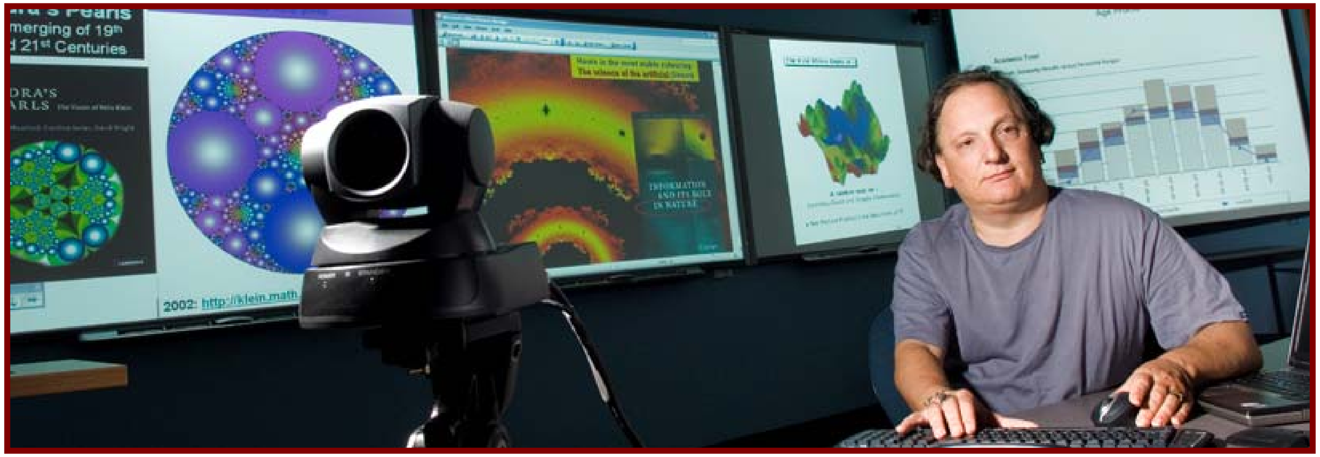

Newcastle
AMSI-AG Room

Convex functions:
Characterizations, Constructions and Counterexamples

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A mathematician is a person who can find analogies between theorems; a better mathematician is one who can see analogies between proofs; and the best mathematician can notice analogies between theories.

(Stefan Banach, 1892-1945)

