

SIAM REVIEW by Jonathan M. Borwein, Professor Laureate
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of **The Princeton Companion to Mathematics** edited by Timothy Gowers
with June Barrow-Green and Imre Leader, associate editors.

Tim Gowers was a 1998 Fields medallist for his marvellous resolution of long-standing problems in Banach space theory --- such as whether it is possible for a Banach space to have no isomorphic hyperplane (it is) --- and in combinatorics; and while he continues such work, in exemplary fashion he has also found time for various more didactic and expository projects such as *Mathematics: A Very Short Introduction* (2002) and the book under review, activity with various media and much else. Both associate editors, June Barrow-Green (Deputy Director for the Centre and Research Fellow in History of Mathematics at the Open University) and the combinatorist Imre Leader (Professor at Trinity College, Cambridge), have distinguished records.

This work, which I shall below refer to as Gowers does as "*the Companion*", is a fine validation of the well known proposition that if you wish a job done right you should ask a busy person to do it. In this case many very busy people have performed an invaluable job very, very well. This handsome, hefty (1008 pages) and attractively priced (\$99.00 / £71.00) Princeton University Press volume received Honorable Mention for the 2008 PROSE Award for Professional and Scholarly Excellence for Single Volume Reference/Science, Association of American Publishers. In his excellent preface Gowers describes the painstaking six-year process which lead to this work and writes that "the central focus of this book is *modern, pure* mathematics," both highlighted terms being lucidly discussed. Since this review is appearing in SIAM I should emphasize that a great deal of less pure mathematics is captured. He also points out that a "Companion is not an Encyclopaedia" and that

"[t]he *Princeton Companion to Mathematics* could be said to be about everything that Russell's definition [of pure Mathematics] leaves out."

Let me complete my review prematurely. Every research mathematician, every university student of mathematics, and every serious amateur of mathematical science should own a least one copy of *the Companion*. Indeed, the sheer weight of the volume suggests that it is advisable to own two: one for work and one at home. You may want to get a copy of *the Companion* for a friend. I bought a copy as an eighty-fifth birthday present for my mathematician father.

Reviews of *the Companion*, both professional and on Amazon (which also has a good selection of superlative comments extracted from professional reviews), have been generally laudatory as indeed they should be. Princeton University Press also maintains a web site at <http://press.princeton.edu/TOCs/c8350.html> (whose scope and intention Gower's describes in his preface) from which the careful potential buyer can make a fully informed decision to purchase. Additionally, Princeton University Press provides the full table of contents, the preface, list of contributors (a most impressive collection which includes mathematical household names such as Atiyah, Connes, Daubechies, Lax, and Tao, as well as many very distinguished

authors most of whose names are probably not familiar to any given reader) and very representative sample articles consistent of the Preface, Contributors, I.2 The Language and Grammar of Mathematics, II.2 Geometry, IV.5 Arithmetic Geometry, IV.21 Numerical Analysis, V.10 Fermat's Last Theorem, VI.61 Jules Henri Poincaré (1854-1912), VII.2 Mathematical Biology, VIII.6 Advice to a Young Mathematician (by Sir Michael Atiyah, Bela Bollobas, and others). From this the reader of this review can already probably glean the structure of the book which is in eight Parts. They comprise:

Part I Introduction contains *What Is Mathematics About?*, *The Language and Grammar of Mathematics*, *Some Fundamental Mathematical Definitions* (32 pages), and *The General Goals of Mathematical Research*.

Part II The Origins of Modern Mathematics has seven entries commencing with *From Numbers to Number Systems and Geometry* and culminating with *The Crisis in the Foundations of Mathematics*.

Part III Mathematical Concepts consists of ninety-nine brief entries arranged alphabetically. These entries are typically between one and three pages. They start with *The Axiom of Choice* and visit topics such as *Calabi-Yau Manifolds*, *Countable and Uncountable Sets*, *Dynamical Systems and Chaos*, *The Fast Fourier Transform*, *Homology and Cohomology*, *The Ising Model*, *The Leech Lattice*, *Matroids*, *Number Fields*, *Probability Distributions*, *Quantum Computation*, *Ricci Flow*, *Special Functions*, before finishing up with *Von Neumann Algebras*, *Wavelets* and Joyce-like revisiting the axioms of set theory with *The Zermelo-Fraenkel Axioms* (on page 314 which should please the Pi lover).

