CHAPTER 1
Aesthetics for the Working Mathematician

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> If my teachers had begun by telling me that mathematics was pure play with presuppositions, and wholly in the air, I might have become a good mathematician, because I am happy enough in the realm of essence. But they were over-worked drudges, and I was largely inattentive, and inclined lazily to attribute to incapacity in myself or to a literary temperament that dullness which perhaps was due simply to lack of initiation. (Santayana, 1944, p. 238)

Most research mathematicians neither think deeply about nor are terribly concerned with either pedagogy or the philosophy of mathematics. Nonetheless, as I hope to indicate, aesthetic notions have always permeated (pure and applied) mathematics. And the top researchers have always been driven by an aesthetic imperative. Many mathematicians over time have talked about the 'elegance' of certain proofs or the 'beauty' of certain theorems, but my analysis goes deeper: I aim to show how the aesthetic imperative interacts with utility and intuition, as well as indicate how it serves to shape my own mathematical experiences. These analyses, rather than being retrospective and passive, will provide a living account of the aesthetic dimension of mathematical work.

> We all believe that mathematics is an art. The author of a book, the lecturer in a classroom tries to convey the structural beauty of mathematics to his readers, to his listeners. In this attempt, he must always fail. Mathematics is logical to be sure; each conclusion is drawn from previously derived statements. Yet the whole of it, the real piece of art, is not linear; worse than that, its perception should be instantaneous. We all have experienced on some rare occasions the feeling of elation in realizing that we have enabled our listeners to see at a moment's glance the whole architecture and all its ramifications. (Emil Artin, in Murty, 1988, p. 60)

I shall similarly argue for aesthetics before utility. Through a suite of examples drawn from my own research and interests, I aim to illustrate how and what this means on the front lines of research. I also will argue that the opportunities to evoke the mathematical aesthetic in research and teaching are almost boundless – at all levels of the curriculum. (An excellent middle-school illustration, for instance, is described in Sinclair, 2001.)

In part, this is due to the increasing power and sophistication of visualisation, geometry, algebra and other mathematical software. Indeed, by drawing on 'hot topics' as well as 'hot methods' (i.e. computer technology), I also provide a contemporary perspective which I hope will complement the more classical contributions to our understanding of the mathematical aesthetic offered by writers such as G. H. Hardy and Henri Poincaré (as discussed in Chapter α).

Webster's dictionary (1993, p. 19) first provides six different meanings of the word 'aesthetic', used as an adjective. However, I want to react to these two definitions of 'aesthetics', used as a noun:

1. The branch of philosophy dealing with such notions as the beautiful, the ugly, the sublime, the comic, etc., as applicable to the fine arts, with a view to establishing the meaning and validity of critical judgments concerning works of art, and the principles underlying or justifying such judgments.
2. The study of the mind and emotions in relation to the sense of beauty.

Personally, for my own definition of the aesthetic, I would require (unexpected) simplicity or organisation in apparent complexity or chaos – consistent with views of Dewey (1934), Santayana (1944) and others. I believe we need to integrate this aesthetic into mathematics education at every level, so as to capture minds for other than utilitarian reasons. I also believe detachment to be an important component of the aesthetic experience: thus, it is important to provide some curtains, stages, scaffolding and picture frames – or at least their mathematical equivalents. Fear of mathematics certainly does not hasten an aesthetic response.

Gauss, Hadamard and Hardy

Three of my personal mathematical heroes, very different individuals from different times, all testify interestingly on the aesthetic and the nature of mathematics.

Gauss

Carl Friedrich Gauss is claimed to have once confessed, "I have had my results for a long time, but I do not yet know how I am to arrive at them" (in Arber, 1954, p. 47). [1] One of Gauss's greatest discoveries, in 1799, was the relationship between the lemniscate sine function and the arithmetic–geometric mean iteration. This was based on a purely computational observation. The young Gauss wrote in his diary that "a whole new field of analysis will certainly be opened up" (*Werke,* X, p. 542; in Gray, 1984, p. 121).

He was right, as it pried open the whole vista of nineteenth-century elliptic and modular function theory. Gauss's specific discovery, based on tables of integrals provided by Scotsman James Stirling, was that the reciprocal of the integral

$$\frac{2}{\pi}\int_0^1 \frac{1}{\sqrt{1-t^4}}\,dt$$

agreed numerically with the limit of the rapidly convergent iteration given by setting $a_0 := 1$, $b_0 := \sqrt{2}$ and then computing:

$$a_{n+1} := \frac{a_n + b_n}{2}, \qquad b_{n+1} := \sqrt{a_n b_n}$$

It transpires that the two sequences $\{a_n\}$, $\{b_n\}$ have a common limit of 1.19814023473559220 74…

Which object, the integral or the iteration, is the more familiar and which is the more elegant – then and now? Aesthetic criteria change with time (and within different cultures) and these changes manifest themselves in the concerns and discoveries of mathematicians. Gauss's discovery of the relationship between the lemniscate function and the arithmetic–geometric mean iteration illustrates how the traditionally preferred 'closed form' (here, the integral form) of equations have yielded centre stage, in terms both of elegance and utility, to recursion. This parallels the way in which biological and computational metaphors (even 'biology envy') have now replaced Newtonian mental images, as described and discussed by Richard Dawkins (1986) in his book *The Blind Watchmaker.*

In fact, I believe that mathematical thought patterns also change with time and that these in turn affect aesthetic criteria – not only in terms of what counts as an interesting problem, but also what methods the mathematician can use to approach these problems, as well as how a mathematician judges their solutions. As mathematics becomes more 'biological', and more computational, aesthetic criteria will continue to change.

