Iterating the Cuntz–Nica–Pimsner construction for compactly aligned product systems over quasi-lattice ordered groups

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Motivation

- Deaconu investigated what he called iterated Toeplitz and Cuntz–Pimsner algebras. Here's his setup:
  (1) Let A be a unital C^*-algebra, and E_1 and E_2 be full finitely generated Hilbert A-bimodules, such that the left action of A on each of E_1 and E_2 is faithful and nondegenerate; and
  (2) Suppose there exists a Hilbert A-bimodule isomorphism $\chi : E_1 \otimes_A E_2 \to E_2 \otimes_A E_1$.

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- Deaconu argues that the balanced tensor product $E_2 \otimes_A \mathcal{T}_{E_1}$ has the structure of a Hilbert $\mathcal{T}_{E_1}$-bimodule (the non-straightforward part is getting the left action of $\mathcal{T}_{E_1}$) — a process he calls 'extending the scalars'.

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- Similarly, $E_2 \otimes_A \mathcal{O}_{E_1}$ has the structure of a Hilbert $\mathcal{O}_{E_1}$-bimodule.

- Deaconu then proves the following isomorphisms:
  (1) $\mathcal{T}_{E_2 \otimes_A \mathcal{T}_{E_1}} \cong \mathcal{T}_{E_1 \otimes_A \mathcal{T}_{E_2}}$;
  (2) $\mathcal{O}_{E_2 \otimes_A \mathcal{O}_{E_1}} \cong \mathcal{O}_{E_1 \otimes_A \mathcal{O}_{E_2}}$;
  (3) $\mathcal{O}_{E_2 \otimes_A \mathcal{T}_{E_1}} \cong \mathcal{T}_{E_1 \otimes_A \mathcal{O}_{E_2}}$ and $\mathcal{O}_{E_1 \otimes_A \mathcal{T}_{E_2}} \cong \mathcal{T}_{E_2 \otimes_A \mathcal{O}_{E_1}}$.

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- Unfortunately, some of the proofs lack detail, and it's not clear where the various hypotheses are used, nor if they are necessary to make the procedure work.

  The results in this talk show that Deaconu's iterative procedure can be extended to quasi-lattice ordered groups that are more general than $(\mathbb{Z}^2, \mathbb{N}^2)$.

  In doing so we will verify the above isomorphisms, showing that many of Deaconu's hypotheses can be relaxed/removed, and hopefully gain a better understanding of what exactly is going on.

Hilbert bimodules

An inner-product A-module is a complex vector space X equipped with a right action of C^*-algebra A, and a map $\langle \cdot, \cdot \rangle_A : X \times X \to A$, linear in its second argument, satisfying the following conditions:

(1) $\langle x, y \rangle_A = \langle y, x \rangle_A^*$;

(2) $\langle x, y \cdot a \rangle_A = \langle x, y \rangle_A a$;

(3) $\langle x, x \rangle_A \geq 0$ in A; and

(4) $\langle x, x \rangle_A = 0$ if and only if $x = 0$.

The formula $\|x\|_X := \|\langle x, x \rangle_A\|_A^{1/2}$ defines a norm on X, and we say that X is a Hilbert A-module if it is complete with respect to this norm.

Quasi-lattice ordered groups

- A quasi-lattice ordered group (G, P) consists of a group G and a subsemigroup P of G such that
  (1) $P \cap P^{-1} = \{e\}$,
  (2) With respect to the partial order on G induced by $p \leq q \Leftrightarrow p^{-1}q \in P$, any two elements $p, q \in G$ which have a common upper bound in P have a least common upper bound in P.
- It follows that if $p, q \in G$ have a least common upper bound in P, then it is unique (and we denote it by $p \vee q$).
- We write $p \vee q = \infty$ if p and q have no common upper bound in P, and $p \vee q < \infty$ otherwise.
- We say that P is directed if $p \vee q < \infty$ for every $p, q \in P$.

Combining quasi-lattice ordered groups

Lemma

Let (G, P) and (H, Q) be quasi-lattice ordered groups. If $\alpha : H \to \mathrm{Aut}(G)$ is a group homomorphism with $\alpha_H(P) \subseteq P$, then the semidirect product $(G \rtimes_\alpha H, P \rtimes_\alpha Q)$ is a quasi-lattice ordered group.

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- In particular, $(G \rtimes_\alpha H, P \rtimes_\alpha Q)$ has the product order i.e.

$$(g, h) \leq (k, l) \iff g \leq k \text{ and } h \leq l.$$

Thus,

$$(g, h) \vee (g', h') = \begin{cases} (g \vee g', h \vee h') & \text{if } g \vee g' < \infty,\ h \vee h' < \infty \\ \infty & \text{otherwise.} \end{cases}$$

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- The condition $\alpha_H(P) \subseteq P$ is not necessary: $(\mathbb{Q} \rtimes_\alpha \mathbb{Q}_+^\times, \mathbb{N} \rtimes_\alpha \mathbb{N}_+^\times)$ where $\alpha_h(g) = hg$ for each $h \in \mathbb{Q}_+^\times$, $g \in \mathbb{Q}$ is quasi-lattice ordered (although not with the product order). Also, $\mathbb{N}$ and $\mathbb{N}_+^\times$ are directed, whilst $\mathbb{N} \rtimes_\alpha \mathbb{N}_+^\times$ is not.

Compactly aligned product systems

Let (G, P) be a quasi-lattice ordered group, and A a C^*-algebra. A product system over P with coefficient algebra A is a semigroup $\mathbf{X} = \bigsqcup_{p \in P} \mathbf{X}_p$ such that:

(1) For each $p \in P$, $\mathbf{X}_p \subseteq \mathbf{X}$ is a Hilbert A-bimodule;

(2) $\mathbf{X}_e$ is equal to the Hilbert A-bimodule ${}_A A_A$;

(3) For each $p, q \in P$ with $p \neq e$, there exists a Hilbert A-bimodule isomorphism $M^{\mathbf{X}}_{p,q} : \mathbf{X}_p \otimes_A \mathbf{X}_q \to \mathbf{X}_{pq}$ satisfying $M^{\mathbf{X}}_{p,q}(x \otimes_A y) = xy$ for each $x \in \mathbf{X}_p$ and $y \in \mathbf{X}_q$; and

(4) Multiplication in $\mathbf{X}$ by elements of $\mathbf{X}_e = A$ implements the left and right actions of A on each $\mathbf{X}_p$; that is $xa = x \cdot a$ and $ax = a \cdot x$ for each $p \in P$, $a \in A$, and $x \in \mathbf{X}_p$.

