phi_n \in \partial_{\epsilon_n} f(x)$. Because f is Fréchet differentiable at x, Theorem 1.4 implies that $\|\phi_n - \phi\| \to 0$. Because $(x_n)_{n=1}^{\infty}$ is bounded, this implies $\phi_n(x_n) - \phi(x_n) \to 0$. From (2.2), we also have $\phi(x_n) - f(x_n) \geq f^*(\phi) - \epsilon_n$ where $\epsilon_n \to 0^+$.

Again, one can check $\phi \in \partial_{\epsilon_n} f(x_n)$ which in turn implies $x_n \in \partial_{\epsilon_n} f^*(\phi)$. Because f^* is Fréchet differentiable at ϕ, Theorem 1.4 implies $\|x_n - x\| \to 0$ as desired. It follows that f is locally uniformly convex, because $\|x_n - x_0\| \to 0$ whenever (2.1) holds for a bounded sequence (x_n). Because $f^{**} = f$, the argument implies f^* is locally uniformly convex as well.

Part (b) follows from (a) because f is supercoercive and bounded on bounded sets if and only if f^* is (see e.g. [7, Theorem 4.4.13]) and since f is Fréchet differentiable when f^* is locally uniformly convex as can be deduced from Proposition 1.5; see [7, Proposition 5.3.6] . □

Remark 3 Theorem 2.3 (a) deserves a comment. In [4, Thm 6.6], consistent with Theorem 2.1 (b), it is shown that a dual norm on X^* is Fréchet differentiable if and only if X is reflexive and the original norm is *strongly convex*: that is, the norm is strictly convex and has the *Kadec-Klee property* (weak and norm convergence agree on unit the sphere). That strong convexity is strictly weaker than local uniform convexity is shown by the norm on $\ell^2(\mathbf{N})$, due to Mark Smith [4, Remark 6.7], given by

$$f(x) := |||x|||^2 := \|Tx\|^2 + (|x| + \|Px\|)^2$$

where

$$Tx := (0, x_2/2, x_3/3, \cdots, x_n/n, \cdots)$$

and

$$Px := (0, x_2, x_3, \cdots, x_n, \cdots).$$

It is easy to check that f is strongly convex since T is a compact operator; but it is not locally uniformly convex since $|||e_n||| \to |||e_1||| = 1$ and $|||(e_n + e_1)/2||| \to 1$ yet $|||e_n - e_1||| \to 2$. Note also that while f^* is Fréchet differentiable, f is not. Thus, we see the necessity of the hypotheses on both f and f^* in Theorem 2.3 (a). The reader will note that we could have also used Yost's [16] example as mentioned earlier to the same end. □

Theorem 2.4 (Fréchet-Legendre functions). *Let X be a reflexive Banach space, and let f be a lower semicontinuous proper convex function on X.*

(a) f is essentially Fréchet smooth if and only if (i) f^* is essentially strictly convex and (ii) every point of range(∂f) *is a strongly exposed point of f^*.*

(b) Moreover, f is Fréchet-Legendre if and only if f^* is.

(c) A continuous and cofinite function f is a Fréchet-Legendre function if and only if it is Fréchet differentiable and locally uniformly convex.

Proof. (a) Suppose f is essentially Fréchet smooth. Then f^* is essentially strictly convex by [2, Theorem 5.4]. Suppose $\phi \in \partial f(x)$. Then $x \in \operatorname{int}\operatorname{dom} f$ by [2, Theorem 5.6] and so f is Fréchet differentiable at x. According to Proposition 1.5, f^* is strongly exposed by x at ϕ. For the converse, f is essentially smooth by [2, Theorem 5.4]. Fix $x_0 \in \operatorname{int}\operatorname{dom} f$. Then f is Gâteaux differentiable at x_0. Let $\phi \in \partial f(x_0)$ and $\phi_n \in \partial f_{\epsilon_n}(x_0)$ where $\epsilon_n \to 0^+$. Then $x_0 \in \partial f^*(\phi)$, $\phi_n \to_{w^*} \phi$ by Theorem 1.4; moreover $f^*(\phi_n) \to f^*(\phi)$. Now let y strongly expose f^* at ϕ. Because $\phi_n \to \phi$ weakly, it now follows that $\limsup (f^* - y)(\phi_n) \le (f^* - y)(\phi)$. Consequently, $\|\phi_n - \phi\| \to 0$. According to Theorem 1.4, f is Fréchet differentiable at x_0.

