C^*-algebras of left cancellative small categories

Jack Spielberg, Arizona State University

Interactions between semigroups and operator algebras,
Newcastle, 24-27 July 2017

Outline

1. C^*-algebras from monoids and categories (my interpretation)
2. regular representations
3. Wiener-Hopf algebras
4. relation with some other constructions

G - discrete group. What should be $C^*(G)$?

G - discrete group. What should be $C^*(G)$?

$$\begin{array}{ccc} G & \xrightarrow{\quad U \quad} & \mathcal{U}(H) \\ \big\downarrow & & \big\downarrow \\ C^*(G) & \underset{\exists !}{\dashrightarrow} & B(H) \end{array}$$

universal for unitary representations.

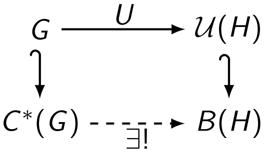
G - discrete group. What should be $C^*(G)$?

$$\begin{array}{ccc} G & \xrightarrow{\quad U \quad} & \mathcal{U}(H) \\ \big\downarrow & & \big\downarrow \\ C^*(G) & \underset{\exists!}{\dashrightarrow} & B(H) \end{array}$$

universal for unitary representations.

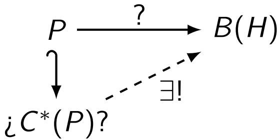
P - discrete monoid. What should be $C^*(P)$?

G - discrete group. What should be $C^*(G)$?



universal for unitary representations.

P - discrete monoid. What should be $C^*(P)$?



— Not clear what is desired.

G - discrete group. What should be $C^*(G)$?

$$\begin{array}{ccc} G & \xrightarrow{\;U\;} & \mathcal{U}(H) \\ \big\downarrow & & \big\downarrow \\ C^*(G) & \overset{}{\underset{\exists!}{\dashrightarrow}} & B(H) \end{array}$$

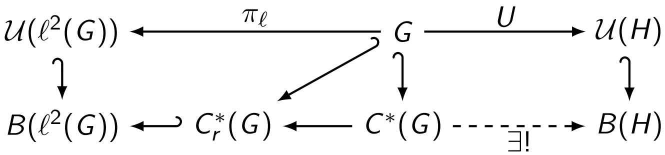
universal for unitary representations.

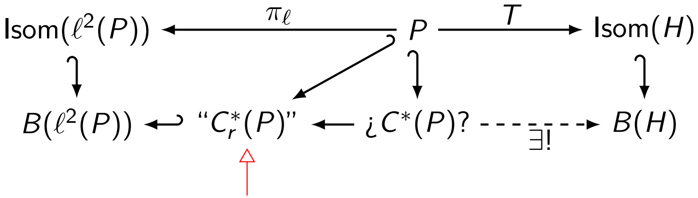
P - discrete monoid. What should be $C^*(P)$?

$$\begin{array}{ccc} P & \xrightarrow{\;T\;} & \mathrm{Isom}(H) \\ \big\downarrow & & \big\downarrow \\ ¿C^*(P)? & \overset{}{\underset{\exists!}{\dashrightarrow}} & B(H) \end{array}$$

universal for representations by isometries. Not generally “right” — too big. Should use a quotient.

$$
\begin{array}{ccccccc}
\mathcal{U}(\ell^2(G)) & \multicolumn{3}{c}{\xleftarrow{\qquad\qquad\pi_\ell\qquad\qquad}} & G & \xrightarrow{\qquad U \qquad} & \mathcal{U}(H) \\
\big\downarrow & & & \swarrow & \big\downarrow & & \big\downarrow \\
B(\ell^2(G)) & \hookleftarrow & C_r^*(G) & \longleftarrow & C^*(G) & \overset{}{\underset{\exists!}{\dashrightarrow}} & B(H)
\end{array}
$$





"$C_r^*(P)$" is a reasonable choice for the "smallest" quotient that is "right".

What universal property/ies should be used to characterize $C^*(P)$?

Example $P = \mathbb{F}_n^+$ free semigroup.

For α, $\beta \in P$ write $\alpha \leq \beta$ if $\beta = \alpha\alpha'$ for some $\alpha' \in P$ (β is an *extension* of α). Nica observed: for $P = \mathbb{F}_n^+$, if α, β have a common extension then they have a unique minimal common extension:

$$\alpha P \cap \beta P \neq \varnothing \implies \exists! \gamma \in P \text{ such that } \alpha P \cap \beta P = \gamma P.$$

(P is called *singly aligned* or *LCM*.) We write $\gamma = \alpha \vee \beta$.

Letting $T_\alpha = \pi_\ell(\alpha) \in \mathrm{Isom}(\ell^2(P))$ we have

$$(*) \quad T_\alpha T_\alpha^* \cdot T_\beta T_\beta^* = \begin{cases} T_\gamma T_\gamma^*, & \text{if } \gamma = \alpha \vee \beta \\ 0, & \text{if } \alpha P \cap \beta P = \varnothing. \end{cases}$$

Nica: *define* $C^*(P)$ to be universal for $(*)$.

Example $P = \mathbb{F}_n^+$ free semigroup.