- We write $\phi_p : A \to \mathcal{L}_A(\mathbf{X}_p)$ for the homomorphism that implements the left action of A on $\mathbf{X}_p$ and $\langle \cdot, \cdot \rangle_A^p$ for the A-valued inner-product on $\mathbf{X}_p$.
- For each $p \in P \setminus \{e\}$ and $q \in P$, we define a homomorphism $\iota_p^{pq} : \mathcal{L}_A(\mathbf{X}_p) \to \mathcal{L}_A(\mathbf{X}_{pq})$ by

$$\iota_p^{pq}(S) := M_{p,q}^{\mathbf{X}} \circ (S \otimes_A \mathrm{id}_{\mathbf{X}_q}) \circ (M_{p,q}^{\mathbf{X}})^{-1}.$$

  For notational simplicity, we write $\iota_e^q : \mathcal{K}_A(\mathbf{X}_e) \cong A \to \mathcal{L}_A(\mathbf{X}_q)$ for ϕ_q for each $q \in P$.
- We say that $\mathbf{X}$ is compactly aligned if for any $p, q \in P$ with $p \vee q$ and any $S \in \mathcal{K}_A(\mathbf{X}_p)$, $T \in \mathcal{K}_A(\mathbf{X}_q)$, we have

$$\iota_p^{p\vee q}(S)\iota_q^{p\vee q}(T) \in \mathcal{K}_A(\mathbf{X}_{p\vee q}).$$

Representations of product systems

- A representation of $\mathbf{X}$ in a C^*-algebra B is a map $\psi : \mathbf{X} \to B$ satisfying the following relations:
  (T1) each $\psi_p := \psi|_{\mathbf{X}_p}$ is a linear map, and ψ_e is a homomorphism;
  (T2) $\psi_p(x)\psi_q(y) = \psi_{pq}(xy)$ for all $p, q \in P$ and $x \in \mathbf{X}_p$, $y \in \mathbf{X}_q$; and
  (T3) $\psi_p(x)^*\psi_p(y) = \psi_e(\langle x, y\rangle_A^p)$ for all $p \in P$ and $x, y \in \mathbf{X}_p$.
- For each $p \in P$ there exists a homomorphism $\psi^{(p)} : \mathcal{K}_A(\mathbf{X}_p) \to B$ such that

$$\psi^{(p)}(\Theta_{x,y}) = \psi_p(x)\psi_p(y)^* \quad \text{for all } x, y \in \mathbf{X}_p.$$

Nica covariance

- We say that ψ is Nica covariant if, for any $p, q \in P$ and $S \in \mathcal{K}_A(\mathbf{X}_p)$, $T \in \mathcal{K}_A(\mathbf{X}_q)$, we have

$$\psi^{(p)}(S)\psi^{(q)}(T) = \begin{cases} \psi^{(p \vee q)}\left(\iota_p^{p \vee q}(S)\iota_q^{p \vee q}(T)\right) & \text{if } p \vee q < \infty \\ 0 & \text{otherwise.} \end{cases}$$

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- Combining the Hewitt–Cohen–Blanchard factorisation theorem with Nica covariance, we get that

$$\psi_p(\mathbf{X}_p)^* \psi_q(\mathbf{X}_q) \subseteq \overline{\operatorname{span}}\{\psi_{p^{-1}(p \vee q)}(\mathbf{X}_{p^{-1}(p \vee q)})\psi_{q^{-1}(p \vee q)}(\mathbf{X}_{q^{-1}(p \vee q)})^*\}$$

if $p \vee q < \infty$, and is $\{0\}$ otherwise.

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- We say that a representation $\psi : \mathbf{X} \to B$ is Cuntz–Pimsner covariant if, for any finite set $F \subseteq P$ and any choice of compact operators $\{T_p \in \mathcal{K}_A(\mathbf{X}_p) : p \in F\}$, we have that

$$\sum_{p \in F} \iota_p^s(T_p) = 0 \in \mathcal{L}_A(\mathbf{X}_s) \quad \text{for large } s \quad \Rightarrow \quad \sum_{p \in F} \psi^{(p)}(T_p) = 0.$$

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- If P is directed and $\phi_p(A) \subseteq \mathcal{K}_A(\mathbf{X}_p)$ for each $p \in P$, then ψ is Cuntz–Pimsner covariant if and only if $\psi^{(p)}(\phi_p(a)) = \psi_e(a)$ for each $p \in P$ and $a \in A$.

The Nica–Toeplitz and Cuntz–Nica–Pimsner algebras

- There exists a C^*-algebra $\mathcal{NT}_{\mathbf{X}}$, which we call $\mathcal{NT}_{\mathbf{X}}$ the Nica–Toeplitz algebra of $\mathbf{X}$, and a Nica covariant representation $i_{\mathbf{X}} : \mathbf{X} \to \mathcal{NT}_{\mathbf{X}}$, that are universal in the following sense:
  (1) the image of $i_{\mathbf{X}}$ generates $\mathcal{NT}_{\mathbf{X}}$; and
  (2) given any other Nica covariant representation $\psi : \mathbf{X} \to B$, there exists a homomorphism $\psi_* : \mathcal{NT}_{\mathbf{X}} \to B$ such that $\psi_* \circ i_{\mathbf{X}} = \psi$.
- Since $i_{\mathbf{X}}$ generates $\mathcal{NT}_{\mathbf{X}}$, it follows that

$$\mathcal{NT}_{\mathbf{X}} = \overline{\mathrm{span}}\left\{ i_{\mathbf{X}}(x) i_{\mathbf{X}}(y)^* : x, y \in \mathbf{X} \right\}.$$

- Similarly, there exists a C^*-algebra $\mathcal{NO}_{\mathbf{X}}$ generated by a universal Cuntz–Nica–Pimsner covariant representation $j_{\mathbf{X}}$, which we call the Cuntz–Nica–Pismner algebra of $\mathbf{X}$.
- $\mathcal{NO}_{\mathbf{X}}$ is a quotient of $\mathcal{NT}_{\mathbf{X}}$, and we write $q_{\mathbf{X}}$ for the quotient map.

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- $\mathbf{X} := \bigsqcup_{p \in P} \mathbf{X}_p := \bigsqcup_{p \in P} \mathbf{Z}_{(p, e_H)}$ is a compactly aligned product system over (G, P) with coefficient algebra A.

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We'll prove the following:

Theorem

Suppose that G is an amenable group. Then there exists a compactly aligned product system $\mathbf{Y}^{\mathcal{NT}}$ over (H, Q) with coefficient algebra $\mathcal{NT}_{\mathbf{X}}$ such that $\mathcal{NT}_{\mathbf{Y}^{\mathcal{NT}}} \cong \mathcal{NT}_{\mathbf{Z}}$.

How does this fit in with Deaconu's procedure

- The data (E_1, E_2, χ) is effectively the same as a product system over $\mathbb{N}^2$. Indeed, if

$$\mathbf{Z}_{(m,n)} := \begin{cases} E_1^{\otimes m} \otimes_A E_2^{\otimes n} & \text{if } m \geq 1 \\ E_2^{\otimes n} & \text{if } m = 0, \end{cases}$$

  then $\mathbf{Z} := \bigsqcup_{(m,n)\in\mathbb{N}^2} \mathbf{Z}_{(m,n)}$ is a product system over $\mathbb{N}^2 = \mathbb{N} \times \mathbb{N}$, with multiplication determined by the isomorphisms

$$M^{\mathbf{Z}}_{(1,0),(0,1)} := \mathrm{id}_{E_1 \otimes_A E_2} \quad \text{and} \quad M^{\mathbf{Z}}_{(0,1),(1,0)} := \chi^{-1}.$$

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- Deaconu's assumption that E_1 and E_2 are finitely generated ensures that $\mathbf{Z}$ is compactly aligned (and so $\mathcal{NT}_{\mathbf{Z}}$ exists).