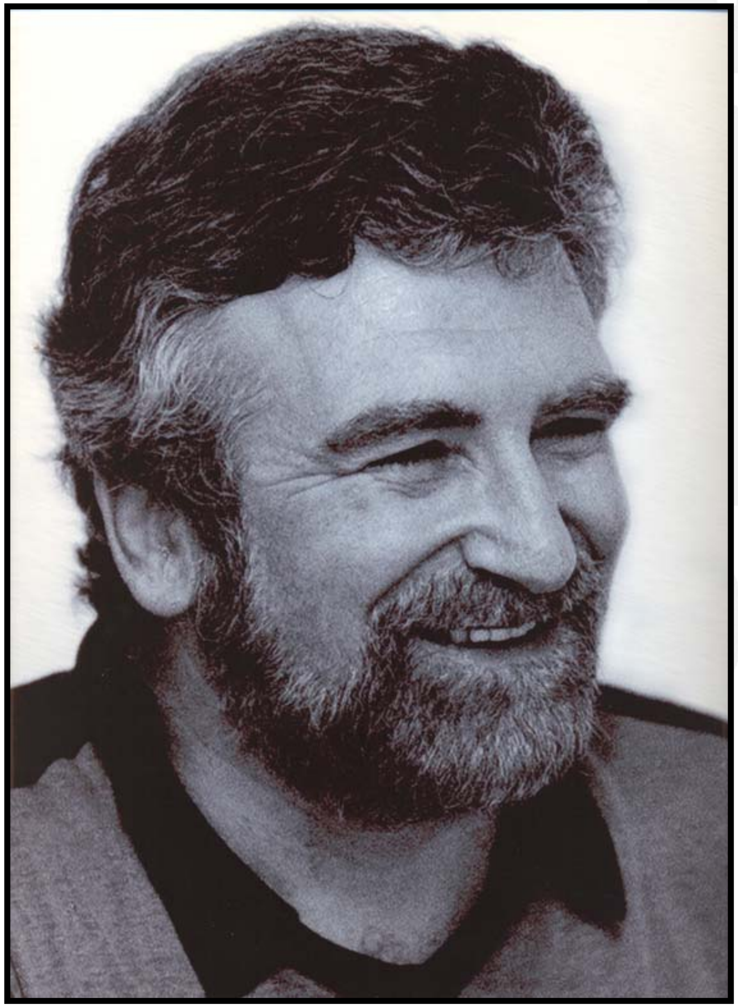
DALHOUSIE University

Drive

15/09/2008

Abstract of CF:CCC Talk

In honour of my friend Boris Mordhukovich



We met in 1990. He said

"How old are you?"

I said *"39 and you?"*

He replied "48."

I left thinking he was 48 and he thinking I was 51.

Some years later Terry Rockafellar corrected our cultural misconnect.

What was it?

Convex Functions: Characterizations, Constructions and Counter examples

(CUP in press)

Convex functions, along with smooth functions, provide the wellspring for much of variational analysis

In this talk I shall look at four open problems in variational analysis, at the convex structure underlying them, and at the convex tools available to make progress with them

In each case, I think better understanding is fundamental to advancing nonsmooth analysis

Contents

1 Why Convex? 1
1.1 Why 'Convex'? . . . 2
1.2 Basic Principles . . . 3
1.3 Some Mathematical Illustrations . . . 9
1.4 Some More Applied Examples . . . 11

2 Convex Functions on Euclidean Spaces 19
2.1 Continuity and Subdifferentials . . . 20
2.2 Differentiability . . . 35
2.3 Conjugate Functions and Duality . . . 46
2.4 Differentiability in Measure and Category . . . 78
2.5 Second-order Differentiability . . . 84
2.6 Support and Extremal Structure . . . 92

3 Finer Structure of Euclidean Spaces 95
3.1 Polyhedral Convex Sets and Functions . . . 96
3.2 Functions of Eigenvalues . . . 101
3.3 Linear and Semidefinite Programming Duality . . . 109
3.4 Selections and Fixed Points . . . 113
3.5 Into the Infinite . . . 119

4 Convex Functions on Banach Spaces 129
4.1 Continuity and Subdifferentials . . . 130
4.2 Differentiability of Convex Functions . . . 152
4.3 Variational Principles . . . 164
4.4 Conjugate Functions and Fenchel Duality . . . 175
4.5 Chebyshev Sets and Proximality . . . 191
4.6 Small Sets and Differentiability . . . 199

5 Duality Between Smoothness and Strict Convexity 215
5.1 Renorming: an Overview . . . 216
5.2 Exposed Points of Convex Functions . . . 239
5.3 Strictly Convex Functions . . . 245
5.4 Moduli of Smoothness and Rotundity . . . 259

5.5 Lipschitz Smoothness . . . 274

6 Further Analytic Topics 283
6.1 Multifunctions and Monotone Operators . . . 284
6.2 Epigraphical Convergence: an Introduction . . . 293
6.3 Convex Integral Functionals . . . 311
6.4 Strongly Rotund Functions . . . 316
6.5 Trace Class Convex Spectral Functions . . . 321
6.6 Deeper Support Structure . . . 327
6.7 Convex Functions on Normed Lattices . . . 339

7 Barriers and Legendre Functions 349
7.1 Essential Smoothness and Essential Strict Convexity . . . 350
7.2 Preliminary Local Boundedness Results . . . 351
7.3 Legendre Functions . . . 355
7.4 Constructions of Legendre Functions in Euclidean Space . . . 360
7.5 Further Examples of Legendre Functions . . . 365
7.6 Zone Consistency of Legendre Functions . . . 369
7.7 Banach Space Constructions . . . 381

8 Convex Functions and Classifications of Banach Spaces 391
8.1 Canonical Examples of Convex Functions . . . 392
8.2 Characterizations of Various Classes of Spaces . . . 398
8.3 Extensions of Convex Functions . . . 408
8.4 Some Other Generalizations and Equivalences . . . 417

