Stabilized Mixed Finite Element Methods for Nearly Incompressible Elasticity Problems

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Workshop on Analysis and its Applications in Honour of Brailey Sims
21 August, 2015
CARMA, University of Newcastle

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A Generalized Saddle Point Problem

Let V, W, P and Q be Hilbert spaces with inner products $(\cdot,\cdot)_V$, $(\cdot,\cdot)_W$, $(\cdot,\cdot)_P$ and $(\cdot,\cdot)_Q$, respectively. Let $a(\cdot,\cdot) : V \times W \to \boldsymbol{R}$, $b_1(\cdot,\cdot) : W \times P \to \boldsymbol{R}$, $b_2(\cdot,\cdot) : V \times Q \to \boldsymbol{R}$, and $c(\cdot,\cdot) : P \times Q \to \boldsymbol{R}$ be bilinear forms. We consider a non-symmetric saddle point problem with penalty: given $f \in W'$ and $g \in Q'$, find $(u,p) \in V \times P$ so that

$$\begin{aligned} a(u,w) + b_1(w,p) &= f(w)\,, \quad w \in W\,, \\ b_2(u,q) - t\, c(p,q) &= g(q)\,, \quad q \in Q\,, \end{aligned} \tag{1}$$

where t is a positive small parameter, and W' and Q' denote the space of continuous linear functionals on W and Q, respectively.

1. Well-posedness of the problem (1) when $t \to 0$.
2. Some relevant resources are by Brezzi, Fortin, Braess, Bernardi, Ciarlet, Canuto and Maday, etc. [BF91, Bra96, CHZ03, BCM88].

A Generalized Saddle Point Problem: Continuity Assumptions

To this end, we assume that the bilinear forms $a(\cdot,\cdot)$, $b_1(\cdot,\cdot)$, $b_2(\cdot,\cdot)$ and $c(\cdot,\cdot)$ satisfy for $v \in V$, $w \in W$, $p \in P$, $q \in Q$:

$$|a(v,w)| \leq \mathsf{a}\|v\|_V\,\|w\|_W\,, \qquad |b_1(w,p)| \leq \mathsf{b}_1\|w\|_W\,\|p\|_P\,,$$
$$|b_2(v,q)| \leq \mathsf{b}_2\|v\|_V\,\|q\|_Q\,, \qquad |c(p,q)| \leq \mathsf{c}\|p\|_P\|q\|_Q.$$

where $\mathsf{a}, \mathsf{b}_1, \mathsf{b}_2$ and c are continuity constants.

A Generalized Saddle Point Problem: Stability Assumptions

We define two kernel spaces $U_W \subset W$ and $U_V \subset V$ as

$$\begin{aligned} U_W &:= \{w \in W : \; b_1(w,p) = 0, \; p \in P\}, \\ U_V &:= \{v \in V : \; b_2(v,q) = 0, \; q \in Q\}, \end{aligned}$$

and assume that for $v \in U_V$, $w \in U_W$, $p \in P$, $q \in Q$:

$$\begin{aligned} \sup_{w \in U_W} \frac{a(v,w)}{\|w\|_W} \geq \alpha \|v\|_V, \quad &\text{and} \quad \sup_{v \in U_V} a(v,w) > 0 \\ \sup_{w \in W} \frac{b_1(w,p)}{\|w\|_W} \geq \beta_1 \|p\|_P, \quad &\text{and} \quad \sup_{v \in V} \frac{b_2(v,q)}{\|v\|_V} \geq \beta_2 \|q\|_Q, \end{aligned}$$

hold for some constants $\alpha, \beta_1, \beta_2 > 0$, where the supremum is taken only over the non-trivial elements of the underlying sets.

A Generalized Saddle Point Problem: Theorem

A theorem due to Nicolaides [Nic82] and Bernardi et al. [BCM88].