For α, $\beta \in P$ write $\alpha \leq \beta$ if $\beta = \alpha\alpha'$ for some $\alpha' \in P$ (β is an *extension* of α). Nica observed: for $P = \mathbb{F}_n^+$, if α, β have a common extension then they have a unique minimal common extension:

$$\alpha P \cap \beta P \neq \varnothing \implies \exists ! \gamma \in P \text{ such that } \alpha P \cap \beta P = \gamma P.$$

(P is called *singly aligned* or *LCM*.) We write $\gamma = \alpha \vee \beta$.

Letting $T_\alpha = \pi_\ell(\alpha) \in \mathrm{Isom}(\ell^2(P))$ we have

$$(*)\ T_\alpha T_\alpha^* \cdot T_\beta T_\beta^* = \begin{cases} T_\gamma T_\gamma^*, & \text{if } \gamma = \alpha \vee \beta \\ 0, & \text{if } \alpha P \cap \beta P = \varnothing. \end{cases}$$

Nica: *define* $C^*(P)$ to be universal for $(*)$.

Q: Why is this "right"?

Example $P = \mathbb{F}_n^+$ free semigroup.

For $\alpha, \beta \in P$ write $\alpha \leq \beta$ if $\beta = \alpha\alpha'$ for some $\alpha' \in P$ (β is an *extension* of α). Nica observed: for $P = \mathbb{F}_n^+$, if α, β have a common extension then they have a unique minimal common extension:

$$\alpha P \cap \beta P \neq \varnothing \implies \exists! \gamma \in P \text{ such that } \alpha P \cap \beta P = \gamma P.$$

(P is called *singly aligned* or *LCM*.) We write $\gamma = \alpha \vee \beta$.

Letting $T_\alpha = \pi_\ell(\alpha) \in \mathrm{Isom}(\ell^2(P))$ we have

$$(*) \quad T_\alpha T_\alpha^* \cdot T_\beta T_\beta^* = \begin{cases} T_\gamma T_\gamma^*, & \text{if } \gamma = \alpha \vee \beta \\ 0, & \text{if } \alpha P \cap \beta P = \varnothing. \end{cases}$$

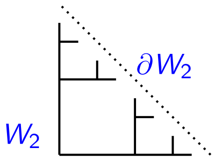
Nica: *define* $C^*(P)$ to be universal for $(*)$.

Q: Why is this "right"? A: $C^*(\mathbb{F}_n^+) = \mathcal{TO}_n$.

There ought to be a more fundamental reason . . .

There ought to be a more fundamental reason . . .

$(\mathbb{F}_n^+, \leq)$ is a tree, W_n. For $\alpha \in \mathbb{F}_n^+$, $\tau^\alpha : \beta \mapsto \alpha\beta$ is an endomorphism.



τ^α, $\sigma^\alpha = (\tau^\alpha)^{-1}$ extend to partial homeomorphisms of $W_n \cup \partial W_n$.

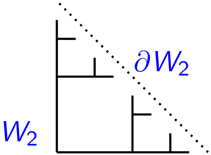
σ^α is the one-sided Bernoulli shift.

Renault: there is a groupoid G with $G^{(0)} = W_n \cup \partial W_n$ — the *groupoid of germs* of $\{\sigma^\alpha, \tau^\alpha\}$. Moreover $C^*(G) = \mathcal{TO}_n$.

We don't have to "guess" at the relation $(*)$.

There ought to be a more fundamental reason . . .

$(\mathbb{F}_n^+, \leq)$ is a tree, W_n. For $\alpha \in \mathbb{F}_n^+$, $\tau^\alpha : \beta \mapsto \alpha\beta$ is an endomorphism.



τ^α, $\sigma^\alpha = (\tau^\alpha)^{-1}$ extend to partial homeomorphisms of $W_n \cup \partial W_n$.

σ^α is the one-sided Bernoulli shift.

Renault: there is a groupoid G with $G^{(0)} = W_n \cup \partial W_n$ — the *groupoid of germs* of $\{\sigma^\alpha, \tau^\alpha\}$. Moreover $C^*(G) = \mathcal{T}\mathcal{O}_n$.

We don't have to "guess" at the relation $(*)$.

Many situations pose a similar problem: which came first, the relations or the algebra? (E.g. graphs, higher rank graphs, arbitrary monoids.) We can generalize the above construction.

Λ - a *small category* (like a monoid, only multiplication is not always defined).

For $\alpha \in \Lambda$ define the *right shift* $\tau^\alpha : \beta \in s(\alpha)\Lambda \mapsto \alpha\beta \in \alpha\Lambda$.

Assume Λ is *left cancellative*: $\alpha\beta = \alpha\gamma \implies \beta = \gamma$. Then τ^α is one-to-one. Let $\sigma^\alpha = (\tau^\alpha)^{-1}$.

How to imitate the construction of ∂W_n?

Λ - a *small category* (like a monoid, only multiplication is not always defined).

For $\alpha \in \Lambda$ define the *right shift* $\tau^\alpha : \beta \in s(\alpha)\Lambda \mapsto \alpha\beta \in \alpha\Lambda$.

Assume Λ is *left cancellative*: $\alpha\beta = \alpha\gamma \implies \beta = \gamma$. Then τ^α is one-to-one. Let $\sigma^\alpha = (\tau^\alpha)^{-1}$.