Part IV Branches of Mathematics occupies pages 315 through 680 and covers 26 topics including *Algebraic Numbers*, *Representation Theory*, *Harmonic Analysis*, *General Relativity and the Einstein Equations*, *Enumerative and Algebraic Combinatorics*, *Numerical Analysis*, and *High-Dimensional Geometry and Its Probabilistic Analogues*.

Part V Theorems and Problems has 36 alphabetic entries of between one and three pages. It starts as it must with *The ABC Conjecture*, and touches upon *The Banach-Tarski Paradox*, *Carleson's Theorem*, *The Classification of Finite Simple Groups*, *Fermat's Last Theorem*, *The Four-Color Theorem*, *Gödel's Theorem*, *The Insolubility of the Halting Problem*, *Mostow's Strong Rigidity Theorem*, *The P versus NP Problem*, *The Poincaré Conjecture*, *The Prime Number Theorem and the Riemann Hypothesis*, *The Resolution of Singularities*, *The Robertson-Seymour Theorem*, and *The Weil Conjectures*.

Part VI Mathematicians are arranged chronologically from *Pythagoras* (ca. 569 B.C.E.-ca. 494 B.C.E.) and *Euclid* (ca. 325 B.C.E.-ca. 265 B.C.E.) through *Abu Ja'far Muhammad ibn Musa al-Khwarizmi* (800-847), *Leonardo of Pisa (known as Fibonacci)* (ca. 1170-ca. 1250) *François Viète* (1540-1603) VI.12 *Pierre Fermat* (160?-1665), *The Bernoullis* (fl. 18th century), *Leonhard Euler* (1707-1783), *Jean-Baptiste Joseph Fourier* (1768-1830), *Carl Friedrich Gauss* (1777-1855), VI.31 *Nicolai Ivanovich Lobachevskii* (1792-1856), *William Rowan Hamilton* (1805-1865) *Eduard Kummer* (1810-1893), *James Joseph Sylvester* (1814-1897), *William*

Burnside (1852-1927), *Jacques Hadamard* (1865-1963), *Emmy Noether* (1882-1935), *Norbert Wiener* (1894-1964), *William Vallance Douglas Hodge* (1903-1975) *Abraham Robinson* (1918-1974) , and finally as the ninety-sixth entry and only living member of the list *Nicolas Bourbaki* (1935-). If your mathematical hero is missing above that is likely to be because of my selection not the editors' oversight. Indeed when I proofread this review I wondered why my own favourite G.H. Hardy was not above and I checked online that he was indeed included.

Part VII The Influence of Mathematics has fourteen entries each of approximately ten pages including: *Mathematics and Chemistry*, *The Mathematics of Traffic in Networks*, *Mathematics and Economic Reasoning*, *Mathematics and Medical Statistics*, and *Mathematics and Art*.

Part VIII Final Perspectives comprises seven essays each between about five and ten pages in length entitled *The Art of Problem Solving*, *"Why Mathematics?" You Might Ask*, *The Ubiquity of Mathematics*, *Numeracy*, *Mathematics: An Experimental Science*, *Advice to a Young Mathematician* (perhaps in homage to Peter Medawar's wonderful 1979 *Advice to a Young Scientist*), and *A Chronology of Mathematical Events*.

The sheer scale and scope of the book which finishes with a very good index should now be fully apparent. In my 2006 Featured *SIAM REVIEW* of *The Oxford Users' Guide to Mathematics* [48:3 (2006), pp. 585-594] I wrote generally of the issues involved with such projects and, despite great sympathy, I found much to be critical of with regards to the roles of both editors and the publisher of the *Users' Guide*. Indeed, I wrote

"... A more thorough review and production process would surely have adequately addressed this last set of issues. I can no better make this point than to quote Simkin and Fiske quoting [in *Science*] the late Stephen J. Gould in a review of Simon Winchester's *Krakatoa*. These reviews do make me question the reliability of the *Professor and the Madman*.