Theorem

Let above assumptions be satisfied. Then for any $f \in W'$ *and* $g \in Q'$*, there exists a unique solution* $(u, p) \in V \times P$ *to the saddle point problem of finding* $(u, p) \in V \times P$ *so that*

$$\begin{aligned} a(u, w) + b_1(w, p) &= f(w), \quad w \in W, \\ b_2(u, q) &= g(q), \quad q \in Q, \end{aligned} \tag{2}$$

which satisfies the following stability estimates:

$$\|u\|_V \leq \beta_2^{-1}(1 + \alpha^{-1}\mathsf{a})\|g\|_{Q'} + \alpha^{-1}\|f\|_{W'}, \quad \|p\|_P \leq \beta_1^{-1}(\|f\|_{W'} + \mathsf{a}\|u\|_V). \tag{3}$$

A Generalized Saddle Point Problem: Theorem

Theorem

Let assumptions of continuity and stability be satisfied, and

$$\delta := \beta_1^{-1}\beta_2^{-1}\mathsf{a}(1+\alpha^{-1}\mathsf{a})t\mathsf{c} < 1\,. \tag{4}$$

Then for any $f \in V'$ and $g \in Q'$, there exists a unique solution $(u,p) \in V \times P$ to the saddle point problem (1) *satisfying the following stability estimates:*

$$\|p\|_P \leq \frac{1}{1-\delta}\|\tilde{p}\|_P\,, \quad \|u\|_V \leq \|\tilde{u}\|_V + \frac{\beta_2(1+\alpha^{-1}\mathsf{a})t\mathsf{c}}{1-\delta}\|\tilde{p}\|_P\,, \tag{5}$$

where $(\tilde{u},\tilde{p})$ is the solution to (2) *and satisfies the bounds*

$$\|\tilde{u}\|_V \leq \beta_2^{-1}(1+\alpha^{-1}\mathsf{a})\|g\|_{Q'} + \alpha^{-1}\|f\|_{W'}\,, \quad \|\tilde{p}\|_P \leq \beta_1^{-1}(\|f\|_{W'} + \mathsf{a}\|\tilde{u}\|_V)\,.$$

A Generalized Saddle Point Problem: Proof

Proof:
Letting $p_0 = 0 \in P$, we define a sequence $\{(u_n, p_n)\}$ for $n \in \mathbb{N}$ by

$$\begin{aligned} a(u_{n+1}, w) + b_1(w, p_{n+1}) &= f(w)\,, \; w \in W \\ b_2(u_{n+1}, q) &= g(q) + tc(p_n, q)\,, \; q \in Q\,. \end{aligned} \quad (6)$$

The sequence is well-defined from Theorem 1, and for $n \in \mathbb{N}$ we have

$$\begin{aligned} a(u_{n+1} - u_n, w) + b_1(w, p_{n+1} - p_n) &= 0\,, \; w \in W \\ b_2(u_{n+1} - u_n, q) &= tc(p_n - p_{n-1}, q)\,, \; q \in Q\,. \end{aligned} \quad (7)$$

Theorem 1 yields the existence and uniqueness of the solution of (7) with the estimates

$$\begin{aligned} \|u_{n+1} - u_n\|_V &\leq \beta_2^{-1}(1 + \alpha^{-1}\mathsf{a})tc\|p_n - p_{n-1}\|_P\,, \\ \|p_{n+1} - p_n\|_P &\leq \beta_1^{-1}\mathsf{a}\|u_{n+1} - u_n\|_V\,, \end{aligned} \quad (8)$$

and hence

$$\|p_{n+1} - p_n\|_P \leq \beta_1^{-1}\beta_2^{-1}\mathsf{a}(1 + \alpha^{-1}\mathsf{a})tc\;\|p_n - p_{n-1}\|_P \leq \delta^n\|p_1\|_P\,. \quad (9)$$

A Generalized Saddle Point Problem: Proof

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A Generalized Saddle Point Problem: Proof

Now taking $n \in \mathbb{N}$ and an integer $m > n$, we have

$$\|p_m - p_n\|_P \leq \sum_{i=n}^{m-1} \|p_{i+1} - p_i\|_P \leq \sum_{i=n}^{m-1} \delta^i \|p_1\|_P \leq \frac{\delta^n}{1-\delta} \|p_1\|_P \,, \tag{10}$$

which shows that $\{p_n\}$ is a Cauchy sequence, and so converges to a $p \in P$.