How to imitate the construction of ∂W_n? One of many ways:

$\mathscr{A}$ = smallest ring of sets such that for all $\alpha \in \Lambda$,

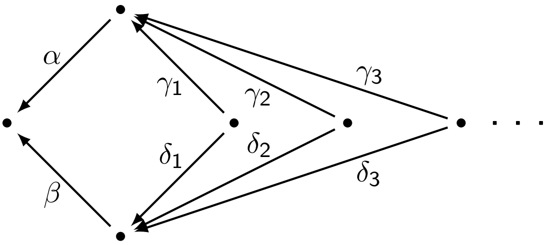
(i) $\alpha\Lambda \in \mathscr{A}$
(ii) $E \in \mathscr{A} \implies \tau^\alpha(E), \sigma^\alpha(E) \in \mathscr{A}$;

$X := \{$*ultrafilters* in $\mathscr{A}\}$

(Equivalently, let $A = \overline{\text{span}}\{\chi_E : E \in \mathscr{A}\} \subseteq \ell^\infty(\Lambda)$, an abelian C^*-algebra. Then $X = \widehat{A}$.)

$x \in X \leftrightarrow \mathcal{U}_x \subseteq \mathscr{A}$ ultrafilter. $\tau^\alpha(\mathcal{U}_x) \neq \varnothing$ iff $s(\alpha)\Lambda \in \mathcal{U}_x$. Then $\tau^\alpha(\mathcal{U}_x)$ is an ultrafilter base; write $\mathcal{U}_{\widehat{\tau^\alpha}(x)}$ for the ultrafilter it generates. $\widehat{\tau^\alpha}$, $\widehat{\sigma^\alpha} = \widehat{\tau^\alpha}^{-1}$ are partial homeomorphisms of X.

What do typical sets in $\mathscr{A}$ look like? Consider the following LCSC Λ:



with relations $\alpha\gamma_i = \beta\delta_i$.

$\tau^\beta(\Lambda) = \beta\Lambda = \{\beta, \beta\delta_1, \beta\delta_2, \ldots\} = \{\beta, \alpha\gamma_1, \alpha\gamma_2, \ldots\}$.

$\sigma^\alpha \circ \tau^\beta(\Lambda) = \{\gamma_1, \gamma_2, \ldots\}$ (Note: β is no longer in the domain.)

Let $\zeta = (\alpha, \beta)$. Then $A(\zeta) = \mathrm{dom}(\varphi_\zeta) = \{\delta_1, \delta_2, \ldots\}$, and $\varphi_\zeta(\delta_i) = \gamma_i$.

More generally,

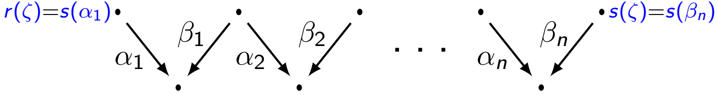
$\beta_n \Lambda = \tau^{\beta_n}(\Lambda)$

$\sigma^{\alpha_n} \circ \tau^{\beta_n}(\Lambda)$

$\dots$

$\sigma^{\alpha_1} \circ \tau^{\beta_1} \circ \dots \circ \sigma^{\alpha_n} \circ \tau^{\beta_n}(\Lambda)$.

Write $\zeta = (\alpha_1, \beta_1, \dots, \alpha_n, \beta_n)$ - a *zigzag*:



$\mathcal{Z}(\Lambda)$ = set of all zigzags; composition by concatenation.

$\varphi_\zeta = \sigma^{\alpha_1} \circ \tau^{\beta_1} \circ \dots \circ \sigma^{\alpha_n} \circ \tau^{\beta_n}$ - *zigzag map*, partial bijection of Λ.

$\varphi_\zeta^{-1} = \varphi_{\overline{\zeta}}$, where $\overline{\zeta} = (\beta_n, \alpha_n, \dots, \beta_1, \alpha_1)$.

$A(\zeta) := \mathrm{dom}(\varphi_\zeta)$ - the *zigzag set* (or *constructible right ideal*).

$\mathcal{D}^{(0)} = \{A(\zeta) \neq \varnothing : \zeta \in \mathcal{Z}(\Lambda)\}$;

closed under intersection: $A(\zeta) = A(\bar{\zeta}\zeta)$, $A(\zeta) \cap A(\theta) = A(\bar{\zeta}\zeta\bar{\theta}\theta)$.

$\mathcal{D} = \{E \setminus \cup_{i=1}^{n} F_i : E, F_i \in \mathcal{D}^{(0)},\ \cup_{i=1}^{n} F_i \subsetneq E\}$.

$\mathscr{A} = \{\sqcup_{j=1}^{m} D_j : D_j \in \mathcal{D}\}$.

$\Phi_\zeta = \widehat{\sigma^{\alpha_1}} \circ \widehat{\tau^{\beta_1}} \circ \cdots \circ \widehat{\sigma^{\alpha_n}} \circ \widehat{\tau^{\beta_n}}$ - partial homeomorphism of X.

$\mathcal{D}^{(0)} = \{A(\zeta) \neq \varnothing : \zeta \in \mathcal{Z}(\Lambda)\}$;

closed under intersection: $A(\zeta) = A(\overline{\zeta}\zeta)$, $A(\zeta) \cap A(\theta) = A(\overline{\zeta}\zeta\overline{\theta}\theta)$.

$\mathcal{D} = \{E \setminus \cup_{i=1}^n F_i : E, F_i \in \mathcal{D}^{(0)},\ \cup_{i=1}^n F_i \subsetneq E\}$.