Hadamard

A constructivist, experimental and aesthetically-driven rationale for mathematics could hardly do better than to start with Hadamard's claim that:

> The object of mathematical rigor is to sanction and legitimize the conquests of intuition, and there was never any other object for it. (in Pólya, 1981, p. 127)

Jacques Hadamard was perhaps the greatest mathematician other than Poincaré to think deeply and seriously about cognition in mathematics. He is quoted as saying, "in arithmetic, until the seventh grade, I was last or nearly last" (in MacHale, 1993, p. 142). Hadamard was co-prover (independently with Charles de la Vallée Poussin, in 1896) of the Prime Number theorem (the number of primes not exceeding n is asymptotic to $n/\log n$), one of the culminating results of nineteenth-century mathematics and one that relied on much preliminary computation and experimentation. He was also the author of *The Psychology of Invention in the Mathematical Field* (1945), a book that still rewards close inspection.

Hardy's *Apology*

Correspondingly, G. H. Hardy, the leading British analyst of the first half of the twentieth century, was also a stylish author who wrote compellingly in defence of pure mathematics. He observed that:

> All physicists and a good many quite respectable mathematicians are contemptuous about proof. (1945/1999, pp. 15-16)

His memoir, entitled *A Mathematician's Apology,* provided a spirited defence of beauty over utility:

> Beauty is the first test. There is no permanent place in the world for ugly mathematics. (1940, p. 84)

That said, although the sentiment behind it being perfectly understandable from an anti-war mathematician in war-threatened Britain, Hardy's claim that real mathematics is almost wholly useless has been over-played and, to my mind, is now very dated, given the importance of cryptography and other pieces of algebra and number theory devolving from very pure study.

In his tribute to Srinivasa Ramanujan entitled *Ramanujan: Twelve Lectures on Subjects Suggested by His Life and Work,* Hardy (1945/1999) offered the so-called 'Skewes number' as a "striking example of a false conjecture" (p. 15). The logarithmic integral function, written Li(x), is specified by:

$$Li(x) = \int_0^x \frac{1}{\log(t)}\, dt$$

Li(x) provides a very good approximation to the number of primes that do not exceed x. For example, Li(10^8) = 5,762,209.375…, while the number of primes not exceeding 10^8 is 5,761,455. It was conjectured that the inequality

> Li(x) > the number of primes not exceeding x

holds for all x and, indeed, it does so for many x. In 1933, Skewes showed the first explicit crossing occurs before $10^{10^{10^{34}}}$. This has been reduced to a relatively tiny number, a mere 10^{1167} (and, most recently, even lower), though one still vastly beyond direct computational reach.

Such examples show forcibly the limits on numerical experimentation, at least of a naïve variety. Many readers will be familiar with the 'law of large numbers' in statistics. Here, we see an instance of what some number theorists (e.g. Guy, 1988) call the 'strong law of small numbers': *all small numbers are special,* many are primes and direct experience is a poor guide. And sadly (or happily, depending on one's attitude), even 10^{1167} may be a small number.

Research Motivations and Goals

As a computational and experimental pure mathematician, my main goal is *insight.* Insight demands speed and, increasingly, parallelism (see Borwein and Borwein, 2001, on the challenges for mathematical computing). The mathematician's 'aesthetic buzz' comes not only from simply contemplating a beautiful piece of mathematics, but, additionally, from achieving insight. The computer, with its capacities for visualisation and computation, can encourage the aesthetic buzz of insight, by offering the mathematician the possibility of visual contact with mathematics and by allowing the mathematician to experiment with, and thus to become intimate with, mathematical ideas, equations and objects.

What is 'easy' is changing and I see an exciting merging of disciplines, levels and collaborators. Mathematicians are more and more able to:

- marry theory and practice, history and philosophy, proofs and experiments;
- match elegance and balance with utility and economy;
- inform all mathematical modalities computationally – analytic, algebraic, geometric and topological.

This is leading us towards what I term an *experimental m**a**thodology* as a philosophy and a practice (Borwein and Corless, 1999). This methodology is based on the following three approaches:

- meshing computation and mathematics, so that intuition is acquired;
- visualisation – three is a lot of dimensions and, nowadays, we can exploit pictures, sounds and haptic stimuli to get a 'feel' for relationships and structures (see also Chapter 7);
- 'exception barring' and 'monster barring' (using the terms of Lakatos, 1976).

Two particularly useful components of this third approach include graphical and randomised checks. For example, comparing $2\sqrt{y} - y$ and $-\sqrt{y}\ln(y)$ (for $0 < y < 1$) pictorially is a much more rapid way to divine which is larger than by using traditional analytic methods. Similarly, randomised checks of equations, inequalities, factorisations or primality can provide enormously secure knowledge or counter-examples when deterministic methods are doomed. As with traditional mathematical methodologies, insight and certainty are still highly valued, yet achieved in different ways.

Pictures and symbols

> If I can give an abstract proof of something, I'm reasonably happy. But if I can get a concrete, computational proof and actually produce numbers I'm much happier. I'm rather an addict of doing things on the computer, because that gives you an explicit criterion of what's going on. I have a visual way of thinking, and I'm happy if I can see a picture of what I'm working with. (John Milnor, in Regis, 1986, p. 78)

I have personally had this experience, in the context of studying the distribution of zeroes of the Riemann zeta function. Consider more explicitly the following image (see Figure 1), which shows the densities of zeroes for polynomials in powers of x with –1 and 1 as coefficients (they are manipulable at: www.cecm.sfu.ca/interfaces/). All roots of polynomials, up to a given degree, with coefficients of either –1 or 1 have been calculated by permuting through all possible combinations of polynomials, then solving for the roots of each. These roots are then plotted on the complex plane (around the origin).