9 Monotone Operators and the Fitzpatrick Function 421
9.1 Monotone Operators and Convex Functions . . . 422
9.2 Cyclic and Acyclic Monotone Operators . . . 432
9.3 Maximality in Reflexive Banach Space . . . 450
9.4 Further Applications . . . 457
9.5 Limiting Examples and Constructions . . . 462
9.6 The Sum Theorem in General Banach Space . . . 467
9.7 More About Operators of Type (NI) . . . 468

10 Further Remarks and Notes 479
10.1 Back to the Finite . . . 480
10.2 Notes on Earlier Chapters . . . 490
10.3 List of Symbols . . . 502

Bibliography 505

Index 526

Convex Functions

- H is (separable) Hilbert space in some renorm ||·||
- C is a norm-closed subset and

$$d_C(x) := \inf_{x \in C} \| x - c \|$$

$$P_C(x) := \arg\min d_C(x)$$

- In the Hilbert case $P_C(x)$ is at most singleton
- In a non-rotund renorm it may be multivalued
- If C is convex it is non-empty

Most of the questions that follow are no easier in arbitrary renorming of Hilbert space than in reflexive Banach space

The Chebyshev problem (Klee 1961)

If every point in H has a unique nearest point in C is C convex?

Existence of nearest points (proximal boundary?)

Do some (many) points in H have a nearest point in C in every renorm of H ?

Second-order expansions in separable Hilbert space

If f is convex and continuous on H does f have a second order Taylor expansion at some (many) points?

Universal barrier functions in infinite dimensions

Is there an analogue for H of the universal barrier function that is so important in Euclidean space?

Convex Functions

The Chebyshev problem (Klee 1961) A set is Chebyshev if every point in H has a unique nearest point in C

Theorem If C is weakly closed and Chebyshev then C is convex. So in Euclidean space Chebyshev iff convex.

Four Euclidean variational proofs (BL 2005, Opt Letters 07, BV 2008)

1. **Brouwer's theorem** (Cheb. implies **sun** implies convex)
2. **Ekeland's theorem** (Cheb. implies **approx. convex** implies convex)
3. **Fenchel duality** (Cheb. iff d_C^2 is Frechet) use f^* smooth implies f convex for

$$\left(\frac{\iota_C + \|\cdot\|^2}{2}\right)^* = \frac{\|\cdot\|^2 + d_C^2}{2}$$

4. Inverse geometry also shows if there is a counter-example it can be a **Klee cavern** (Asplund) the closure of the complement of a convex body. **WEIRD**

- **Counterexamples** exist in incomplete inner product spaces. #2 seems most likely to work in Hilbert space.
- **Euclidean case** is due to Motzkin-Bunt

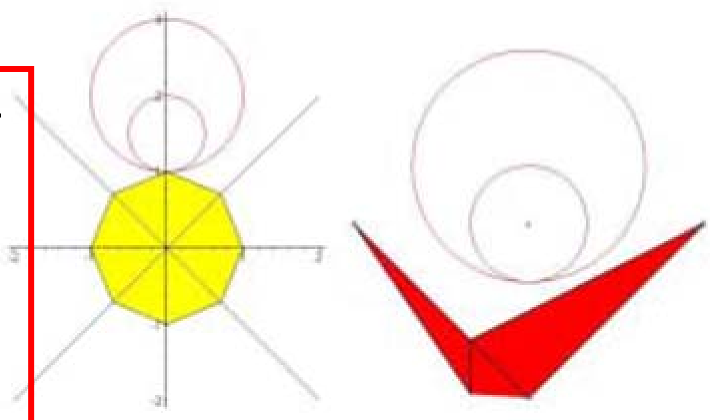


FIGURE 1. Suns and approximate convexity.

Existence of nearest points

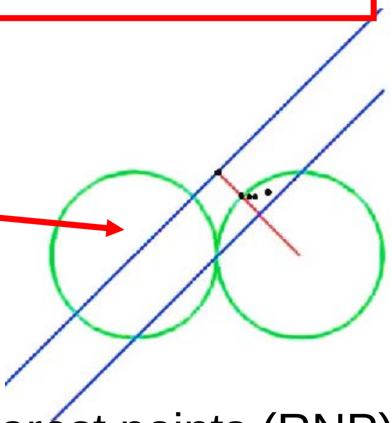
Do some (many) points in H have a nearest point in C in every renorm of H ?