1. The stability estimate for p is obtained by taking $n = 0$ in (10).
2. Using the first inequality of (8) and the estimate (9), the sequence $\{u_n\}$ is shown to be a Cauchy sequence, and stability estimate for u is obtained similarly as for p.
3. The uniqueness also follows similarly.

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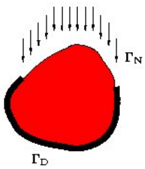
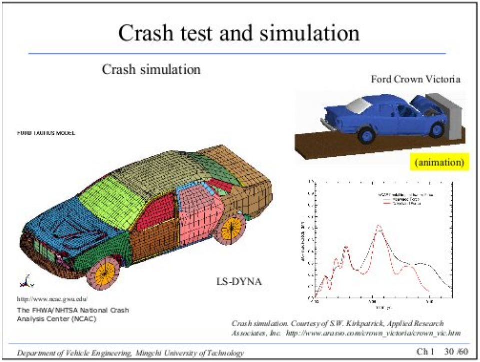
The Boundary Value Problem of Elasticity

Consider an elastic body in a bounded polyhedral domain Ω in $\mathbb{R}^d$, $d \in \{2, 3\}$. We want to compute the deformation and stress on the elastic body under a body force $\boldsymbol{f}$ on Ω and a surface force $\boldsymbol{g}_N$ on a part Γ_N of the boundary of Ω. The elastic body is supposed to be fixed on a part Γ_D of its boundary, where $\partial\Omega = \Gamma_D \cup \Gamma_N$. Useful in manufacture engineering.





Measured (black) and computed (red) impact on the wall is plotted.

Boundary Value Problem of Linear Elasticity

Let the material body be an isotropic linear elastic body. The deformation is governed by the equilibrium equation and Saint-Venant Kirchhoff material law:

$$\begin{aligned} -\operatorname{div}\boldsymbol{\sigma} &= \boldsymbol{f} \text{ in } \Omega && \text{Momentum Balance Law} \\ \boldsymbol{\sigma} &= \mathcal{C}\boldsymbol{\epsilon}(\boldsymbol{u}) && \text{Hooke's Constitutive Equation} \end{aligned} \tag{11}$$

- $\boldsymbol{\sigma} \in [L^2(\Omega)]^{d\times d}$ is the Cauchy stress, $\boldsymbol{\epsilon}(\boldsymbol{u}) := \frac{1}{2}(\nabla \boldsymbol{u} + [\nabla \boldsymbol{u}]^t)$ is the strain
- $\mathcal{C}$ is the Hooke's tensor and $\mathcal{C}$ applied to a tensor $\boldsymbol{d} \in [L^2(\Omega)]^{d\times d}$ yields

$$\mathcal{C}\boldsymbol{d} := \lambda(\operatorname{tr}\boldsymbol{d})\mathbf{1} + 2\mu\,\boldsymbol{d},$$

  where λ and μ are Lamé constants.
- The boundary conditions are: $\boldsymbol{u} = \mathbf{0}$ on Γ_D and $\boldsymbol{\sigma}\,\boldsymbol{n} = \boldsymbol{g}_N$ on Γ_N. And $\boldsymbol{n}$ is the outer normal vector on the boundary of Ω.