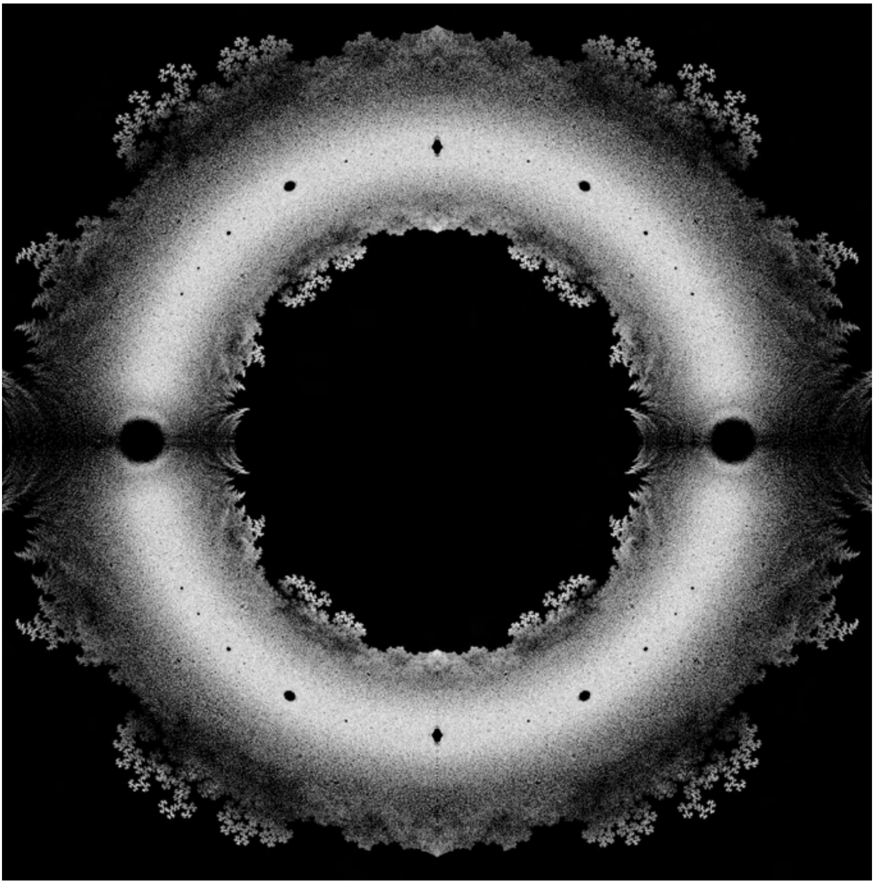

Figure 1: Density of zeroes for polynomials with coefficients of –1 and 1

In this case, graphical output from a computer allows a level of insight no amount of numbers could.

Some colleagues and I have been building educational software with these precepts embedded, such as *LetsDoMath* (see: www.mathresources.com). The intent is to challenge students honestly (e.g. through allowing subtle explorations within John Conway's 'Game of Life'), while making things tangible (e.g. 'Platonic solids' offers virtual manipulables that are more robust and expressive than the standard classroom solids).

Evidently, though, symbols are often more reliable than pictures. The picture opposite purports to give evidence that a solid can fail to be polyhedral at only one point. It shows the steps up to pixel level of inscribing a

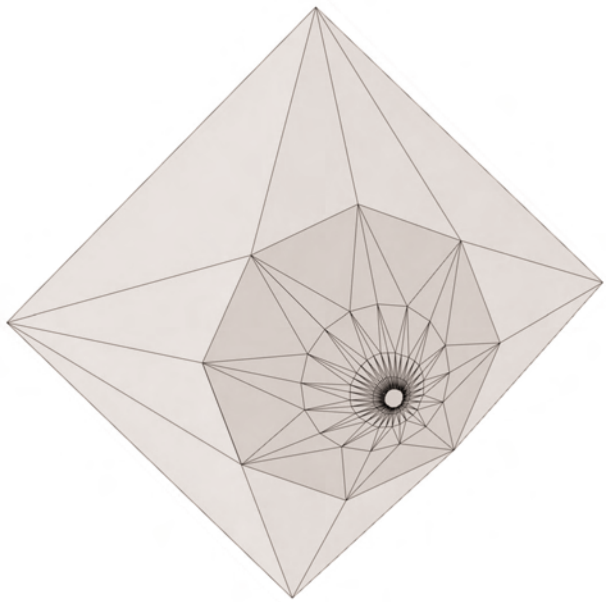

Figure 2: A misleading picture

regular 2^{n+1}-gon at height 2^{1-n}. However, ultimately, such a construction fails and produces a right circular cone. The false evidence in this picture held back a research project for several days – and might have derailed it.

Two Things about $\sqrt{2}$ and One Thing about π

Remarkably, one can still find new insights in the oldest areas. I discuss three examples of this. The first involves a new proof of the irrationality of $\sqrt{2}$ and the way in which it provides insight into a previously known result. The second invokes the strange interplay between rational and irrational numbers. Finally, the third instance reveals how the computer can make opaque some properties that were previously transparent, and *vice versa*.

Irrationality

Below is a graphical representation of Tom Apostol's (2000) lovely new geometric proof of the irrationality of $\sqrt{2}$. This example may seem routine at first, with respect to the literature on the mathematical aesthetic. Writers such as Hardy (1940), King (1992) and Wells (1990) have also talked about the beauty of quadratics such as $\sqrt{2}$. These writers have emphasised aesthetic criteria (such as economy and unexpectedness) that contribute to that judgement of beauty. On the other hand, Apostol's new proof, prefigured in others, shows how aesthetics can also serve to *motivate* mathematical inquiry.

PROOF Consider the *smallest* right-angled isosceles triangle with integer sides. Circumscribe a circle of length equal to the vertical side and construct the tangent to the circle where the hypotenuse cuts it (see Figure 3). The *smaller* isosceles triangle once again has integer sides.