- We will see that the subspace

$$\mathbf{Y}_1^{\mathcal{NT}} := \overline{\operatorname{span}}\{ i_{\mathbf{Z}}(\mathbf{Z}_{P\times\{1\}}) i_{\mathbf{Z}}(\mathbf{Z}_{P\times\{0\}})^* \} \subseteq \mathcal{NT}_{\mathbf{Z}}$$

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- Our theorem then says that

$$\mathcal{T}_{E_2\otimes_A \mathcal{T}_{E_1}} \cong \mathcal{T}_{\mathbf{Y}_1^{\mathcal{NT}}} \cong \mathcal{NT}_{\mathbf{Y}^{\mathcal{NT}}} \cong \mathcal{NT}_{\mathbf{Z}}.$$

By symmetry, $\mathcal{T}_{E_1\otimes_A \mathcal{T}_{E_2}}$ is also isomorphic to $\mathcal{NT}_{\mathbf{Z}}$, and so we get

$$\mathcal{T}_{E_2\otimes_A \mathcal{T}_{E_1}} \cong \mathcal{T}_{E_1\otimes_A \mathcal{T}_{E_2}}.$$

Constructing the product system $\mathbf{Y}^{\mathcal{NT}}$

The first step in our procedure is checking that the inclusion of $\mathbf{X}$ in $\mathbf{Z}$ induces an inclusion between their respective Nica–Toeplitz algebras.

Proposition

There exists a homomorphism $\phi_{\mathbf{X}}^{\mathcal{NT}} : \mathcal{NT}_{\mathbf{X}} \to \mathcal{NT}_{\mathbf{Z}}$ *such that* $\phi_{\mathbf{X}}^{\mathcal{NT}} \circ i_{\mathbf{X}} = i_{\mathbf{Z}}$*. If* G *is an amenable group, then* $\phi_{\mathbf{X}}^{\mathcal{NT}}$ *is injective.*

In summary, for every $p \in P$, the following diagram commutes.

$$
\begin{array}{ccc}
\mathbf{X}_p & \xrightarrow{\;i_{\mathbf{Z}_{(p,e_H)}}\;} & \mathcal{NT}_{\mathbf{Z}} \\
{\scriptstyle i_{\mathbf{X}_p}}\big\downarrow & \nearrow{\scriptstyle \phi_{\mathbf{X}}^{\mathcal{NT}}} & \\
\mathcal{NT}_{\mathbf{X}} =: \mathbf{Y}_{e_H}^{\mathcal{NT}} & &
\end{array}
$$

We use this injective homomorphism to construct a collection of Hilbert $\mathcal{NT}_{\mathbf{X}}$-modules $\{\mathbf{Y}_q^{\mathcal{NT}} : q \in Q\}$. We set $\mathbf{Y}_{e_H}^{\mathcal{NT}} := {}_{\mathcal{NT}_{\mathbf{X}}}(\mathcal{NT}_{\mathbf{X}})_{\mathcal{NT}_{\mathbf{X}}}$.

Proposition

Suppose G is an amenable group. For each $q \in Q \setminus \{e_H\}$, define

$$\mathbf{Y}_q^{\mathcal{NT}} := \overline{\operatorname{span}}\{i_{\mathbf{Z}_{(e_G,q)}}(\mathbf{Z}_{(e_G,q)})\phi_{\mathbf{X}}^{\mathcal{NT}}(\mathcal{NT}_{\mathbf{X}})\} = \overline{\operatorname{span}}\{i_{\mathbf{Z}}(\mathbf{Z}_{P\rtimes_\alpha\{q\}})i_{\mathbf{Z}}(\mathbf{Z}_{P\rtimes_\alpha\{e_H\}})^*\} \subseteq \mathcal{NT}_{\mathbf{Z}}.$$

Then $\mathbf{Y}_q^{\mathcal{NT}}$ has the structure of a Hilbert $\mathcal{NT}_{\mathbf{X}}$-module, with

$$y \cdot b := y\phi_{\mathbf{X}}^{\mathcal{NT}}(b) \quad \text{for each } y \in \mathbf{Y}_q^{\mathcal{NT}},\ b \in \mathcal{NT}_{\mathbf{X}}$$

and

$$\langle y, w\rangle_{\mathcal{NT}_{\mathbf{X}}}^q := (\phi_{\mathbf{X}}^{\mathcal{NT}})^{-1}(y^*w) \quad \text{for each } y, w \in \mathbf{Y}_q^{\mathcal{NT}}.$$

Sketch Proof.

Need to check that if $y, w \in \mathbf{Y}_q^{\mathcal{NT}}$, then $y^* w \in \phi_{\mathbf{X}}^{\mathcal{NT}}(\mathcal{NT}_{\mathbf{X}})$. By linearity and continuity, it suffices to consider when $y = i_{\mathbf{Z}_{(e_G,q)}}(x)\phi_{\mathbf{X}}^{\mathcal{NT}}(a)$ and $w = i_{\mathbf{Z}_{(e_G,q)}}(z)\phi_{\mathbf{X}}^{\mathcal{NT}}(b)$ for some $x, z \in \mathbf{Z}_{(e_G,q)}$ and $a, b \in \mathcal{NT}_{\mathbf{X}}$. Relation (T3) gives

$$\begin{aligned}
\left(i_{\mathbf{Z}_{(e_G,q)}}(x)\phi_{\mathbf{X}}^{\mathcal{NT}}(a)\right)^* i_{\mathbf{Z}_{(e_G,q)}}(z)\phi_{\mathbf{X}}^{\mathcal{NT}}(b) &= \phi_{\mathbf{X}}^{\mathcal{NT}}(a)^* i_{\mathbf{Z}_{(e_G,q)}}(x)^* i_{\mathbf{Z}_{(e_G,q)}}(z)\phi_{\mathbf{X}}^{\mathcal{NT}}(b) \\
&= \phi_{\mathbf{X}}^{\mathcal{NT}}(a^*) i_{\mathbf{Z}_{(e_G,e_H)}}\left(\langle x, z\rangle_A^{(e_G,q)}\right)\phi_{\mathbf{X}}^{\mathcal{NT}}(b) \\
&= \phi_{\mathbf{X}}^{\mathcal{NT}}\left(a^* i_{\mathbf{X}_{e_G}}\left(\langle x, z\rangle_A^{(e_G,q)}\right)b\right).
\end{aligned}$$

□

The homomorphism $\phi_{\mathbf{X}}^{\mathcal{NT}}$ also gives a left action of $\mathcal{NT}_{\mathbf{X}}$ on $\mathbf{Y}_q^{\mathcal{NT}}$ by adjointable operators. Hence, each $\mathbf{Y}_q^{\mathcal{NT}}$ is a Hilbert $\mathcal{NT}_{\mathbf{X}}$-bimodule, and it can be shown that the collection of these bimodules gives a compactly aligned product system.