$\mathscr{A} = \{\sqcup_{j=1}^m D_j : D_j \in \mathcal{D}\}$.

$\Phi_\zeta = \widehat{\sigma^{\alpha_1}} \circ \widehat{\tau^{\beta_1}} \circ \cdots \circ \widehat{\sigma^{\alpha_n}} \circ \widehat{\tau^{\beta_n}}$ - partial homeomorphism of X.

Define two groupoids with unit space X:

$$G_1(\Lambda) = \text{groupoid of germs of } \{\Phi_\zeta : \zeta \in \mathcal{Z}(\Lambda)\}$$
$$= \mathcal{Z}(\Lambda) * X / \sim_1,$$

where $(\zeta, x) \sim_1 (\zeta', x')$ if $x = x'$ and $\Phi_\zeta = \Phi_{\zeta'}$ near x.

$$G_2(\Lambda) = \mathcal{Z}(\Lambda) * X / \sim_2,$$

where $(\zeta, x) \sim_2 (\zeta', x')$ if $x = x'$ and $\varphi_\zeta|_E = \varphi_{\zeta'}|_E$ for some $E \in \mathcal{U}_x$.

Theorem. If Λ has no inverses then $\sim_1 = \sim_2$.

(There are other sufficient conditions.)

Example. If Λ is a group then $G_1(\Lambda) = \{\text{pt}\}$ and $G_2(\Lambda) = \Lambda$.

Definition. $\mathcal{T}(\Lambda) = C^*(G_2(\Lambda))$ (or $\mathcal{T}_i(\Lambda) = C^*(G_i(\Lambda))$, $i = 1, 2$) - the *Toeplitz algebra* of Λ.

The example shows that $G_2(\Lambda)$ is generally the more important one.

Theorem. If Λ has no inverses then $\sim_1 = \sim_2$.

(There are other sufficient conditions.)

Example. If Λ is a group then $G_1(\Lambda) = \{\text{pt}\}$ and $G_2(\Lambda) = \Lambda$.

Definition. $\mathcal{T}(\Lambda) = C^*(G_2(\Lambda))$ (or $\mathcal{T}_i(\Lambda) = C^*(G_i(\Lambda))$, $i = 1, 2$) - the *Toeplitz algebra* of Λ.

The example shows that $G_2(\Lambda)$ is generally the more important one.

Theorem. $\mathcal{T}_i(\Lambda)$ is universal for representations of Λ by partial isometries $\{T_\alpha : \alpha \in \Lambda\} \subseteq B(H)$ satisfying the following relations. Let $T_\zeta = T^*_{\alpha_1} T_{\beta_1} \cdots T^*_{\alpha_n} T_{\beta_n}$ (where $\zeta = (\alpha_1, \beta_1, \ldots, \alpha_n, \beta_n)$).

(i) $T_{\zeta_1} T_{\zeta_2} = T_{\zeta_1 \zeta_2}$

(ii) $T_{\overline{\zeta}} = T^*_\zeta$

(iii) if $A(\zeta) = \cup_{i=1}^n A(\zeta_i)$ then $T^*_\zeta T_\zeta = \bigvee_{i=1}^n T^*_{\zeta_i} T_{\zeta_i}$

(iv)$_1$ if $\Phi_\zeta = \text{id}_{\widehat{A(\zeta)}}$ then $T_\zeta = T^*_\zeta T_\zeta$

or (iv)$_2$ if $\varphi_\zeta = \text{id}_{A(\zeta)}$ then $T_\zeta = T^*_\zeta T_\zeta$

(The relations again.)

(i) $T_{\zeta_1} T_{\zeta_2} = T_{\zeta_1 \zeta_2}$

(ii) $T_{\overline{\zeta}} = T_\zeta^*$

(iii) if $A(\zeta) = \cup_{i=1}^n A(\zeta_i)$ then $T_\zeta^* T_\zeta = \bigvee_{i=1}^n T_{\zeta_i}^* T_{\zeta_i}$

(iv)$_1$ if $\Phi_\zeta = \mathrm{id}_{\widehat{A(\zeta)}}$ then $T_\zeta = T_\zeta^* T_\zeta$

or (iv)$_2$ if $\varphi_\zeta = \mathrm{id}_{A(\zeta)}$ then $T_\zeta = T_\zeta^* T_\zeta$

The key aspect of these relations is the following

Theorem. Let $\{p_E : E \in \mathcal{D}^{(0)}\}$ be projections in a C^*-algebra B. There is a ring homomorphism $\mu : \mathscr{A} \to \mathcal{P}(B)$ with $\mu(E) = p_E$ for $E \in \mathcal{D}^{(0)}$ if and only if (iii) and $p_{E_1 \cap E_2} = p_{E_1} p_{E_2}$ (which follows from (i) and (ii)).

The Regular Representation

Lemma. $\pi_\ell : \Lambda \to B(\ell^2(\Lambda))$ extends to a representation (also called π_ℓ) of $\mathcal{T}(\Lambda) = C^*(G_2(\Lambda))$:

$$\begin{array}{ccc} \Lambda & \xrightarrow{\pi_\ell} & \mathrm{P.I.}(\ell^2(\Lambda)) \\ \big\downarrow & & \big\downarrow \\ \mathcal{T}(\Lambda) & \xrightarrow[\pi_\ell]{} & B(\ell^2(\Lambda)) \end{array}$$

(The map $\Lambda \to \mathcal{T}(\Lambda)$ is given by $\alpha \mapsto T_\alpha$.)