Theorem (Lau-Konjagin, 76-86) A norm on a reflexive space is Kadec-Klee iff for every norm-closed C in X best approximations exist generically (densely) in X\ C.

Nicest proof is via dense existence of Frechet subderivatives

$$\varphi \in \partial_F d_C(x)$$

The KK property forces approximate minimizers to line up.



– There are non KK norms with proximal points dense in bdry C

– If C is closed and bounded then there are some points with nearest points (RNP)

– So a counterexample has to be a weird unbounded set in a rotten renorm (BF89, BZ 2005)

A norm is **Kadec-Klee** norm if weak and norm topologies agree on the unit sphere.

Hence all LUR norms are Kadec-Klee.

Second-order derivatives in separable Hilbert space

If f is continuous and convex on H does f have a (weak) second-order Taylor expansion at some (many) points?

Theorem (Alexandrov) In Euclidean space the points at which a continuous convex function admits a second-order Taylor expansion are full measure

- In Banach space, this is known to fail pretty completely unless one restricts the class of functions, say to nice integral functionals
- Is it possible in separable Hilbert space (BV 2009) that every such f has at least one point with a second-order Gateaux expansion?
- The goal is to build good jets and save as much as possible of extensions of lovely Euclidean results like

$$\partial\left[\frac{1}{2}\Delta_t^2 f(x)\right] = \Delta_t[\partial f](x).$$

Universal barrier functions in infinite dimensions

– Is there an analogue for H of the universal barrier function that is so important in Euclidean space?

Theorem (Nesterov-Nemirovskii) For any open convex set A in n-space, the function

$$F(x) := \lambda_N((A-x)^o)$$

is an essentially smooth, log-convex barrier function for A.

– This relies heavily on the existence of **Haar** measure (Lebesgue).
– Amazingly for A the semidefinite matrix cone we recover – log det, etc

In Hilbert space the only really nice examples I know are similar to:

$$\phi(T) := \mathrm{trace}(T) - \log(\det(I+T))$$

is a strictly convex Frechet differentiable barrier function for the Hilbert-Schmidt operators with I+T > 0.

We (JB-JV) are able to build barriers in great generality but not "universally".

The Chebyshev problem (Klee 1961)

If every point in H has a unique nearest point in C is C convex?

I HAVE A SUGGESTION FOR THESE TWO: DISTORTION

Existence of nearest points (proximal boundary?)

Do some (many) points in H have a nearest point in C in every renorm of H

Second-order expansions in separable Hilbert space

If f is convex continuous on H does f have a second order Taylor expansion at some (many) points?

I THINK PROGRESS FOR THESE TWO WILL BE INCREMENTAL

Universal barrier functions in infinite dimensions

Is there an analogue for H of the universal barrier function that is so important in Euclidean space?

A Banach space X is **distortable** if there is a renorm and $\lambda>1$ such that, for all infinite-dimensional subspaces $Y \subseteq X$,