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$$\Phi_q^{\mathcal{NT}}(b)(y) = \phi_{\mathbf{X}}^{\mathcal{NT}}(b)y \quad \text{for each } b \in \mathcal{NT}_{\mathbf{X}},\ y \in \mathbf{Y}_q^{\mathcal{NT}}.$$

The homomorphism $\phi_{\mathbf{X}}^{\mathcal{NT}}$ also gives a left action of $\mathcal{NT}_{\mathbf{X}}$ on $\mathbf{Y}_q^{\mathcal{NT}}$ by adjointable operators. Hence, each $\mathbf{Y}_q^{\mathcal{NT}}$ is a Hilbert $\mathcal{NT}_{\mathbf{X}}$-bimodule, and it can be shown that the collection of these bimodules gives a compactly aligned product system.

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Suppose G is an amenable group. For each $q \in Q \setminus \{e_H\}$, there exists a homomorphism $\Phi_q^{\mathcal{NT}} : \mathcal{NT}_{\mathbf{X}} \to \mathcal{L}_{\mathcal{NT}_{\mathbf{X}}}\left(\mathbf{Y}_q^{\mathcal{NT}}\right)$ such that

$$\Phi_q^{\mathcal{NT}}(b)(y) = \phi_{\mathbf{X}}^{\mathcal{NT}}(b)y \quad \text{for each } b \in \mathcal{NT}_{\mathbf{X}},\ y \in \mathbf{Y}_q^{\mathcal{NT}}.$$

Proposition

Suppose G is an amenable group. Then $\mathbf{Y}^{\mathcal{NT}} := \bigsqcup_{q \in Q} \mathbf{Y}_q^{\mathcal{NT}}$ (with multiplication given by multiplication in $\mathcal{NT}_{\mathbf{Z}}$) is a compactly aligned product system over (H, Q) with coefficient algebra $\mathcal{NT}_{\mathbf{X}}$.

Sketch Proof.

The basic idea is to show that $\mathbf{Y}_q^{\mathcal{NT}}\mathbf{Y}_n^{\mathcal{NT}} \subseteq \mathbf{Y}_{qn}^{\mathcal{NT}}$. By linearity and continuity it suffices to show that if $y = i_{\mathbf{Z}_{(p,q)}}(x) i_{\mathbf{Z}_{(r,e_H)}}(z)^*$ and $w = i_{\mathbf{Z}_{(m,n)}}(u) i_{\mathbf{Z}_{(s,e_H)}}(v)^*$, then $yw \in \mathbf{Y}_{qn}^{\mathcal{NT}}$.
Using the Nica covariance of $i_{\mathbf{Z}}$ and the fact that $(G \rtimes_\alpha H, P \rtimes_\alpha Q)$ has the product order, we get that $yw = 0$ if $r \vee m = \infty$, whilst if $r \vee m$,

$$\begin{aligned} &i_{\mathbf{Z}_{(p,q)}}(x) i_{\mathbf{Z}_{(r,e_H)}}(z)^* i_{\mathbf{Z}_{(m,n)}}(u) i_{\mathbf{Z}_{(s,e_H)}}(v)^* \\ &\qquad \in \overline{\mathrm{span}}\{ i_{\mathbf{Z}}\big(\mathbf{Z}_{(p\alpha_q(r^{-1}(r\vee m)),qn)}\big) i_{\mathbf{Z}}\big(\mathbf{Z}_{(s\alpha_{n^{-1}}(m^{-1}(r\vee m)),e_H)}\big)^* \} \subseteq \mathbf{Y}_{qn}^{\mathcal{NT}} \end{aligned}$$

where we used the fact that

$$\begin{aligned} (p,q)(r,e_H)^{-1}\big((r,e_H)\vee(m,n)\big) &= (p\alpha_q(r^{-1}(r\vee m)), qn) \\ (s,e_H)(m,n)^{-1}\big((r,e_H)\vee(m,n)\big) &= (s\alpha_{n^{-1}}(m^{-1}(r\vee m)), e_H). \end{aligned}$$

□

Establishing the isomorphism $\mathcal{NT}_{\mathbf{Y}^{\mathcal{NT}}} \cong \mathcal{NT}_{\mathbf{Z}}$

- The idea is to use the universal properties of $\mathcal{NT}_{\mathbf{Y}^{\mathcal{NT}}}$ and $\mathcal{NT}_{\mathbf{Z}}$ to induce homomorphisms between the two C^*-algebras, and then check that these homomorphisms are mutually inverse.

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- The homomorphism from $\mathcal{NT}_{\mathbf{Y}^{\mathcal{NT}}}$ to $\mathcal{NT}_{\mathbf{Z}}$ is easy to get. The key point is that once we identify $\mathcal{NT}_{\mathbf{X}}$ with $\phi_{\mathbf{X}}^{\mathcal{NT}}(\mathcal{NT}_{\mathbf{X}})$, each fibre $\mathbf{Y}_q^{\mathcal{NT}}$ sits inside $\mathcal{NT}_{\mathbf{Z}}$.

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Proposition

The inclusion of $\mathbf{Y}_q^{\mathcal{NT}}$ in $\mathcal{NT}_{\mathbf{Z}}$ is a Nica covariant representation. Hence, there exists a homomorphism $\Omega'^{\mathcal{NT}} : \mathcal{NT}_{\mathbf{Y}^{\mathcal{NT}}} \to \mathcal{NT}_{\mathbf{Z}}$ such that $\Omega'^{\mathcal{NT}} \circ i_{\mathbf{Y}_q^{\mathcal{NT}}} = \mathrm{incl}_{\mathbf{Y}_q^{\mathcal{NT}}}$ for each $q \in Q$.

- The homomorphism from $\mathcal{NT}_{\mathbf{Z}}$ to $\mathcal{NT}_{\mathbf{Y}^{\mathcal{NT}}}$ is slightly more difficult to get. The key point is that

$$i_{\mathbf{Z}}(\mathbf{Z}_{(p,q)}) \subseteq \mathbf{Y}_q^{\mathcal{NT}}$$

for any $(p,q) \in P \rtimes_\alpha Q$.

To see this observe that if $z \in \mathbf{Z}_{(p,q)}$ and $z = z' \cdot \langle z', z' \rangle_A^{(p,q)}$ is its Hewitt–Cohen–Blanchard factorisation, then

$$i_{\mathbf{Z}_{(p,q)}}(z) = i_{\mathbf{Z}_{(p,q)}}(z') i_{\mathbf{Z}_{(e_G, e_H)}}\big(\langle z', z' \rangle_A^{(p,q)}\big) \in \mathbf{Y}_q^{\mathcal{NT}}.$$

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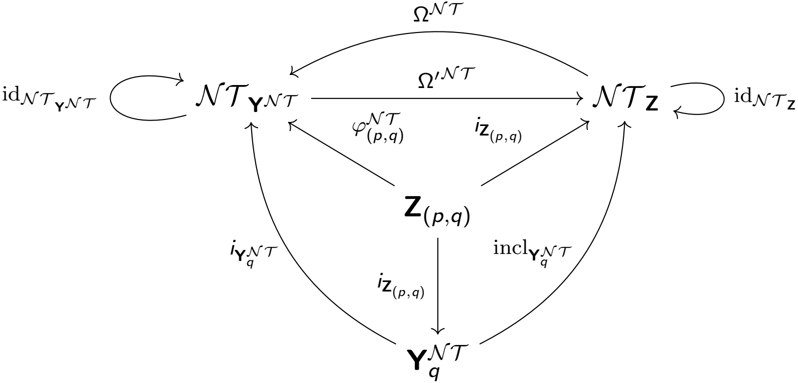
Proposition