Definition 1. $\mathcal{T}_\ell(\Lambda) := \pi_\ell(\mathcal{T}(\Lambda))$ - the *regular Toeplitz algebra*.

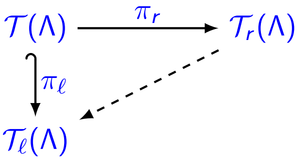
For $x \in X$ there is an induced representation Ind_x of $C^*(G_2(\Lambda))$ on $\ell^2(G_2(\Lambda)x)$; $\pi_r = \bigoplus_{x \in X} \mathrm{Ind}_x$ is the *regular representation* of $C^*(G_2(\Lambda))$: $\quad C^*_r(G_2(\Lambda)) = \pi_r(C^*(G_2(\Lambda)))$.

Definition 2. $\mathcal{T}_r(\Lambda) = C^*_r(G_2(\Lambda))$ - the *reduced Toeplitz algebra*.

For $x \in X$ there is an induced representation Ind_x of $C^*(G_2(\Lambda))$ on $\ell^2(G_2(\Lambda)x)$; $\pi_r = \bigoplus_{x \in X} \mathrm{Ind}_x$ is the *regular representation* of $C^*(G_2(\Lambda))$: $\quad C^*_r(G_2(\Lambda)) = \pi_r(C^*(G_2(\Lambda)))$.

Definition 2. $\mathcal{T}_r(\Lambda) = C^*_r(G_2(\Lambda))$ - the *reduced Toeplitz algebra*.

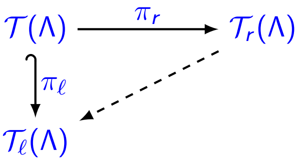
Proposition. π_ℓ factors through $\mathcal{T}_r(\Lambda)$:



For $x \in X$ there is an induced representation Ind_x of $C^*(G_2(\Lambda))$ on $\ell^2(G_2(\Lambda)x)$; $\pi_r = \bigoplus_{x \in X} \mathrm{Ind}_x$ is the *regular representation* of $C^*(G_2(\Lambda))$: $\quad C_r^*(G_2(\Lambda)) = \pi_r(C^*(G_2(\Lambda)))$.

Definition 2. $\mathcal{T}_r(\Lambda) = C_r^*(G_2(\Lambda))$ - the *reduced Toeplitz algebra*.

Proposition. π_ℓ factors through $\mathcal{T}_r(\Lambda)$:



Which of $\mathcal{T}_\ell$ and $\mathcal{T}_r$ is "the" reduced C^*-algebra of Λ?

Do we have to choose?

Some conditions giving an isomorphism $\mathcal{T}_r \to \mathcal{T}_\ell$.

(1) Λ is *finitely aligned* if whenever $E \subseteq \Lambda$ finite, there exists $F \subseteq \Lambda$ finite such that

$$\bigcap_{\alpha \in E} \alpha\Lambda = \bigcup_{\beta \in F} \beta\Lambda.$$

F is the set of *minimal common extensions* of E (well-defined up to right multiplication by invertibles). (Recall that $\mathbb{F}_n^+$ is *singly aligned*: $|F| = 0$ or 1.)

Theorem. If Λ is finitely aligned then $\mathcal{T}_r \to \mathcal{T}_\ell$ is an isomorphism.

(2) The groupoid $G_2(\Lambda)$ is not necessarily Hausdorff (but is always *ample*, i.e. étale with totally disconnected unit space).

Theorem. If $G_2(\Lambda)$ is Hausdorff then $\mathcal{T}_r \to \mathcal{T}_\ell$ is an isomorphism.

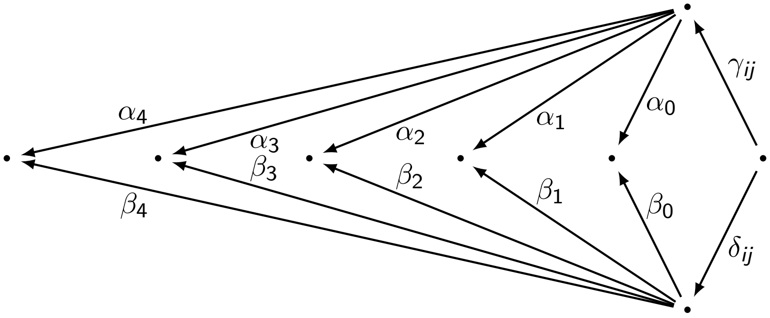
For example,

Theorem. If Λ is a subcategory of a groupoid then $G_2(\Lambda)$ is Hausdorff.

Corollary. If (G, P) is an ordered group (not necessarily pointed) then $\mathcal{T}_r \to \mathcal{T}_\ell$ is an isomorphism.

In general, $\mathcal{T}_r \to \mathcal{T}_\ell$ is not an isomorphism.