"In his review of Winchester's previous book, *The Map That Changed the World*, Stephen Jay Gould wrote: "I don't mean to sound like an academic sourpuss, but I just don't understand the priorities of publishers who spare no expense to produce an elegantly illustrated and beautifully designed book and then permit the text to wallow in simple, straight-out factual errors, all easily corrected for the minimal cost of one scrutiny of the galleys by a reader with professional expertise" ...

"With *Krakatoa*, the publisher clearly spared considerable expense, and this new book also wallows in errors. Perhaps, given our popular culture's appetite for sensationalized disasters, a modern publisher would rather not see all those pesky details corrected." "

Even an academic sourpuss should be pleased with the *Companion's* publishers', editors' and authors' attention to detail and to many judicious decisions---about level of exposition, level of detail, what to include and what to omit, and much more--- which have led to a well integrated and highly readable volume. Gowers writes

"[T]he editorial process has been a very active one: we have not just commissioned the articles and accepted whatever we have been sent. Some drafts have had to be completely discarded and new articles written in the light of editorial comments. Others have needed substantial changes, which have sometimes been made by the authors and sometimes by the

editors. A few articles were accepted with only trivial changes, but these were a very small minority.”

I have described in my 2006 review how hard it is to produce such a volume---let alone to do it so splendidly---and how easy it is to find fault in any project with such audacious goals. This I know full well from my own more prosaic efforts as a co-author of *The Collins-Smithsonian Dictionary of Mathematics*. Thus, in attempt to limit bias, I left a copy in my office for several months and sampled it with students or colleagues who dropped in for a chat or with a query. I found little missing. Indeed, the only item I did not find during this process but thought I should have found was “Turing test” and that term is perhaps not fairly within the compass of modern pure mathematics. I finish by quoting again from Gower’s own preface.

“6 Who Is *The Companion* Aimed At?

The original plan for *The Companion* was that all of it should be accessible to anybody with a good background in high school mathematics (including calculus). However, it soon became apparent that this was an unrealistic aim: there are branches of mathematics that are so much easier to understand when one knows at least some university-level mathematics that it does not make good sense to attempt to explain them at a lower level. On the other hand, there are other parts of the subject that decidedly *can* be explained to readers without this extra experience. So in the end we abandoned the idea that the book should have a uniform level of difficulty.

Accessibility has, however, remained one of our highest priorities, and throughout the book we have tried to discuss mathematical ideas at the lowest level that is practical. *In particular, the editors have tried very hard not to allow any material into the book that they do not themselves understand, which has turned out to be a very serious constraint* [my emphasis]. Some readers will find some articles too hard and other readers will find other articles too easy, but we hope that all readers from advanced high school level onwards will find that they enjoy a substantial proportion of the book.

What can readers of different levels hope to get out of *The Companion*? If you have embarked on a university level mathematics course, you may find that you are presented with a great deal of difficult and unfamiliar material without having much idea why it is important and where it is all going. Then you can use *The Companion* to provide yourself with some perspective on the subject. (For example, many more people know what a ring is than can give a good reason for caring about rings. But there are very good reasons, which you can read about in RINGS, IDEALS, AND MODULES [III.81] and ALGEBRAIC NUMBERS [IV.1].) If you are coming to the end of the course, you may be interested in doing research in mathematics. But undergraduate courses typically give you very little idea of what research is actually like. So how do you decide which areas of mathematics truly interest you at the research level? It is not easy, but the decision can make the difference between becoming disillusioned and ultimately not getting a Ph.D., and going on to a successful career in mathematics. This book, especially part IV, tells you what mathematicians of many different kinds are thinking about at the research level, and may help you to make a more informed decision.

If you are already an established research mathematician, then your main use for this book will probably be to understand better what your colleagues are up to. Most non-mathematicians are very surprised to learn how extraordinarily specialized mathematics has become. Nowadays it is not uncommon for a very good mathematician to be completely unable to understand the papers of another mathematician, even from an area that appears to be quite close. This is not a healthy state of affairs: anything that can be done to improve the level of communication among mathematicians is a good idea. The editors of this book have learned a huge amount from reading the articles carefully, and we hope that many others will avail themselves of the same opportunity.”

Judging by the sales numbers shown on Amazon, however such are actually computed, a great many copies are with readers if perhaps not all yet read.

Everyone involved with this project deserve our deep gratitude.

Jonathan Borwein, Newcastle NSW, May 6, 2009