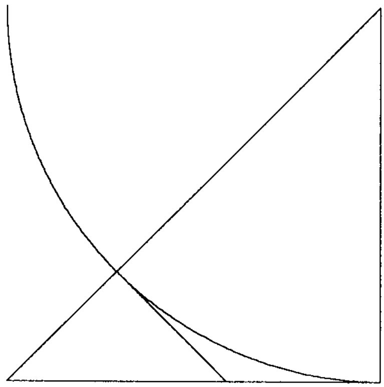

Figure 3: The square root of two is irrational

The proof is lovely because it offers new insight into a result that was first proven over two thousand years ago. It also verges on being a 'proof without words' (Nelsen, 1993), proofs which are much admired – yet infrequently encountered and not always trusted – by mathematicians (see Brown, 1999). Apostol's work demonstrates how mathematicians are not only motivated to find ground-breaking results, but that they also strive for better ways to say things or to show things, as Gauss was surely doing when he worked out his fourth, fifth and sixth proof of the law of quadratic reciprocity.

Rationality

By a variety of means, including the one above, we know that the square root of two is irrational. But mathematics is always full of surprises: $\sqrt{2}$ can also *make* things rational (a case of two wrongs making a right?).

$$\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^{(\sqrt{2}\sqrt{2})} = \sqrt{2}^{2} = 2.$$

Hence, by the principle of the excluded middle:

$$\text{Either} \quad \sqrt{2}^{\sqrt{2}} \in Q \quad \text{or} \quad \sqrt{2}^{\sqrt{2}} \notin Q.$$

In either case, we can deduce that there are irrational numbers a and b with a^b rational. But how do we know which ones? One may build a whole mathematical philosophy project around this. Yet, as *Maple* (the computer algebra system) confirms:

setting $\alpha := \sqrt{2}$ and $\beta := 2\ln_2 3$ yields $\alpha^\beta = 3$.

This illustrates nicely that verification is often easier than discovery. (Similarly, the fact that multiplication is easier than factorisation is at the base of secure encryption schemes for e-commerce.)

π and two integrals

Even *Maple* knows $\pi \neq 22/7$, since:

$$0 < \int_0^1 \frac{(1-x)^4\, x^4}{1+x^2}\, dx = \frac{22}{7} - \pi.$$

Nevertheless, it would be prudent to ask 'why' *Maple* is able to perform the evaluation and whether to trust it. In contrast, *Maple* struggles with the following *sophomore's dream:*

$$\int_0^1 \frac{1}{x^x}\, dx = \sum_{n=1}^{\infty} \frac{1}{n^n}.$$

Students asked to confirm this typically mistake numerical validation for symbolic proof.

Again, we see that computing adds reality, making the abstract concrete, and makes some hard things simple. This is strikingly the case with Pascal's Triangle. Figure 4 (from: www.cecm.sfu.ca/interfaces/) affords an emphatic example where deep fractal structure is exhibited in the elementary binomial coefficients.

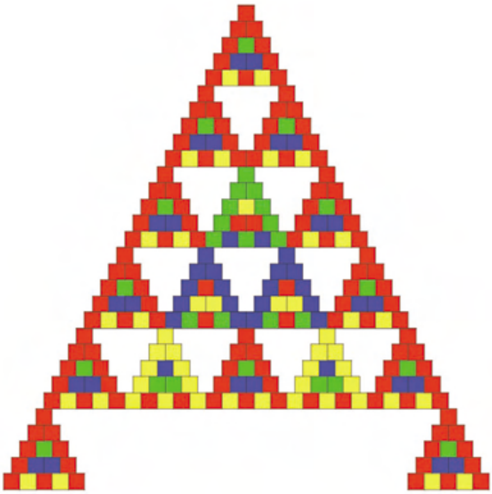

Figure 4: Thirty rows of Pascal's triangle (modulo five)

Berlinski (1997) comments on some of the effects of such visual–experimental possibilities in mathematics:

> The computer has in turn changed the very nature of mathematical experience, suggesting for the first time that mathematics, like physics, may yet become an empirical discipline, a place where things are discovered because they are seen. (p. 39)

Berlinski (1995) had earlier suggested, in his book *A Tour of the Calculus,* that there will be long-term effects:

> The body of mathematics to which the calculus gives rise embodies a certain swashbuckling style of thinking, at once bold and dramatic, given over to large intellectual gestures and indifferent, in large measure, to any very detailed description of the world. It is a style that has shaped the physical but not the biological sciences, and its success in Newtonian mechanics, general relativity, and quantum mechanics is among the miracles of mankind. But the era in thought that the calculus made possible is coming to an end. Everyone feels this is so, and everyone is right. (p. xiii)

π and Its Friends

My research on π with my brother, Peter Borwein, also offers aesthetic and empirical opportunities. In this example, my personal fascinations provide compelling illustrations of an aesthetic imperative in my own work. I first discuss the algorithms I have co-developed to compute the digits of π. These algorithms, which consist of simple algebraic equations, have made it possible for researchers to compute its first 2^{36} digits. I also discuss some of the methods and algorithms I have used to gain insight into relationships involving π.

A quartic algorithm (Borwein and Borwein, 1984)

The next algorithm I present grew out of work of Ramanujan. Set $a_0 = 6 - 4\sqrt{2}$ and $y_0 = \sqrt{2} - 1$. Iterate:

$$(1) \qquad y_{k+1} = \frac{1-(1-y_k^4)^{1/4}}{1+(1-y_k^4)^{1/4}}$$

$$(2) \qquad a_{k+1} = a_k(1+y_{k+1})^4 \, b - 2^{2k+3}\, y_{k+1}(1+y_{k+1}+y_{k+1}^2)$$

Then the sequence $\{a_k\}$ converges *quartically* to 1/π.