$$\sup\{|y|/|x| \mid x, y \in Y, \|x\| = \|y\| = 1\} > \lambda.$$

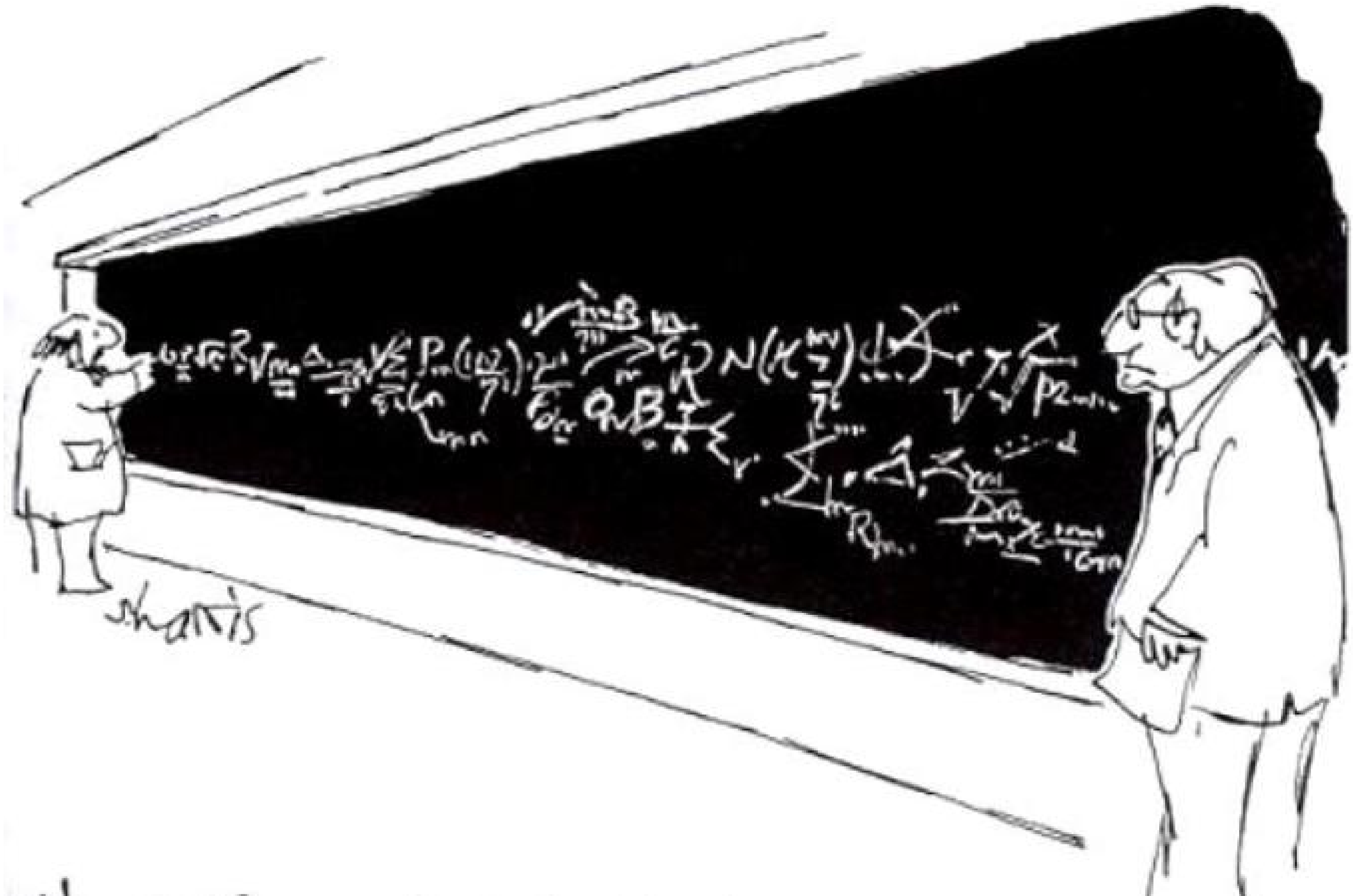
X is **arbitrarily distortable** if this can be done for all $\lambda>1$.

Theorem (Odell and Schlumprecht 93,94) Separable infinite-dimensional Hilbert space is arbitrarily distortable

Distortability of $l_2(N)$ is equivalent to existence of two separated sets in the sphere both intersecting every infinite-dimensional closed subspace of $l_2(N)$. Indeed, there is a sequence of (**asymptotically orthogonal**) subsets $(C_i)_{i=1}^{\infty}$ of the unit sphere such that (a) each set C_i intersects each infinite-dimensional closed subspace of and (b) as $i,j \to \infty$

$$\sup\{|\langle x, y\rangle| \mid x \in C_i, y \in C_j\} \to 0$$

These are such surprising sequences of sets that they should shed insight on the two proximality questions



"*Here's* where you made your mistake."

REFERENCES


Enigma

J.M. Borwein and Qiji Zhu, *Techniques of Variational Analysis,* CMS- Springer, 2005.

J.M. Borwein and A.S. Lewis, *Convex Analysis and Nonlinear Optimization. Theory and Examples*, CMS-Springer, Second extended edition, 2005.

J.M. Borwein and J.D. Vanderwerff, *Convex functions, constructions, characterizations and counterexamples*, Cambridge University Press, 2009.

"The object of mathematical rigor is to sanction and legitimize the conquests of intuition, and there was never any other object for it."

- J. Hadamard quoted at length in E. Borel, *Lecons sur la theorie des fonctions*, 1928.