Example. Let $p > 1$ be odd. In the following, $i \in \mathbb{Z}$, $j \in \mathbb{Z}/p\mathbb{Z}$:



with relations

(i) $\alpha_0\gamma_{ij} = \beta_0\delta_{ij}$, for $i \in \mathbb{Z}$, $j \in \mathbb{Z}/p\mathbb{Z}$

(ii) $\alpha_1\gamma_{ij} = \begin{cases} \beta_1\delta_{i,j+1}, \text{ if } i \equiv 1 \pmod 3, \ j \in \mathbb{Z}/p\mathbb{Z} \\ \beta_1\delta_{ij}, \text{ if } i \not\equiv 1 \pmod 3, \ j \in \mathbb{Z}/p\mathbb{Z} \end{cases}$

(iii) $\alpha_2\gamma_{ij} = \begin{cases} \beta_2\delta_{i,j+1}, & \text{if } i \equiv 2 \pmod 3, \ j \in \mathbb{Z}/p\mathbb{Z} \\ \beta_2\delta_{ij}, & \text{if } i \not\equiv 2 \pmod 3, \ j \in \mathbb{Z}/p\mathbb{Z} \end{cases}$

(iv) $\alpha_3\gamma_{ij} = \begin{cases} \beta_3\delta_{i+3,j}, \text{ if } i \equiv 0 \pmod 3, \ j \in \mathbb{Z}/p\mathbb{Z} \\ \beta_3\delta_{ij}, \text{ if } i \not\equiv 0 \pmod 3, \ j \in \mathbb{Z}/p\mathbb{Z} \end{cases}$

(v) $\alpha_4\gamma_{ij} = \beta_4\delta_{ij}$, if $i \not\equiv 0 \pmod 3$, $j \in \mathbb{Z}/p\mathbb{Z}$.

$$\theta_1 = \sigma^{\alpha_0}\tau^{\beta_0}\sigma^{\beta_1}\tau^{\alpha_1} \qquad \theta_1(\gamma_{ij}) = \begin{cases} \gamma_{i,j+1} & \text{if } i \equiv 1 \pmod 3 \\ \gamma_{ij} & \text{if } i \not\equiv 1 \pmod 3, \end{cases}$$

$$\theta_2 = \sigma^{\alpha_0}\tau^{\beta_0}\sigma^{\beta_2}\tau^{\alpha_2} \qquad \theta_2(\gamma_{ij}) = \begin{cases} \gamma_{i,j+1} & \text{if } i \equiv 2 \pmod 3 \\ \gamma_{ij} & \text{if } i \not\equiv 2 \pmod 3. \end{cases}$$

$$\theta_3 = \sigma^{\alpha_0}\tau^{\beta_0}\sigma^{\beta_3}\tau^{\alpha_3} \qquad \theta_3(\gamma_{ij}) = \begin{cases} \gamma_{i+3,j} & \text{if } i \equiv 0 \pmod 3 \\ \gamma_{ij} & \text{if } i \not\equiv 0 \pmod 3, \end{cases}$$

$$\theta_4 = \sigma^{\alpha_0}\tau^{\beta_0}\sigma^{\beta_4}\tau^{\alpha_4} \qquad \theta_4(\gamma_{ij}) = \gamma_{ij} \text{ if } i \not\equiv 0 \pmod 3.$$

Let $A := \operatorname{dom}\theta_1 = \operatorname{dom}\theta_2 = \operatorname{dom}\theta_3 = \{\gamma_{ij} : \text{ all } i, j\}$

$B := \operatorname{dom}\theta_4 = \{\gamma_{ij} : i \not\equiv 0 \pmod 3\}$.

On $A \setminus B$:

$\theta_1 = \theta_2 = \mathrm{id}$,

θ_3 has no fixed points.

On B:

θ_1 and θ_2 have only fixed points and orbits of length p,

$\theta_1(\mu) = \mu$ iff $\theta_2(\mu) \neq \mu$,

$\theta_3 = \mathrm{id}$.

On $\ell^2(\Lambda)$: there exists $c > 0$ with

$$(*) \quad |\langle \pi_\ell(T_{\theta_1})\xi, \xi\rangle| + |\langle \pi_\ell(T_{\theta_2})\xi, \xi\rangle| + 1 - \mathrm{Re}\langle \pi_\ell(T_{\theta_3})\xi, \xi\rangle \geq c\|\xi\|^2,$$

for all $\xi \in \ell^2(\Lambda)$. (In fact, $c = \frac{1}{2}(1 - \cos\frac{\pi}{p})$.)

Moreover B defines $x \in X$ such that if π_ℓ is replaced by Ind_x then the lefthand side of $(*)$ equals 0. It follows that Ind_x is not weakly contained in π_ℓ, and hence $\mathcal{T}_r \to \mathcal{T}_\ell$ is not an isomorphism.

The C^*-algebra in this example is (very) type I (and Λ is nearly a 2-graph). One can identify the vertices of Λ to obtain a monoid with the same properties (and a more complicated C-algebra). (However, this monoid cannot be embedded in a group.)

The C^*-algebra in this example is (very) type I (and Λ is nearly a 2-graph). One can identify the vertices of Λ to obtain a monoid with the same properties (and a more complicated C-algebra). (However, this monoid cannot be embedded in a group.)

In general it is not clear how to describe $\mathcal{T}_\ell(\Lambda)$ by generators and relations.

Some remarks on amenability.