There are nineteen pairs of simple algebraic equations (1, 2) as *k* ranges from 0 to 18. After seventeen years, this still gives me an aesthetic buzz. Why? With less than one page of equations, I have a tool for computing a number that differs from π (the most celebrated transcendental number) only after seven hundred billion digits. It is not only the economy of the tool that delights me, but also the stirring idea of 'almost-ness' – that even after seven hundred billion digits we still cannot nail π. The difference might seem trivial, but mathematicians know that it is not and they continue to improve their algorithms and computational tools.

This iteration has been used since 1986, with the Salamin–Brent scheme, by David Bailey (at the Lawrence Berkeley Labs) and by Yasumasa Kanada (in Tokyo). In 1997, Kanada computed over 51 billion digits on a Hitachi supercomputer (18 iterations, 25 hrs on 210 cpus). His penultimate world record was 2^{36} digits in April, 1999. A billion (2^{30}) digit computation has been performed on a single Pentium II PC in less than nine days. The present record is 1.24 trillion digits, computed by Kanada in December 2002 using quite different methods, and is described in my new book, co-authored with David Bailey (2003).

The fifty-billionth decimal digit of π *or* of 1/π is 04<u>2</u>! And after eighteen billion digits, the string 0123456789 has finally appeared and so Brouwer's famous intuitionist example *now* converges. [2] (Details such as this about π can be found at: www.cecm.sfu.ca/personal/jborwein/pi_cover.html.) From a probability perspective, such questions may seem uninteresting, but they continue to motivate and amaze mathematicians.

A further taste of Ramanujan

G. N. Watson, in discussing his response to similar formulae of the wonderful Indian mathematical genius Srinivasa Ramanujan, describes:

> a thrill which is indistinguishable from the thrill which I feel when I enter the Sagrestia Nuova of the Capelle Medicee and see before me the austere beauty of [the four statues representing] 'Day,' 'Night,' 'Evening,' and 'Dawn' which Michelangelo has set over the tombs of Giuliano de' Medici and Lorenzo de' Medici. (in Chandrasekhar, 1987, p. 61)

One of these is Ramanujan's remarkable formula, based on the elliptic and modular function theory initiated by Gauss.

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{k=0}^{\infty} \frac{(4k)!(1103 + 26390k)}{(k!)^4\, 396^{4k}}.$$

Each term of this series produces an additional *eight* correct digits in the result – and only the ultimate multiplication by $\sqrt{2}$ is not a *rational* operation. Bill Gosper used this formula to compute seventeen million terms of the continued fraction for π in 1985. This is of interest, because we still cannot prove that the continued fraction for π is unbounded. Again, everyone *knows* that this is true.

That said, Ramanujan preferred related explicit forms for approximating π, such as the following:

$$\frac{\log(640320^3)}{\sqrt{163}} = 3.141592653589793\underline{0164} \approx \pi.$$

This equation is correct until the underlined places. *Inter alia,* the number e^{π} is the easiest transcendental to fast compute (by elliptic methods). One 'differentiates' $e^{-\pi t}$ to obtain algorithms such as the one above for π, via the arithmetic–geometric mean.

Integer relation detection

I make a brief digression to describe what integer relation detection methods do. (These may be tried at: www.cecm.sfu.ca/projects/IntegerRelations/.) I then apply them to π (see Borwein and Lisonek, 2000).

> DEFINITION A vector $(x_1, x_2, \ldots, x_n)$ of real numbers *possesses an integer relation,* if there exist integers a_i (not all zero) with:
>
> $$a_1x_1 + a_2x_2 + \ldots + a_nx_n = 0$$
>
> PROBLEM Find a_i if such integers exist. If not, obtain lower 'exclusion' bounds on the size of possible a_i.
>
> SOLUTION For $n = 2$, *Euclid's algorithm* gives a solution. For $n \geq 3$, Euler, Jacobi, Poincaré, Minkowski, Perron and many others sought methods. The *first general algorithm* was found (in 1977) by Ferguson and Forcade. Since 1977, one has many variants: I will mainly be talking about two algorithms, LLL ('Lenstra, Lenstra and Lovász'; also available in *Maple* and *Mathematica*) and PSLQ ('Partial sums using matrix LQ decomposition', 1991; *parallelised,* 1999).

Integer relation detection was recently ranked among:

> the 10 algorithms with the greatest influence on the development and practice of science and engineering in the 20th century. (Dongarra and Sullivan, 2000, p. 22)

It could be interesting for the reader to compare these algorithms with the theorems on the list of the most 'beautiful' theorems picked out by Wells (1990) in his survey, in terms of criteria such as applicability, unexpectedness and fruitfulness.

Determining whether or not a number is algebraic is one problem that can be attacked using integer relation detection. Asking about algebraicity is handled by computing α to sufficiently high precision ($O(n = N^2)$) and applying LLL or PSLQ to the vector $(1, \alpha, \alpha^2, \ldots, \alpha^{N-1})$. Solution integers a_i are coefficients of a polynomial likely satisfied by α. If one has computed α to $n + m$ digits and run LLL using n of them, one has m digits to confirm the result heuristically. I have never seen this method return an honest 'false positive' for $m > 20$, say. If no relation is found, exclusion bounds are obtained, saying, for example, that any polynomial of degree less than N must have the Euclidean norm of its coefficients in excess of L (often astronomical). If we know or suspect an identity exists, then integer relations methods are very powerful. Let me illustrate this in the context of approximating π.