(b) follows because $f^{**} = f$.

(c) follows because in the cofinite case f is locally uniformly convex if f and f^* are Fréchet differentiable as shown in Part (a) of Theorem 2.3. $\square$

We conclude this section by describing how the relationship between strict convexity and essential strict convexity for coercive functions characterizes reflexive spaces. In this construction we will use the concept of a *Markushevich basis* (M-basis) which for a separable space X is a system $\{x_n, x_n^*\} \subset X \times X^*$ such that: (i) $x_n^*(x_m) = 1$ if $m = n$ and 0 otherwise; (ii) the norm closed span of $\{x_n\}_{n=1}^\infty = X$ and (iii) $\{x_n^*\}$ separates points in X. A concise introduction to M-bases can be found in [12, Chapter 6].

Theorem 2.5 (Essentially strictly convex functions)**.** *Let X be a Banach space. Then X is reflexive if and only if every proper coercive lower semicontinuous strictly convex function on X is essentially strictly convex.*

Proof. $\Rightarrow$: Suppose X is reflexive and let $f : X \to (-\infty, \infty]$ be a coercive proper lower semicontinuous strictly convex function. Then $0 \in \operatorname{int}\operatorname{dom} f^*$ according to a Moreau-Rockafellar theorem (see [7, Theorem 4.4.10]). In particular, $\operatorname{int}\operatorname{dom} f^* \neq \emptyset$. According to [2, Lemma 5.1], ∂f^* is single-valued on its domain since $f = (f^*)^*$ is strictly convex. Now [2, Theorem 5.6] ensures that f^* is essentially smooth (since condition (ii) therein is satisfied). Consequently Theorem 1.2 shows that f is essentially strictly convex.

$\Leftarrow$: Suppose X is not reflexive. Consider the separable case first. Because X is not reflexive there is a bounded basic sequence (x_n) and $\phi \in X^*$ so that $\lim_n \phi(x_n) > 0$. (See, for example, the Eberlein-Šmulian theorem presented in [11, p. 41]). This basic sequence can be extended to a Markushevich basis on all of X (see [12, Theorem 6.42]). Call this M-basis (y_n, y_n^*). We may scale it so $\|y_n\| \leq 1$ for all n, and so that $\phi(y_n) = \epsilon$ for infinitely many n. Replacing ϕ with $\lambda\phi$ for appropriate $\lambda > 0$ we may assume $\phi(y_n) = 1$ for infinitely many n, and re-scaling the M-basis we may assume $|\phi(y_n)| \leq 1$ for all n.

Now let $\|\!|\cdot|\!\|$ be the equivalent norm on X whose ball B is defined by $B := \{x \in B_X : |\phi(x)| \leq 1\}$. With this norm, we have $\|\!|\phi|\!\| = 1$ and $\phi(y_n) = 1$ for infinitely many n. Let us relabel this M-basis as $\big\{\{u_n\} \cup \{v_n\}, \{u_n^*\} \cup \{v_n\}^*\big\}_{n\in\mathbb{N}}$ where $\phi(u_n) = 1$ for each n.

We now define a continuous strictly convex coercive function via

$$f(x) := \|\!|x|\!\| + (u_1^*(x) - 1)^2 + \sum_{i=2}^{\infty} \frac{(\tilde{u}_i^*(x))^2}{2^i} + \sum_{i=1}^{\infty} \frac{(\tilde{v}_i^*(x))^2}{2^i},$$

where $\tilde{u}_i^* := \frac{u_i^*}{\|u_i^*\|}$ and $\tilde{v}_i^* := \frac{v_i^*}{\|v_i^*\|}$.