1. If $G_2(\Lambda)$ is amenable (in the sense of Anantharaman-Delaroche and Renault) then $\mathcal{T}(\Lambda) = \mathcal{T}_r(\Lambda)$, but it need not be the case that $\mathcal{T}_r(\Lambda) \to \mathcal{T}_\ell(\Lambda)$ is an isomorphism.

2. If (G, P) is a quasi-lattice ordered group, Nica defines *amenable* to mean that $C^*(P) \to \pi_\ell(C^*(P))$ is an isomorphism. In this case $\mathcal{T}_r(\Lambda) = \mathcal{T}_\ell(\Lambda)$, so this is equivalent to $\mathcal{T}(\Lambda) = \mathcal{T}_r(\Lambda)$ - an ostensibly weaker condition than groupoid amenability.

3. In the quasi-lattice ordered case Nica showed that an equivalent condition is that the conditional expectation $C^*(P) \to \overline{\text{span}}\{T_\alpha T_\alpha^* : \alpha \in P\}$ be faithful. For general Λ, one ought to use $\overline{\text{span}}\{T_\zeta^* T_\zeta : \zeta \in \mathcal{Z}(\Lambda)\}$ (i.e. $C_0(X)$). Then this is equivalent to the condition that $\mathcal{T}(\Lambda) = \mathcal{T}_r(\Lambda)$.

4. Amenability of Λ is a much stronger condition. (Related to *independence*, *right reversibility*, ...)

Wiener-Hopf algebras

(G, P) - ordered group

$J : \ell^2(P) \hookrightarrow \ell^2(G)$

$L : G \to \mathcal{U}(\ell^2(G))$ - left regular representation

For $t \in G$, $W_t := J^* L_t J \in B(\ell^2(P))$ - *Wiener-Hopf operator*

(compression of L_t to $\ell^2(P)$)

$W_t \neq 0$ iff $t \in PP^{-1}$

$\mathcal{W} := C^*(\{W_t : t \in G\})$, $\mathcal{W}_0 := C^*(\{W_\alpha : \alpha \in P\})$

Theorem (Nica): If (G, P) is quasi-lattice ordered (and $P \cap P^{-1} = \{e\}$) then $\mathcal{W} = \mathcal{W}_0$.

(Recall: (G, P) is qlo if for $t \in G$,
$(tP \cap P \neq \varnothing) \implies (\exists \alpha \in P$ s.t. $tP \cap P = \alpha P)$.)

More generally. . .

Y - countable groupoid

$\Lambda \subseteq Y$ - subcategory with $\Lambda^0 = Y^0$

((Y, Λ) is an *ordered groupoid*)

$J : \ell^2(\Lambda) \hookrightarrow \ell^2(Y)$

$L : Y \to \text{P.I.}(\ell^2(Y))$ - left regular representation

For $t \in Y$, $W_t := J^* L_t J \in B(\ell^2(\Lambda))$ - *Wiener-Hopf operator*

$W_t \neq 0$ iff $t \in \Lambda\Lambda^{-1}$

$\mathcal{W} := C^*(\{W_t : t \in Y\})$, $\mathcal{W}_0 := C^*(\{W_\alpha : \alpha \in \Lambda\})$

More generally. . .

Y - countable groupoid

$\Lambda \subseteq Y$ - subcategory with $\Lambda^0 = Y^0$

((Y, Λ) is an *ordered groupoid*)

$J : \ell^2(\Lambda) \hookrightarrow \ell^2(Y)$

$L : Y \to \text{P.I.}(\ell^2(Y))$ - left regular representation

For $t \in Y$, $W_t := J^* L_t J \in B(\ell^2(\Lambda))$ - *Wiener-Hopf operator*

$W_t \neq 0$ iff $t \in \Lambda\Lambda^{-1}$

$\mathcal{W} := C^*(\{W_t : t \in Y\})$, $\mathcal{W}_0 := C^*(\{W_\alpha : \alpha \in \Lambda\})$

¿ $\mathcal{W} = \mathcal{W}_0$?

Definition. (Y, Λ) is *finitely aligned* if for $t \in Y$,
$(t\Lambda \cap \Lambda \neq \varnothing) \implies (\exists F \subseteq \Lambda$ finite s.t. $t\Lambda \cap \Lambda = \bigcup_{\alpha \in F} \alpha\Lambda)$.
Note that if (Y, Λ) is finitely aligned then Λ is finitely aligned (as LCSC).

Theorem. Let (Y, Λ) be an ordered groupoid. Suppose that Λ is a finitely aligned LCSC. Then $\mathcal{W} = \mathcal{W}_0$ iff (Y, Λ) is finitely aligned.

Key point: If Λ is finitely aligned, then for all ζ, $A(\zeta) = \cup_{i=1}^{n} \alpha_i \Lambda$ (i.e. every constructible right ideal is a finite union of principal right ideals).

In the general (nonfinitely aligned) case,

Proposition. Let $t \in Y$. Then $W_t \in \mathcal{W}_0$ iff there is a finite set $F \subseteq \mathcal{Z}(\Lambda)$ such that

(i) for $\zeta \in F$, $\varphi_\zeta = t|_{A(\zeta)}$ (i.e. $\alpha_1^{-1}\beta_1 \cdots \alpha_n^{-1}\beta_n = t$)

(ii) $\Lambda \cap t^{-1}\Lambda = \bigcup_{\zeta \in F} A(\zeta)$.

Corollaries.