The map $\varphi^{\mathcal{NT}} : \mathbf{Z} \to \mathcal{NT}_{\mathbf{Y}^{\mathcal{NT}}}$ defined by $\varphi^{\mathcal{NT}}_{(p,q)} := i_{\mathbf{Y}_q^{\mathcal{NT}}} \circ i_{\mathbf{Z}_{(p,q)}}$ is a Nica covariant representation of $\mathbf{Z}$. Hence, there exists a homomorphism $\Omega^{\mathcal{NT}} : \mathcal{NT}_{\mathbf{Z}} \to \mathcal{NT}_{\mathbf{Y}^{\mathcal{NT}}}$ such that $\Omega^{\mathcal{NT}} \circ i_{\mathbf{Z}_{(p,q)}} = i_{\mathbf{Y}_q^{\mathcal{NT}}} \circ i_{\mathbf{Z}_{(p,q)}}$.

Theorem

The homomorphisms $\Omega'^{\mathcal{NT}} : \mathcal{NT}_{\mathbf{Y}^{\mathcal{NT}}} \to \mathcal{NT}_{\mathbf{Z}}$ *and* $\Omega^{\mathcal{NT}} : \mathcal{NT}_{\mathbf{Z}} \to \mathcal{NT}_{\mathbf{Y}^{\mathcal{NT}}}$ *are mutually inverse. Hence,* $\mathcal{NT}_{\mathbf{Y}^{\mathcal{NT}}}$ *and* $\mathcal{NT}_{\mathbf{Z}}$ *are isomorphic.*

In summary, the maps in the following diagram exist, and make the diagram commutative.



Straightforward corollaries

Katsura has numerous results relating the coefficent algebra of a Hilbert bimodule to its Toeplitz algebra. Repeatedly decomposing $\mathbb{N}^k$ as $\mathbb{N} \times \mathbb{N}^{k-1}$, we may view Nica–Toeplitz algebras of product systems over $\mathbb{N}^k$ as a k-times iterated Toeplitz algebras.

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Corollary

Let $\mathbf{Z}$ be a compactly aligned product system over $\mathbb{N}^k$ with coefficient algebra A. Then

(1) *The homomorphism $i_{\mathbf{Z}_0}$ induces an isomorphism $K_*(A) \cong K_*(\mathcal{NT}_{\mathbf{Z}})$;*

(2) *If A is separable and the fibres $\{\mathbf{Z}_{e_i} : 1 \leq i \leq k\}$ are countably generated as Hilbert A-modules, then $i_{\mathbf{Z}_0}$ induces a KK-equivalence between A and $\mathcal{NT}_{\mathbf{Z}}$;*

(3) *A is exact if and only if $\mathcal{NT}_{\mathbf{Z}}$ is exact; and*

(4) *A is nuclear if and only if $\mathcal{NT}_{\mathbf{Z}}$ is nuclear.*

Iterating the Cuntz–Nica–Pimsner construction

We now aim to prove the following analogous decomposition result for Cuntz–Nica–Pimsner algebras.

Theorem

Suppose that G is an amenable group and Q is directed. If A acts faithfully on each fibre of $\mathbf{Z}$ and compactly on each $\mathbf{Z}_{(e_G,q)}$, then there exists a compactly aligned product system $\mathbf{Y}^{\mathcal{NO}}$ over (H, Q) with coefficient algebra $\mathcal{NO}_{\mathbf{X}}$ such that $\mathcal{NO}_{\mathbf{Y}^{\mathcal{NO}}}$ and $\mathcal{NO}_{\mathbf{Z}}$ are isomorphic.

Constructing the product system $\mathbf{Y}^{\mathcal{NO}}$

First things first, we need to know that $\mathcal{NO}_{\mathbf{Z}}$ contains a faithful copy of $\mathcal{NO}_{\mathbf{X}}$.

Proposition

Suppose A acts faithfully on each fibre of $\mathbf{Z}$. Then the inclusion of $\mathbf{X}$ in $\mathbf{Z}$ induces a homomorphism $\phi_{\mathbf{X}}^{\mathcal{NO}} : \mathcal{NO}_{\mathbf{X}} \to \mathcal{NO}_{\mathbf{Z}}$ such that $\phi_{\mathbf{X}}^{\mathcal{NO}} \circ j_{\mathbf{X}} = j_{\mathbf{Z}}$ for each $x \in \mathbf{X}$. If G is an amenable group, then $\phi_{\mathbf{X}}^{\mathcal{NO}}$ is injective.

Proof Sketch.

The idea is to show that if $\{T_p \in \mathcal{K}_A(\mathbf{X}_p) : p \in F\}$ is a finite collection of compact operators such that $\sum_{p\in F} \iota_p^s(T_p) = 0 \in \mathcal{L}_A(\mathbf{X}_s)$ for large s then $\sum_{p\in F} j_{\mathbf{Z}}^{((p,e_H))}(T_p) = 0$. Since $j_{\mathbf{Z}}$ is Cuntz–Pimsner covariant, it suffices to show that

$$\sum_{p\in F} \iota_{(p,e_H)}^{(s,t)}(T_p) = 0 \in \mathcal{L}_A(\mathbf{Z}_{(s,t)}) \quad \text{for large } (s,t).$$

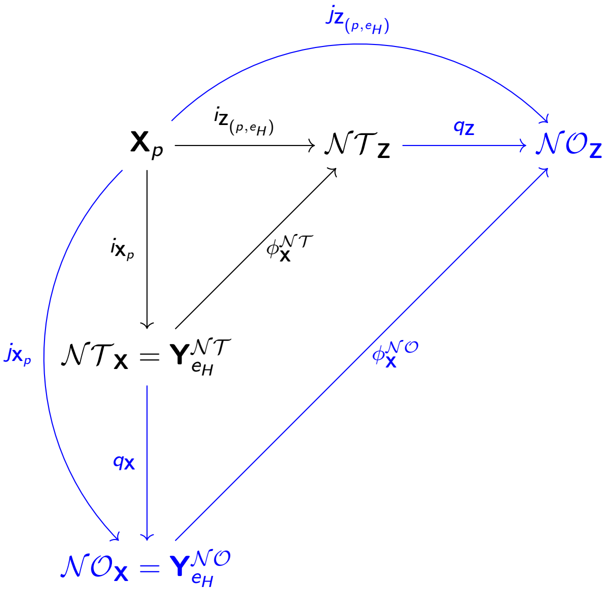
Since $\mathbf{Z}_{(s,t)} \cong \mathbf{Z}_{(s,e_H)} \otimes_A \mathbf{Z}_{(e_G,t)} \cong \mathbf{X}_s \otimes_A \mathbf{Z}_{(e_G,t)}$, we get that

$$\sum_{p\in F} \iota_{(p,e_H)}^{(s,t)}(T_p) = M^{\mathbf{Z}}_{(s,e_H),(e_G,t)} \circ \left(\sum_{p\in F} \iota_p^s(T) \otimes_A \mathrm{id}_{\mathbf{Z}_{(e_G,t)}} \right) \circ \left(M^{\mathbf{Z}}_{(s,e_H),(e_G,t)} \right)^{-1}.$$

We then get what we need since $(G \rtimes_\alpha H, P \rtimes_\alpha Q)$ has the product order.

