Semigroup C*-algebras. The independence condition, nuclearity and amenability

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Semigroup C*-algebras. Nuclearity and amenability

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- G is amenable;
- $C^*(G)$ is nuclear;
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Murphy showed that this C*-algebra is not nuclear.

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- Goal: Find an explanation!

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When is this map an isomorphism?

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Theorem (Norling)

For a subsemigroup P of a group, $C^*_\lambda(S) \to C^*_\lambda(P)$ is an isomorphism if and only if P satisfies independence.

The independence condition. Examples

Every right LCM monoid P (i.e., $\mathcal{J}_P^\times = \{pP\}$) satisfies independence:
Let $pP = \bigcup_{i=1}^n p_i P$. Then $p = p \cdot e \in pP$, so $p \in p_i P$ for some $1 \leq i \leq n$.
But $p_i P$ is a right ideal, so $p \in p_i P$ implies $pP \subseteq p_i P$. Hence $pP = p_i P$.

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For a ring R of algebraic integers in a number field, $R \rtimes R^\times$ always satisfies independence (because R is a Dedekind domain).

The independence condition. Counterexamples

Let $R = \mathbb{Z}[i\sqrt{3}]$. R is not integrally closed in $\mathbb{Q}[i\sqrt{3}]$. Its integral closure is $\mathbb{Z}[\frac{1}{2}(1 + i\sqrt{3})]$. And $R \rtimes R^{\times}$ does not satisfy independence.

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A similar argument shows that for every numerical semigroup of the form $\mathbb{N} \setminus F$, where $\emptyset \neq F$ is finite, the independence condition does not hold.

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Theorem (L)

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$v_a^* v_a = v_b^* v_b = 1 \Rightarrow \chi(v_a) = \chi(v_b) \in \mathbb{T}$.
But then, $0 = \chi(v_a v_a^* v_b v_b^*) = |\chi(v_a)|^2 |\chi(v_b)|^2 = 1$.

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Nuclearity and amenability

Theorem

Let $P \subseteq G$, and let P satisfy independence. Then $C^*_\lambda(P) \cong C^*_\lambda(I_l(P)) \cong C^*_\lambda(G \ltimes \Omega_P)$ and $C^*(P) = C^*(I_l(P)) \cong C^*(G \ltimes \Omega_P)$.

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Then (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Rightarrow$ (iv).

Nuclearity and amenability

Theorem

Let $P \subseteq G$, and let P satisfy independence. Then $C^*_\lambda(P) \cong C^*_\lambda(I_l(P)) \cong C^*_\lambda(G \ltimes \Omega_P)$ and $C^*(P) = C^*(I_l(P)) \cong C^*(G \ltimes \Omega_P)$.

Theorem

Let $P \subseteq G$. Assume that P satisfies independence. Consider

(i) $C^*(P)$ is nuclear.

(ii) $C^*_\lambda(P)$ is nuclear.

(iii) $G \ltimes \Omega_P$ is amenable.

(iv) The left regular representation $C^*(P) \to C^*_\lambda(P)$ is an isomorphism.

Then (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Rightarrow$ (iv).
In particular, (i) – (iv) are true if G is amenable.

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Examples

- This explains why $C^*(\mathbb{N} * \mathbb{N}) \cong C^*_\lambda(\mathbb{N} * \mathbb{N})$ is nuclear:

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- For every right-angled Artin monoid A_Γ^+, $C^*_\lambda(A_\Gamma^+)$ is nuclear.

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- For every right-angled Artin monoid A_Γ^+, $C^*_\lambda(A_\Gamma^+)$ is nuclear.
  Q: Do right-angled Artin monoids embed into amenable groups?
- $C^*_\lambda(R \rtimes R^\times)$ is nuclear for every integral domain R: $R \rtimes R^\times$ embeds into the amenable group $K \rtimes K^\times$, where K is the quotient field of R.