Now $\phi \in \partial f(u_1)$ since $f(u_1) = 1$ and $\phi(u_1) = 1$ and $\phi(x) \leq \|x\|$ for all x. Also, $\phi(u_1 + nu_n) = n + 1$, and so $\phi \in \partial\|u_1 + nu_n\|$. Let

$$g(x) := f(x) - \|x\|,$$

then $g'(u_1 + nu_n) = \frac{2\tilde{u}_n^*(nu_n)\tilde{u}_n^*}{2^n}$. Therefore,

$$\Lambda_n \in \partial f(u_1 + nu_n) \quad \text{where } \Lambda_n = \phi + \frac{2\tilde{u}_n^*(nu_n)\tilde{u}_n^*}{2^n}.$$

Observe that $\|u_1 + nu_n\| \to \infty$ while $\|\Lambda_n - \phi\| \leq \frac{n}{2^{n-1}}$. Thus, $(\partial f)^{-1}$ is not locally bounded at ϕ, and accordingly f is not essentially strictly convex.

If X nonseparable and not reflexive, then it contains a separable subspace Y that is not reflexive. Construct $f : Y \to \mathbb{R}$ as above, and define the desired function as $\tilde{f}(x) := f(x)$ if $x \in Y$ and $\tilde{f}(x) := +\infty$ if $x \notin Y$, making sure subgradients above are extended so as to stay close. Now take a norm- preserving extension of ϕ ; then likewise extend $\Lambda_n - \phi$ and add that extension to the extension of ϕ to get an extension of Λ_n which is close to the extension of ϕ. □

Remark 4 Notice that while the function f constructed above is above is continuous in the separable case, in general it is not. Indeed whenever a coercive continuous strictly convex function exists on X, one can use it to construct a strictly convex norm on the space (see e.g., [6]). □

3 Concluding comments

We have, we think, made a clear case for the role of reflexivity in studying Legendre functions (Gâteaux or Fréchet). It is easy to show with our definitions that, under appropriate constraint qualifications, the sum and infimal convolution of Legendre functions is again Legendre.

Outside of reflexive space, as we have seen the conjugate of a Legendre function need not be Legendre. Also, as shown in Theorem 2.5, outside of reflexive space a coercive lower semicontinuous strictly convex function may–perhaps counter-intuitively—fail to be essentially strictly convex. Thus, Euclidean intuition is of little help and it is unrealistic to expect a full duality to hold in more general spaces.

That said, it would be highly desirable to find a generalization of the notion of Legendre function that covers the case of important convex functions which are nowhere continuous such as the strictly convex (negative) *Boltzmann-Shannon entropy*

$$x \mapsto \int_0^1 x(t)\, \log x(t)\, d\lambda$$

defined on $L^1([0,1],\lambda)$ with λ Lebesgue measure, which has weakly compact lower level sets and an everywhere Gateaux differentiable conjugate function

$$x^* \mapsto \int_0^1 e^{x^*(t)-1}\, d\lambda$$

on $L^\infty([0,1],\lambda)$ as discussed in [7, §6.4]. We should also point the reader to algorithmic applications of Legendre functions in reflexive space as described in [3, 7].

Finally, another impressive illustration of the value of Fréchet differentiability is afforded by [7, Corollary 4.5.2 (a)]:

Theorem 3.1 (Convexity of pre-conjugates)**.** *Suppose* $f : X \to (-\infty, +\infty]$ *is such that* f^{**} *is proper and that* f^* *is Fréchet differentiable at all* x^* *in* $\mathrm{dom}(\partial f^*)$ *and* f *is lower semicontinuous. Then* f *is convex.*

An elegant application of Theorem 3.1 is to show that a finite dimensional (or just weakly closed) Chebyshev set is convex—this is the Motzkin-Bunt theorem [7, §4.5]. In general Fréchet differentiability plays an important role in the study of the distance function to a non-convex set in reflexive space [9, §5.3].

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