To get injectivity use the gauge invariant uniqueness theorem. * □

In summary, for every $p \in P$, the following diagram is commutative:



- We use the injective homomorphism $\phi_{\mathbf{X}}^{\mathcal{NO}}$ to construct a product system over (H, Q) with coefficient algebra $\mathcal{NO}_{\mathbf{X}}$. The idea is to make use of the collection of Hilbert

  $\mathcal{NT}_{\mathbf{X}}$-bimodules $\left\{\mathbf{Y}_q^{\mathcal{NT}} : q \in Q\right\}$ exhibited earlier and apply the canonical quotient homomorphisms $q_{\mathbf{X}} : \mathcal{NT}_{\mathbf{X}} \to \mathcal{NO}_{\mathbf{X}}$ and $q_{\mathbf{Z}} : \mathcal{NT}_{\mathbf{Z}} \to \mathcal{NO}_{\mathbf{Z}}$ at the appropriate places. For each $q \in Q \setminus \{e_H\}$, we set $\mathbf{Y}_q^{\mathcal{NO}} := q_{\mathbf{Z}}\left(\mathbf{Y}_q^{\mathcal{NT}}\right) \subseteq \mathcal{NO}_{\mathbf{Z}}$.

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- The key point is that we have the following (easily verified) relationship between $\phi_{\mathbf{X}}^{\mathcal{NO}}$ and $\phi_{\mathbf{X}}^{\mathcal{NT}}$:

$$\phi_{\mathbf{X}}^{\mathcal{NO}} \circ q_{\mathbf{X}} = q_{\mathbf{Z}} \circ \phi_{\mathbf{X}}^{\mathcal{NT}}.$$

- There exists a well-defined map from $\mathbf{Y}_q^{\mathcal{NO}} \times \mathcal{NO}_{\mathbf{X}} \to \mathbf{Y}_q^{\mathcal{NO}}$ given by

$$(q_{\mathbf{Z}}(y), q_{\mathbf{X}}(a)) \mapsto q_{\mathbf{Z}}(y \cdot a) \quad \text{for all } y \in \mathbf{Y}_q^{\mathcal{NT}},\ a \in \mathcal{NT}_{\mathbf{X}}.$$

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- There exists a well-defined map $\langle \cdot, \cdot \rangle_{\mathcal{NO}_{\mathbf{X}}}^q : \mathbf{Y}_q^{\mathcal{NO}} \times \mathbf{Y}_q^{\mathcal{NO}} \to \mathcal{NO}_{\mathbf{X}}$ given by

$$\langle q_{\mathbf{Z}}(y), q_{\mathbf{Z}}(w) \rangle_{\mathcal{NO}_{\mathbf{X}}}^q = q_{\mathbf{X}}\big(\langle y, w \rangle_{\mathcal{NT}_{\mathbf{X}}}^q\big) \quad \text{for all } y, w \in \mathbf{Y}_q^{\mathcal{NT}}.$$

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$$\Phi_q^{\mathcal{NO}}(q_{\mathbf{X}}(a))(q_{\mathbf{Z}}(y)) = q_{\mathbf{Z}}(\Phi_q^{\mathcal{NT}}(a)(y)) \quad \text{for all } y \in \mathbf{Y}_q^{\mathcal{NT}},\ a \in \mathcal{NT}_{\mathbf{X}}$$

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- For each $q, t \in Q \setminus \{e_H\}$ there is a well-defined Hilbert $\mathcal{NO}_{\mathbf{X}}$-bimodule isomorphism $M_{q,t}^{\mathbf{Y}^{\mathcal{NO}}} : \mathbf{Y}_q^{\mathcal{NO}} \otimes_{\mathcal{NO}_{\mathbf{X}}} \mathbf{Y}_t^{\mathcal{NO}} \to \mathbf{Y}_{qt}^{\mathcal{NO}}$ given by
$$M_{q,t}^{\mathbf{Y}^{\mathcal{NO}}}(q_{\mathbf{Z}}(y) \otimes_{\mathcal{NO}_{\mathbf{X}}} q_{\mathbf{Z}}(y)) = q_{\mathbf{Z}}\big(M_{q,t}^{\mathbf{Y}^{\mathcal{NT}}}(y \otimes_{\mathcal{NT}_{\mathbf{X}}} w)\big) \quad \text{for all } y \in \mathbf{Y}_q^{\mathcal{NT}},\ w \in \mathbf{Y}_t^{\mathcal{NT}}.$$

- With $\mathbf{Y}_{e_H}^{\mathcal{NO}} := {}_{\mathcal{NO}_{\mathbf{X}}}(\mathcal{NO}_{\mathbf{X}})_{\mathcal{NO}_{\mathbf{X}}}$, the semigroup $\mathbf{Y}^{\mathcal{NO}} := \bigsqcup_{q \in Q} \mathbf{Y}_q^{\mathcal{NO}}$ (multiplication is given by multiplication in $\mathcal{NO}_{\mathbf{Z}}$) is a compactly aligned product system over (H, Q) with coefficient algebra $\mathcal{NO}_{\mathbf{X}}$.

$$\begin{array}{ccc} \mathbf{Y}_q^{\mathcal{NT}} \times \mathcal{NT}_{\mathbf{X}} & \xrightarrow{(y,a)\mapsto y\cdot a} & \mathbf{Y}_q^{\mathcal{NT}} \\ \downarrow{\scriptstyle q_{\mathbf{Z}}\times q_{\mathbf{X}}} & & \downarrow{\scriptstyle q_{\mathbf{Z}}} \\ \mathbf{Y}_q^{\mathcal{NO}} \times \mathcal{NO}_{\mathbf{X}} & \xrightarrow[(y,a)\mapsto y\cdot a]{} & \mathbf{Y}_q^{\mathcal{NO}} \end{array}$$

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Establishing the isomorphism $\mathcal{NO}_{\mathbf{Y}^{\mathcal{NO}}} \cong \mathcal{NO}_{\mathbf{Z}}$

- Firstly, we need to know that the Cuntz–Nica–Pimsner algebra of the product system $\mathbf{Y}^{\mathcal{NO}}$ exists (i.e. the homomorphisms $\Phi_q^{\mathcal{NO}} : \mathcal{NO}_{\mathbf{X}} \to \mathcal{L}_{\mathcal{NO}_{\mathbf{X}}}(\mathbf{Y}_q^{\mathcal{NO}})$ are injective).

Proposition

Suppose G is an amenable group, A acts faithfully on each fibre of $\mathbf{Z}$, so that the product system $\mathbf{Y}^{\mathcal{NO}}$ exists. Then $\Phi_q^{\mathcal{NO}} : \mathcal{NO}_{\mathbf{X}} \to \mathcal{L}_{\mathcal{NO}_{\mathbf{X}}}\left(\mathbf{Y}_q^{\mathcal{NO}}\right)$ is injective.

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- Additional standing hypothesis: To avoid repeating ourselves, we'll assume from now on that G is amenable and A acts faithfully on each fibre of $\mathbf{Z}$.