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Standard Weak Formulation

1. $L^2(\Omega)$ is the space of square-integrable functions defined on Ω with the inner product and norm being denoted by $(\cdot,\cdot)_0$ and $\|\cdot\|_0$, respectively, and $L_0^2(\Omega) := \{p \in L^2(\Omega) : \int_\Omega p\,dx = 0\}$.
2. Let $H^1(\Omega) = \{u \in L^2(\Omega) : \frac{\partial u}{\partial x_i} \in L^2(\Omega),\ i = 1,\cdots,d\}$ be a Hilbert space with norm $\|u\|_{H^1(\Omega)} = \sqrt{\int_\Omega (u^2 + \|\nabla u\|^2)\,dx}$, and $H_D^1(\Omega) = \{u \in H^1(\Omega) : u_{|\Gamma_D} = 0\}$.
3. We need the space of vector functions $\boldsymbol{V} := [H_D^1(\Omega)]^d$ for displacements with inner product $(\cdot,\cdot)_1$ and norm $\|\cdot\|_1$ defined in the standard way; that is, $(\boldsymbol{u},\boldsymbol{v})_1 := \sum_{i=1}^d (u_i, v_i)_1$, with the norm being induced by this inner product.

A Mixed Formulation of Elasticity Equations

Note that for an identity matrix $\boldsymbol{I}$ and pressure $p = \lambda \operatorname{div} \boldsymbol{u}$:

$$\mathcal{C}\boldsymbol{\epsilon}(\boldsymbol{u}) = 2\mu\boldsymbol{\epsilon}(\boldsymbol{u}) + \lambda \operatorname{div} \boldsymbol{u}\boldsymbol{I} = 2\mu\boldsymbol{\epsilon}(\boldsymbol{u}) + p\boldsymbol{I}.$$

A mixed variational formulation of linear elastic problem is found by using p as an addition unknown. Thus given $\ell \in [L^2(\Omega)]^d$, we want to find $(\boldsymbol{u}, p) \in \boldsymbol{V} \times L^2_0(\Omega)$ such that

$$\begin{aligned} A(\boldsymbol{u}, \boldsymbol{v}) + B(\boldsymbol{v}, p) &= \ell(\boldsymbol{v}), \quad \boldsymbol{v} \in \boldsymbol{V}, \\ B(\boldsymbol{u}, q) - \frac{1}{\lambda} C(p, q) &= 0, \quad q \in L^2_0(\Omega). \end{aligned}$$

A Mixed Formulation of Elasticity Equations

$$
\begin{aligned}
A(\boldsymbol{u}, \boldsymbol{v}) &:= 2\mu \int_\Omega \boldsymbol{\epsilon}(\boldsymbol{u}) : \boldsymbol{\epsilon}(\boldsymbol{v}) \, dx \\
B(\boldsymbol{v}, q) &:= \int_\Omega \operatorname{div} \boldsymbol{v} \, q \, dx, \\
C(p, q) &:= \int_\Omega p \, q \, dx, \\
\ell(\boldsymbol{v}) &:= \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \, dx + \int_{\Gamma_N} \boldsymbol{g}_N \cdot \boldsymbol{v} \, d\sigma.
\end{aligned}
$$

Well-posedness from the standard theory of saddle point problem [BF91].

Finite Element Space for Displacement

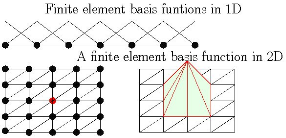
1. A quasi-uniform triangulation $\mathcal{T}_h$ of the polygonal or polyhedral domain Ω, where $\mathcal{T}_h$ consists of simplices, either triangles or tetrahedra.
2. S_h is the standard linear finite element space defined on the triangulation $\mathcal{T}_h$

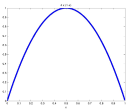
$$S_h := \left\{ v \in H^1(\Omega) : \, v_{|_T} \in \mathcal{P}_1(T), \; T \in \mathcal{T}_h \right\}$$

   and B_h is the space of bubble functions

$$B_h := \left\{ b_T \in \mathcal{P}_3(T) : \, b_{T|_{\partial T}} = 0, \text{ and } \int_T b_T \, dx > 0, \; T \in \mathcal{T}_h \right\},$$