Machin's formula

We use *Maple* to look for the linear dependence of the following quantities:

[arctan(1), arctan(1/5), arctan(1/239)]

and 'recover' [1, –4, 1]. In other words, we can establish the following equation:

$$\pi/4 = 4\arctan(1/5) - \arctan(1/239).$$

Machin's formula was used on all serious computations of π from 1706 (a hundred digits) to 1973 (a million digits), as well as more abstruse but similar formulae used in creating Kanada's present record. After 1980, the methods described above started to be used instead.

Dase's formula

Again, we use *Maple* to look for the linear dependence of the following quantities:

[π/4, arctan(1/2), arctan(1/5), arctan(1/8)].

and recover [–1, 1, 1, 1]. In other words, we can establish the following equation:

$$\pi/4 = \arctan(1/2) + \arctan(1/5) + \arctan(1/8).$$

This equation was used by Dase to compute two hundred digits of π in his head in perhaps the greatest feat of mental arithmetic ever – 1/8 is apparently better than 1/239 (as in Machin's formula) for this purpose.

Who was Dase? Another burgeoning component of modern research and teaching life is having access to excellent data bases, such as the MacTutor History Archive maintained at: www-history.mcs.st-andrews.ac.uk (alas, not all sites are anywhere near so accurate and informative as this one). One may find details there on almost all of the mathematicians appearing in this chapter. I briefly illustrate its value by showing verbatim what it says about Dase.

> Zacharias Dase (1824–1861) had incredible calculating skills but little mathematical ability. He gave exhibitions of his calculating powers in Germany, Austria and England. While in Vienna in 1840 he was urged to use his powers for scientific purposes and he discussed projects with Gauss and others.
>
> Dase used his calculating ability to calculate π to 200 places in 1844. This was published in Crelle's Journal for 1844. Dase also constructed 7 figure log tables and produced a table of factors of all numbers between 7 000 000 and 10 000 000.

> Gauss requested that the Hamburg Academy of Sciences allow Dase to devote himself full-time to his mathematical work but, although they agreed to this, Dase died before he was able to do much more work.

Pentium farming

I finish this sub-section with another result obtained through integer relations methods or, as I like to call it, 'Pentium farming'. Bailey, Borwein and Plouffe (1997) discovered a series for π (and corresponding ones for some other *polylogarithmic* constants), which somewhat disconcertingly allows one to compute hexadecimal digits of π *without* computing prior digits. (This feels like magic, being able to tell the seventeen-millionth digit of π, say, without having to calculate the ones before it; it is like seeing God reach her hand deep into π.)

The algorithm needs very little memory and no multiple precision. The running time grows only slightly faster than linearly in the order of the digit being computed. The key, found by PSLQ as described above, is:

$$\pi = \sum_{k=0}^{\infty} \left(\frac{1}{16}\right)^k \left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6}\right).$$

Knowing an algorithm would follow, Bailey, Borwein and Plouffe spent several months hunting by computer for such a formula. Once found, it is easy to prove in *Mathematica,* in *Maple* or by hand – and provides a very nice calculus exercise.

This was a most successful case of *reverse mathematical engineering* and is entirely practicable. In September 1997, Fabrice Bellard (at INRIA) used a variant of this formula to compute one hundred and fifty-two binary digits of π, starting at the *trillionth* (10^{12}) place. This took twelve days on twenty work-stations working in parallel over the internet. In August 1998, Colin Percival (Simon Fraser University, age 17) finished a 'massively parallel' computation of the *five-trillionth bit* (using twenty-five machines at roughly ten times the speed of Bellard). In *hexadecimal notation,* he obtained:

07E45 733CC790B5B5979.

The corresponding binary digits of π starting at the forty-trillionth bit are:

0 0000 1111 1001 1111.

By September 2000, the quadrillionth bit had been found to be the digit 0 (using 250 cpu years on a total of one thousand, seven hundred and thirty-four machines from fifty-six countries). Starting at the 999,999,999,999,997th bit of π, we find:

11100 0110 0010 0001 0110 1011 0000 0110.

Solid and Discrete Geometry – and Number Theory

Although my own primary research interests are in numerical, classical and functional analysis, I find that the fields of solid and discrete geometry, as well as number theory, offer many examples of the kinds of concrete, insightful ideas I value. In the first example, I argue for the computational affordances available to study of solid geometry. I then discuss the genesis of an elegant proof in discrete geometry. Finally, I illustrate a couple of deep results in partition theory.

de Morgan

Augustus de Morgan, one of the most influential educators of his period and first president of the London Mathematical Society, wrote:

> Considerable obstacles generally present themselves to the beginner, in studying the elements of Solid Geometry, from the practice which has hitherto uniformly prevailed in this country, of never submitting to the eye of the student, the figures on whose properties he is reasoning, but of drawing perspective representations of them upon a plane. [...] I hope that I shall never be obliged to have recourse to a perspective drawing of any figure whose parts are not in the same plane. (in Rice, 1999, p. 540)

His comment illustrates the importance of concrete experiences with mathematical objects, even when the ultimate purpose is to abstract. There is a sense in which insight lies in physical manipulation. I imagine that de Morgan would have been happier using JavaViewLib (see: www.cecm.sfu.ca/interfaces/). This is Konrad Polthier's modern version of Felix Klein's famous set of geometric models. Correspondingly, a modern interactive version of Euclid is provided by *Cinderella* (a software tool which is largely comparable with *The Geometer's Sketchpad;* the latter is discussed in detail in Chapter 7 of this volume). Klein, like de Morgan, was equally influential as an educator and as a researcher.

Sylvester's theorem

Sylvester's theorem is worth mentioning because of its elegant visual proof, but also because of Sylvester's complex relationship to geometry: "The early study of Euclid made me a hater of geometry" (quoted in MacHale, 1993, p. 135). James Joseph Sylvester, who was the second president of the London Mathematical Society, may have hated Euclidean geometry, but discrete geometry (now much in fashion under the name 'computational geometry', offering another example of very useful pure mathematics) was different. His strong, emotional preference nicely illustrates how the aesthetic is involved in a mathematician's choice of fields.