Proposition

The representation $\varphi^{\mathcal{NO}} : \mathbf{Z} \to \mathcal{NO}_{\mathbf{Y}^{\mathcal{NO}}}$ *given by* $\varphi^{\mathcal{NO}}_{(p,q)} := j_{\mathbf{Y}_q^{\mathcal{NO}}} \circ j_{\mathbf{Z}_{(p,q)}}$ *is Cuntz–Nica–Pimsner covariant, and so induces a homomorphism* $\Omega^{\mathcal{NO}} : \mathcal{NO}_{\mathbf{Z}} \to \mathcal{NO}_{\mathbf{Y}^{\mathcal{NO}}}$ *such that* $\Omega^{\mathcal{NO}} \circ j_{\mathbf{Z}_{(p,q)}} = j_{\mathbf{Y}_q^{\mathcal{NO}}} \circ j_{\mathbf{Z}_{(p,q)}}$.

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Proposition

Suppose that A *acts compactly on each* $\mathbf{Z}_{(e_G,q)}$ *(this implies that* $\mathcal{NO}_{\mathbf{X}}$ *acts compactly on each fibre of* $\mathbf{Y}^{\mathcal{NO}}$*) and* Q *is directed. Then the inclusion of* $\mathbf{Y}^{\mathcal{NO}}_q$ *in* $\mathcal{NO}_{\mathbf{Z}}$ *is a Cuntz–Nica–Pimsner covariant representation of* $\mathbf{Y}^{\mathcal{NO}}$*, and so induces a homomorphism* $\Omega'^{\mathcal{NO}} : \mathcal{NO}_{\mathbf{Y}^{\mathcal{NO}}} \to \mathcal{NO}_{\mathbf{Z}}$ *such that* $\Omega'^{\mathcal{NO}} \circ j_{\mathbf{Y}^{\mathcal{NO}}_q} = \mathrm{incl}_{\mathbf{Y}^{\mathcal{NO}}_q}$.

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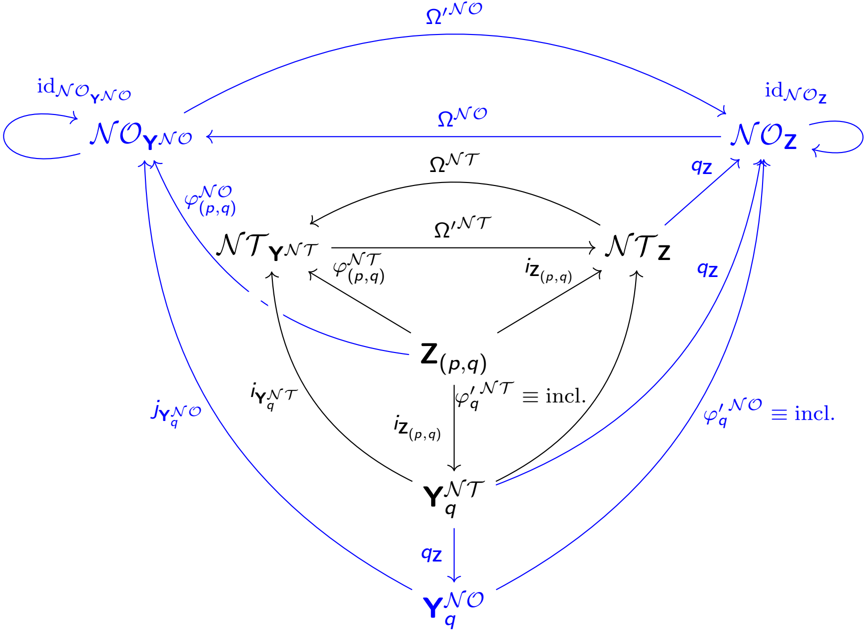
Theorem

The homomorphisms $\Omega^{\mathcal{NO}}$ *and* $\Omega'^{\mathcal{NO}}$ *are mutually inverse. Hence,* $\mathcal{NO}_{\mathbf{Z}} \cong \mathcal{NO}_{\mathbf{Y}^{\mathcal{NO}}}$.



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$\mathrm{id}_{\mathcal{NO}_{\mathbf{Z}}}$
$\mathcal{NO}_{\mathbf{Y}^{\mathcal{NO}}}$
$\Omega^{\mathcal{NO}}$
$\mathcal{NO}_{\mathbf{Z}}$
$\Omega^{\mathcal{NT}}$
$q_{\mathbf{Z}}$
$\varphi^{\mathcal{NO}}_{(p,q)}$
$\Omega'^{\mathcal{NT}}$
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$\mathcal{NT}_{\mathbf{Z}}$
$\varphi^{\mathcal{NT}}_{(p,q)}$
$i_{\mathbf{Z}_{(p,q)}}$
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$q_{\mathbf{Z}}$
$\mathbf{Y}_q^{\mathcal{NO}}$

Using this theorem we can view the Cuntz–Nica–Pimsner algebra of a compactly aligned product system over $\mathbb{N}^k$ (with faithful and compact left actions) as a k-times iterated Cuntz–Pimsner algebra.

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Corollary

Let $\mathbf{Z}$ be a product system over $\mathbb{N}^k$ with coefficient algebra A. Suppose that A acts faithfully and compactly on the fibres $\{\mathbf{Z}_{e_i} : 1 \leq i \leq k\}$. Then

(1) *A is exact if and only if $\mathcal{NO}_{\mathbf{Z}}$ is exact;*

(2) *If A is nuclear, then $\mathcal{NO}_{\mathbf{Z}}$ is nuclear;*

(3) *If each $\mathbf{Z}_{e_i}$ is countably generated as a Hilbert A-module and A is separable, nuclear, and satisfies the universal coefficient theorem, then $\mathcal{NO}_{\mathbf{Z}}$ satisfies the universal coefficient theorem.*

Relative Cuntz–Nica–Pimsner algebras

We now assume that α is trivial so that

$$(G \rtimes_\alpha H, P \rtimes_\alpha Q) = (G \times H, P \times Q) \cong (H \times G, Q \times P).$$

If we also assume that

- H is amenable;
- A acts faithfully on each fibre of $\mathbf{Z}$;

then we can swap the roles of G and H (and P and Q) in our earlier results to get the following:

(1) A compactly aligned product systems $\mathbf{V} := \bigsqcup_{q \in Q} \mathbf{V}_q := \bigsqcup_{q \in Q} \mathbf{Z}_{(e_G, q)}$ over (H, Q) with coefficient algebra A;

(2) An injective homomorphism $\phi_{\mathbf{V}}^{\mathcal{NO}} : \mathcal{NO}_{\mathbf{V}} \to \mathcal{NO}_{\mathbf{Z}}$ such that $\phi_{\mathbf{V}}^{\mathcal{NO}} \circ j_{\mathbf{V}} = j_{\mathbf{Z}}$;

(3) A compactly aligned product system $\mathbf{W}^{\mathcal{NO}}$ over (G, P) with coefficient algebra $\mathcal{NO}_{\mathbf{V}}$, with fibres given by

$$\mathbf{W}_p^{\mathcal{NO}} := \overline{\operatorname{span}}\{j_{\mathbf{Z}_{(p,e_H)}}(x)\phi_{\mathbf{V}}^{\mathcal{NO}}(b) : x \in \mathbf{Z}_{(p,e_H)}, b \in \mathcal{NO}_{\mathbf{V}}\}$$

(4) We can also show that each of the homomorphisms $\Phi_q^{\mathcal{NT}} : \mathcal{NT}_{\mathbf{X}} \to \mathcal{L}_{\mathcal{NT}_{\mathbf{X}}}\left(\mathbf{Y}_q^{\mathcal{NT}}\right)$ is injective (and so we know that $\mathcal{NO}_{\mathbf{Y}^{\mathcal{NT}}}$ exists).