3. Our finite element space for the displacement is $\boldsymbol{V}_h = (S_h \oplus B_h)^d \cap \boldsymbol{V}$.





Finite Element Space for Pressure

Let $\{\phi_1, \ldots, \phi_N\}$ be the finite element basis of S_h. Starting with the basis of S_h, we construct a dual space Q_h spanned by the basis $\{\mu_1, \ldots, \mu_N\}$ so that the basis functions of S_h and Q_h satisfy a condition of biorthogonality relation

$$\int_\Omega \mu_i \, \phi_j \, dx = c_j \delta_{ij} \, , \quad c_j \neq 0, \; 1 \leq i, j \leq N \, , \tag{12}$$

where δ_{ij} is the Kronecker symbol. The finite element trial and test spaces for pressure are $S_h^0 \subset L_0^2(\Omega) \cap S_h$ and $Q_h^0 \subset L_0^2(\Omega) \cap Q_h$.

$$S_h^0 = \left\{ \phi_h \in S_h : \int_\Omega \phi_h \, dx = 0 \right\} \quad \text{and} \quad Q_h^0 = \left\{ q_h \in Q_h : \int_\Omega q_h \, dx = 0 \right\}.$$

Finite Element Problem

1. Our finite element problem is to find $(\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_h \times S_h^0$ such that

$$\begin{aligned} A(\boldsymbol{u}_h, \boldsymbol{v}_h) + B_1(\boldsymbol{v}_h, p_h) &= \ell(\boldsymbol{v}_h)\,, \quad \boldsymbol{v}_h \in \boldsymbol{V}_h\,, \\ B_2(\boldsymbol{u}_h, q_h) - \frac{1}{\lambda} C(p_h, q_h) &= 0\,, \quad q_h \in Q_h^0. \end{aligned} \tag{13}$$

2. With the choice of a biorthogonal system S_h and Q_h the matrix associated with the bilinear form $C(\cdot,\cdot)$ is diagonal. Note that $C(\phi_i, \mu_j) = \delta_{ij}$.

Finite Element Method

We show the existence and uniqueness of the solution of the mixed formulation (16) using Theorem 2.

1. **Continuity** $A(\cdot,\cdot)$ on $\boldsymbol{V}_h \times \boldsymbol{V}_h$, of $B_1(\cdot,\cdot)$ on $\boldsymbol{V}_h \times S_h^0$, and $B_2(\cdot,\cdot)$ on $\boldsymbol{V}_h \times Q_h^0$ and of $C(\cdot,\cdot)$ on $S_h^0 \times Q_h^0$ are continuous.
2. **Coercivity** By using the Korn's inequality the ellipticity of the bilinear form $A(\cdot,\cdot)$ holds on $\boldsymbol{V}_h \times \boldsymbol{V}_h$.
3. **Inf-sup condition** There exists a constant $\beta_1 > 0$ and $\beta_2 > 0$ independent of the mesh-size such that

$$\sup_{\boldsymbol{v}_h \in \boldsymbol{V}_h} \frac{B_1(\boldsymbol{v}_h, \mu_h)}{\|\boldsymbol{v}_h\|_1} \geq \beta \|\mu_h\|_0 \,, \quad \mu_h \in S_h^0 \tag{14}$$

$$\sup_{\boldsymbol{v}_h \in \boldsymbol{V}_h} \frac{B_2(\boldsymbol{v}_h, q_h)}{\|\boldsymbol{v}_h\|_1} \geq \beta \|q_h\|_0 \,, \quad q_h \in Q_h^0. \tag{15}$$

Finite Element Method

The following theorem holds [Nic82, BCM88, BF91, Bra01].

Theorem

The discrete problem (16) *has exactly one solution* $(\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_h \times S_h^0$, *and there exists a constant* c *independent of Lamé parameter* λ *such that*

$$\|\boldsymbol{u}_h\|_1 + \|p_h\|_0 \leq c\|\boldsymbol{f}\|_0 \, .$$

Furthermore, if $(\boldsymbol{u}, p)$ *is the solution to the problem* (12), *we have the following error estimate uniform with respect to* λ*:*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_1 + \|p - p_h\|_0 \leq c_1 \inf_{\boldsymbol{v}_h \in \boldsymbol{V}_h} \|\boldsymbol{u} - \boldsymbol{v}_h\|_1 + c_2 \inf_{q_h \in S_h^0} \|p - q_h\|_0 \, ,$$

where the constants c_1 *and* c_2 *are independent of the mesh-size.*

Using the standard approximation properties of the spaces $\boldsymbol{V}_h$ and S_h^0, we see that the approximation to the displacement converges to the exact solution with $O(h)$ in H^1-norm.