Sylvester (1893) came up with the following conjecture, which he posed in *The Educational Times:*

THEOREM Given n non-collinear points in the plane, then there is always at least one (*elementary* or *proper*) line going through exactly two points of the set.

Sylvester's conjecture was, so it seems, forgotten for fifty years. It was first established – 'badly', in the sense that the proof is much more complicated – by T. Grünwald (Gallai) in 1933 (see editorial comment in Steinberg, 1944) and also by Paul Erdös. Erdös, an atheist, named 'the Book' the place where God keeps aesthetically perfect proofs. L. Kelly's proof (given below), which Erdös accepted into 'the Book', was actually published by Donald Coxeter (1948) in the *American Mathematical Monthly.* This is a fine example of how the archival record may rapidly get obscured.

PROOF Consider the point *closest* to a line it is not on and then suppose that line has three points on it (the horizontal line). The middle of those three points is clearly *closer* to the other line.

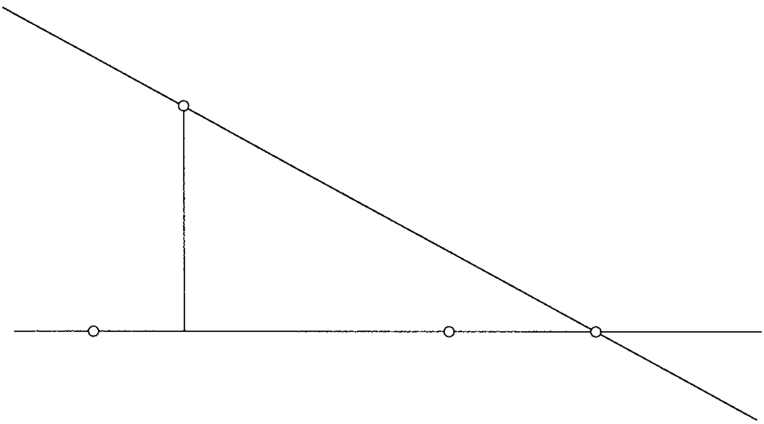

Figure 5. Kelly's proof from 'the Book'

As with Apostol's proof of the irrationality of $\sqrt{2}$, we can see the power of the right *minimal configuration.* Aesthetic appeal often comes from having this characteristic: that is, its appeal stems from being able to reason about an unknown number of objects by identifying a restricted view that captures all the possibilities. This is a process that is not so very different from that powerful method of proof known as mathematical induction.

Another example worth mentioning in this context (one that belongs in 'the Book') is Niven's (1947) marvellous (simple and short), half-page proof that π is *irrational* (see: www.cecm.sfu.ca/personal/jborwein/pi.pdf).

Partitions and patterns

Another subject that can be made highly accessible through experimental methods is additive number theory, especially *partition theory.* The number of *additive partitions* of q, $P(q)$, is generated by the following equation:

$$P(q) := \prod_{n \geq 1} (1 - q^n)^{-1}$$

Thus, P(5) = 7, since:

5 = 4 + 1 = 3 + 2 = 3 + 1 + 1 = 2 + 2 + 1

= 2 + 1 + 1 + 1 = 1 + 1 + 1 + 1 + 1

QUESTION How hard is P(q) to compute? Consider this question as it might apply in 1900 (for Major MacMahon, the father of our modern combinatorial analysis) and in 2000 (for *Maple*).

ANSWER Seconds for *Maple,* months for MacMahon. It is interesting to ask if development of the beautiful asymptotic analysis of partitions by Hardy, Ramanujan and others would have been helped or impeded by such facile computation.

Ex-post-facto algorithmic analysis can be used to facilitate independent student discovery of *Euler's pentagonal number theorem.*

$$\prod_{n \geq 1} (1 - q^n) = \sum_{n=-\infty}^{\infty} (-1)^n \, q^{(3n+1)n/2}$$

Ramanujan used MacMahon's table of P(q) to intuit remarkable and deep congruences, such as:

$P(5n + 4) \equiv 0 \pmod 5$

$P(7n + 5) \equiv 0 \pmod 7$

and

$P(11n + 6) \equiv 0 \pmod{11}$

from data such as:

$$\begin{aligned} P(q) \;&=\; 1 + q + 2q^2 + 3q^3 + 5q^4 + 7q^5 + 11q^6 + 15q^7 + 22q^8 + 30q^9 \\ &+ 42q^{10} + 56q^{11} + 77q^{12} + 101q^{13} + 135q^{14} + 176q^{15} + 231q^{16} \\ &+ 297q^{17} + 385q^{18} + 490q^{19} + 627q^{20}b + 792q^{21}b + 792q^{21}b \\ &+ 1002q^{22} + 1255q^{23} + \ldots \end{aligned}$$

Nowadays, if introspection fails, we can recognise the *pentagonal numbers* occurring above in Sloane and Plouffe's on-line *Encyclopaedia of Integer Sequences* (see: www.research.att.com/personal/njas/sequences/eisonline.html). Here, we see a very fine example of *Mathematics: the Science of Patterns,* which is the title of Keith Devlin's (1994) book. And much more may similarly be done.

Some Concluding Discussion

In recent years, there have been revolutionary advances in cognitive science – advances that have a profound bearing on our understanding of mathematics. (More serious curricular insights should come from neuro-biology – see Dehaene *et al.,* 1999.) Perhaps the most profound of these new insights are the following, presented in Lakoff and Nuñez (2000).