We then have the following theorem:

Theorem

Suppose that Q is directed, and A acts compactly on each $\mathbf{Z}_{(e_G,q)}$, then $\mathcal{NO}_{\mathbf{Y}^{\mathcal{NT}}} \cong \mathcal{NT}_{\mathbf{W}^{\mathcal{NO}}}$.

Sketch Proof.

(1) Show that the map $\vartheta : \mathbf{Z} \to \mathcal{NT}_{\mathbf{W}^{\mathcal{NO}}}$ defined by $\vartheta_{(p,q)} := i_{\mathbf{W}_p^{\mathcal{NO}}} \circ j_{\mathbf{Z}_{(p,q)}}$ is a Nica covariant representation of $\mathbf{Z}$.

(2) Show that (subject to some hypotheses) restricting the induced homomorphism $\Xi : \mathcal{NT}_{\mathbf{Z}} \to \mathcal{NT}_{\mathbf{W}^{\mathcal{NO}}}$ to $\mathbf{Y}^{\mathcal{NT}} \subseteq \mathcal{NT}_{\mathbf{Z}}$ gives a Cuntz–Nica–Pimsner covariant representation of $\mathbf{Y}^{\mathcal{NT}}$, which we denote by Ψ (the idea is that Ξ plays the same role as the inclusion map). Let $\omega : \mathcal{NO}_{\mathbf{Y}^{\mathcal{NT}}} \to \mathcal{NT}_{\mathbf{W}^{\mathcal{NO}}}$ denote the induced homomorphism.

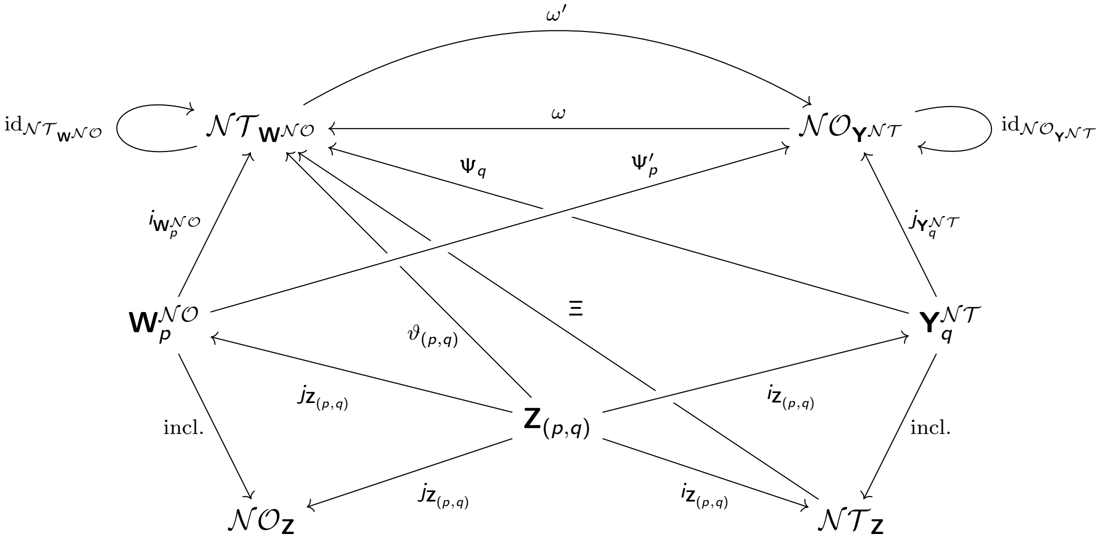
(3) Show that $\vartheta' : \mathbf{V} \to \mathcal{NO}_{\mathbf{Y}^{\mathcal{NT}}}$ defined by $\vartheta'_q := j_{\mathbf{Y}_q^{\mathcal{NT}}} \circ i_{\mathbf{Z}_{(e_G,q)}}$ is a Cuntz–Nica–Pimsner covariant representation. Let $\Psi'_{e_G} : \mathcal{NO}_{\mathbf{V}} \to \mathcal{NO}_{\mathbf{Y}^{\mathcal{NT}}}$ denote the induced homomorphism.

(4) Use Ψ'_{e_G} to construct, for each $p \in P \setminus \{e_G\}$, a norm decreasing map $\Psi'_p : \mathbf{W}_p^{\mathcal{NO}} \to \mathcal{NO}_{\mathbf{Y}^{\mathcal{NT}}}$ such that $\Psi'_p \circ j_{\mathbf{Z}_{(p,q)}} = j_{\mathbf{Y}_q^{\mathcal{NT}}} \circ i_{\mathbf{Z}_{(p,q)}}$.

(5) Check that the collection of maps $\{\Psi'_p : p \in P\}$ gives a Nica covariant representation of $\mathbf{W}^{\mathcal{NO}}$, and let $\omega' : \mathcal{NT}_{\mathbf{W}^{\mathcal{NO}}} \to \mathcal{NO}_{\mathbf{Y}^{\mathcal{NT}}}$ denote the induced homomorphism.

(6) Verify that ω and ω' are mutually inverse.

$\square$


ω'
$\mathrm{id}_{\mathcal{NT}_{\mathbf{W}^{\mathcal{NO}}}}$
$\mathcal{NT}_{\mathbf{W}^{\mathcal{NO}}}$
ω
$\mathcal{NO}_{\mathbf{Y}^{\mathcal{NT}}}$
$\mathrm{id}_{\mathcal{NO}_{\mathbf{Y}^{\mathcal{NT}}}}$
Ψ_q
Ψ'_p
$i_{\mathbf{W}_p^{\mathcal{NO}}}$
$j_{\mathbf{Y}_q^{\mathcal{NT}}}$
$\mathbf{W}_p^{\mathcal{NO}}$
Ξ
$\mathbf{Y}_q^{\mathcal{NT}}$
$\vartheta_{(p,q)}$
$j_{\mathbf{Z}_{(p,q)}}$
$i_{\mathbf{Z}_{(p,q)}}$
incl.
$\mathbf{Z}_{(p,q)}$
incl.
$j_{\mathbf{Z}_{(p,q)}}$
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- Can we use this procedure to compute the K-theory of k-graphs algebras (row finite, no sources) when $k \geq 3$ (idea: view $C^*(\Lambda)$ as an iterated Cuntz–Pimsner algebra and apply the Pimsner–Voiculescu exact sequence)?
- Can we prove that the K-theory of a k-graph algebra (row finite, no sources, $k \geq 3$) does not depend on the factorisation scheme (i.e. just depends on the skeleton)?