Finite Element Method: Getting rid of bubble functions

1. If we do not include the bubble functions, the inf-sup conditions for bilinear forms $B_1(\cdot,\cdot)$ and $B_2(\cdot,\cdot)$ do not hold.
2. In this case the saddle point problem is modified as

$$\begin{aligned} A(\boldsymbol{u}_h, \boldsymbol{v}_h) + B_1(\boldsymbol{v}_h, p_h) &= \ell(\boldsymbol{v}_h)\,, \quad \boldsymbol{v}_h \in \boldsymbol{V}_h\,, \\ B_2(\boldsymbol{u}_h, q_h) - g(p_h, q_h) &= 0\,, \quad q_h \in Q_h^0, \end{aligned} \tag{16}$$

where $g(p_h, q_h) = \frac{1}{\lambda} C(p_h, q_h) - G(p_h, q_h)$.

3. We need to choose $G(p_h, q_h)$ appropriately to stabilise the system.

Finite Element Method: Getting rid of bubble functions

1. One example of $G(p_h, q_h)$ is

$$G(p_h, q_h) = G(p_h - \Pi_h p_h, q_h - \Pi_h q_h),$$

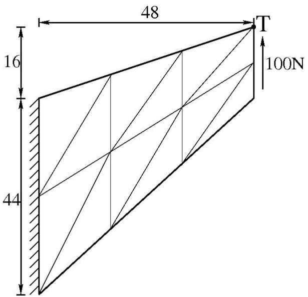
   where Π_h is a suitable projection operator. [Bochev, Dohrman and Gunzburger 2006].

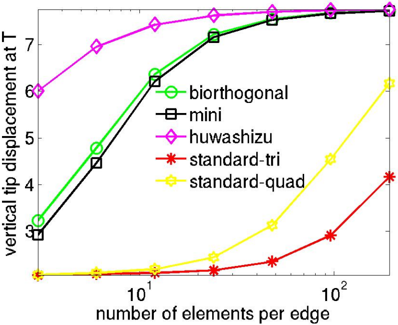
2. Now we need a g-biorthogonal system defined as

$$g(\phi_i, \mu_j) = -\frac{1}{\lambda} C(\phi_i, \mu_j) + G(\phi_i, \mu_j) = c_j \delta_{ij}$$

   to get a diagonal matrix as before. [L' 2014].

Numerical Results



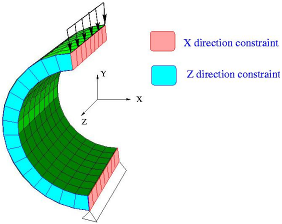


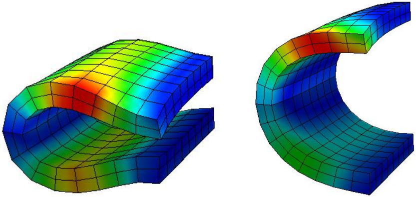
Cook's membrane problem with initial triangulation (left) and the vertical tip displacement versus number of elements per edge

Note: Hu-Washizu and standard-quad are computed on quadrilateral mesh with the equal number of nodes.

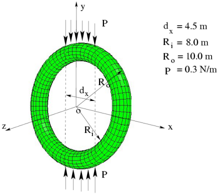
Numerical Results

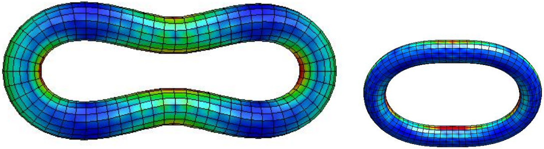
Nearly incompressible cylindrical (Mooney-Rivlin) shell under bending force





A nearly incompressible (neo-Hookean) torus under compression





Thank You

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