1. *The embodiment of mind* The detailed nature of our bodies, our brains and our everyday functioning in the world structures human concepts and human reason. This includes mathematical concepts and mathematical reason. (See also Chapter 6.)
2. *The cognitive unconscious* Most thought is unconscious – not repressed in the Freudian sense, but simply inaccessible to direct conscious introspection. We cannot look directly at our conceptual systems and at our low-level thought processes. This includes most mathematical thought.
3. *Metaphorical thought* For the most part, human beings conceptualise abstract concepts in concrete terms, using ideas and modes of reasoning grounded in sensori-motor systems. The mechanism by which the abstract is comprehended in terms of the concept is called *conceptual metaphor.* Mathematical thought also makes use of conceptual metaphor: for instance, when we conceptualise numbers as points on a line.

Lakoff and Nuñez subsequently observe:

> What is particularly ironic about this is it follows from the empirical study of numbers as a product of mind that it is natural for people to believe that numbers are not a product of mind! (p. 81)

I find their general mathematical schema pretty persuasive but their specific accounting of mathematics forced and unconvincing (see also Schiralli and Sinclair, 2003). Compare this with a more traditional view, one that I most certainly espouse:

> The price of metaphor is eternal vigilance. (Arturo Rosenblueth and Norbert Wiener, in Lewontin, 2001, p. 1264)

Form follows function

> The waves of the sea, the little ripples on the shore, the sweeping curve of the sandy bay between the headlands, the outline of the hills, the shape of the clouds, all these are so many riddles of form, so many problems of morphology, and all of them the physicist can more or less easily read and adequately solve [...] (Thompson, 1917/1968, p. 10)

A century after biology started to think physically, how will mathematical thought patterns change?

> The idea that we could make biology mathematical, I think, perhaps is not working, but what is happening, strangely enough, is that maybe mathematics will become biological! (Chaitin, 2002)

To appreciate Greg Chaitin's comment, one has only to consider the metaphorical or actual origin of current 'hot topics' in mathematics research: simulated annealing ('protein folding'); genetic algorithms ('scheduling problems'); neural networks ('training computers'); DNA computation ('travelling salesman problems'); quantum computing ('sorting algorithms').

Humanistic philosophy of mathematics

However extreme the current paradigm shifts are and whatever the outcome of these discourses, mathematics is and will remain a uniquely human undertaking. Indeed, Reuben Hersh's (1995) full argument for a humanist philosophy of mathematics, as paraphrased below, becomes all the more convincing in this setting.

1. *Mathematics is human* It is part of and fits into human culture. It does not match Frege's concept of an abstract, timeless, tenseless and objective reality (see Resnik, 1980, and Chapter 8).
2. *Mathematical knowledge is fallible* As in science, mathematics can advance by making mistakes and then correcting or even re-correcting them. The 'fallibilism' of mathematics is brilliantly argued in Imre Lakatos's (1976) *Proofs and Refutations.*
3. *There are different versions of proof or rigour* Standards of rigour can vary depending on time, place and other things. Using computers in formal proofs, exemplified by the computer-assisted proof of the four-colour theorem in 1977, is just one example of an emerging, non-traditional standard of rigour.
4. *Aristotelian logic is not always necessarily the best way of deciding* Empirical evidence, numerical experimentation and probabilistic proof can all help us decide what to believe in mathematics.
5. *Mathematical objects are a special variety of a social–cultural–historical object* Contrary to the assertions of certain post-modern detractors, mathematics cannot be dismissed as merely a new form of literature or religion. Nevertheless, many mathematical objects can be seen as shared ideas, like *Moby Dick* in literature or the Immaculate Conception in religion.

The recognition that 'quasi-intuitive' methods may be used to gain good mathematical insight can dramatically assist in the learning and discovery of mathematics. Aesthetic and intuitive impulses are shot through our subject and honest mathematicians will acknowledge their role.

Some Final Observations

> When we have before us, for instance, a fine map, in which the line of coast, now rocky, now sandy, is clearly indicated, together with the windings of the rivers, the elevations of the land, and the distribution of the population, we have the simultaneous suggestion of so many facts, the sense of mastery over so much reality, that we gaze at it with delight, and need no practical motive to keep us studying it, perhaps for hours together. A map is not naturally thought of as an æsthetic object; it is too exclusively expressive. (Santayana, 1896/1910, p. 209)

This Santayana quotation was my earliest, and still favourite, encounter with aesthetic philosophy. It may be old fashioned and un-deconstructed in tone, but to me it rings true. He went on:

> And yet, let the tints of it be a little subtle, let the lines be a little delicate, and the masses of land and sea somewhat balanced, and we really have a beautiful thing; a thing the charm of which consists almost entirely in its meaning, but which nevertheless pleases us in the same way as a picture or a graphic symbol might please. Give the symbol a little intrinsic worth of form, line, and color, and it attracts like a magnet all the values of the things it is known to symbolize. It becomes beautiful in its expressiveness. (p. 210)

However, in conclusion, and to avoid possible accusations of mawkishness at the close, I also quote Jerry Fodor (1985):

> It is, no doubt, important to attend to the eternally beautiful and to believe the eternally true. But it is more important not to be eaten. (p. 4)

Notes

[1] This quotation is commonly attributed to Gauss, but it has proven remarkably resistant to being tracked down. Arber, the citation I give here, a philosopher of biology, acknowledges in a footnote (p. 47) that, "the present writer has been unable to trace this dictum to its original source". Interestingly, even the St. Andrews history of mathematics site cites Arber. See also Dunnington (1955/2004).

[2] In *Brouwer's Cambridge Lectures on Intuitionism,* the editor van Dalen (1981, p. 95) comments in a footnote:

> 3. The first use of undecidable properties of effectively presented objects (such as the decimal expansion of π) occurs in Brouwer (1908 [/1975]).