1. Let $t \in \Lambda\Lambda^{-1}$. If $t \in \Lambda^{-1}\Lambda$ then $W_t \in \mathcal{W}_0$.
2. If Y is abelian then $\mathcal{W} = \mathcal{W}_0$.

Definition. Λ is *right reversible* if $\Lambda\alpha \cap \Lambda\beta \neq \varnothing$, for all $\alpha,\ \beta \in \Lambda$.

Lemma. Let (Y, Λ) be an ordered groupoid. Then Λ is right reversible iff $\Lambda\Lambda^{-1} \subseteq \Lambda^{-1}\Lambda$.

3. If Λ is right reversible then $\mathcal{W} = \mathcal{W}_0$.

In the general (nonfinitely aligned) case,

Proposition. Let $t \in Y$. Then $W_t \in \mathcal{W}_0$ iff there is a finite set $F \subseteq \mathcal{Z}(\Lambda)$ such that

(i) for $\zeta \in F$, $\varphi_\zeta = t|_{A(\zeta)}$ (i.e. $\alpha_1^{-1}\beta_1 \cdots \alpha_n^{-1}\beta_n = t$)

(ii) $\Lambda \cap t^{-1}\Lambda = \bigcup_{\zeta \in F} A(\zeta)$.

Corollaries.

1. Let $t \in \Lambda\Lambda^{-1}$. If $t \in \Lambda^{-1}\Lambda$ then $W_t \in \mathcal{W}_0$.
2. If Y is abelian then $\mathcal{W} = \mathcal{W}_0$.

Definition. Λ is *right reversible* if $\Lambda\alpha \cap \Lambda\beta \neq \varnothing$, for all $\alpha,\ \beta \in \Lambda$.

Lemma. Let (Y, Λ) be an ordered groupoid. Then Λ is right reversible iff $\Lambda\Lambda^{-1} \subseteq \Lambda^{-1}\Lambda$.

3. If Λ is right reversible then $\mathcal{W} = \mathcal{W}_0$.

Proposition. There exist examples of ordered groupoids (Y, Λ) with Λ not finitely aligned, both such that $\mathcal{W} = \mathcal{W}_0$, and such that $\mathcal{W} \neq \mathcal{W}_0$.

Comparison with other semigroup algebras — joint with E. Bedos, S. Kaliszewski, J. Quigg

X. Li (JFA 2012) described five C^*-algebras associated to a left cancellative monoid. We adapt to the case of a left cancellative small category Λ.

Definition. Consider the universal C^*-algebra generated by partial isometries $\{v_\alpha : \alpha \in \Lambda\}$ and projections $\{p_E : E \in \mathcal{D}^{(0)} \cup \{\varnothing\}\}$ with some relations.

Relations. (1) $v_\alpha^* v_\alpha = p_{s(\alpha)\Lambda}$

(2) $v_\alpha v_\beta = v_{\alpha\beta}$ if $s(\alpha) = s(\beta)$, (and $= 0$ otherwise)

(3) $p_\varnothing = 0$

(4) $p_E p_F = p_{E\cap F}$

(5) $v_\alpha p_E v_\alpha^* = p_{\tau^\alpha(E)}$.

1. $C^*(\Lambda)$: use (1) - (5).

1. $C^*(\Lambda)$: use (1) - (5).

Let $\widetilde{\mathcal{D}^{(0)}} = \{\cup_{i=1}^n E_i : E_i \in \mathcal{D}^{(0)} \cup \{\varnothing\}\}$.

(4)$^{(\cup)}$, (5)$^{(\cup)}$ - same as (4), (5) but using $\widetilde{\mathcal{D}^{(0)}}$

(6) $p_{E\cup F} = p_E \vee p_F, \; E, F \in \widetilde{\mathcal{D}^{(0)}}$

2. $C^{*\,(\cup)}(\Lambda)$: use (1) - (3), (4)$^{(\cup)}$, (5)$^{(\cup)}$, (6).

1. $C^*(\Lambda)$: use (1) - (5).

Let $\widetilde{\mathcal{D}^{(0)}} = \{\cup_{i=1}^n E_i : E_i \in \mathcal{D}^{(0)} \cup \{\varnothing\}\}$.

(4)$^{(\cup)}$, (5)$^{(\cup)}$ - same as (4), (5) but using $\widetilde{\mathcal{D}^{(0)}}$

(6) $p_{E\cup F} = p_E \vee p_F$, $E, F \in \widetilde{\mathcal{D}^{(0)}}$

2. $C^{*\,(\cup)}(\Lambda)$: use (1) - (3), (4)$^{(\cup)}$, (5)$^{(\cup)}$, (6).

(7) If $\varphi_\zeta = \mathrm{id}_{A(\zeta)}$ then $v^*_{\alpha_1} v_{\beta_1} \cdots v^*_{\alpha_n} v_{\beta_n} = p_{A(\zeta)}$ (where $\zeta = (\alpha_1, \beta_1, \ldots, \alpha_n, \beta_n)$)

3. $C^*_s(\Lambda)$: use (1) - (3), (7).

1. $C^*(\Lambda)$: use (1) - (5).

Let $\widetilde{\mathcal{D}^{(0)}} = \{\cup_{i=1}^n E_i : E_i \in \mathcal{D}^{(0)} \cup \{\varnothing\}\}$.

(4)$^{(\cup)}$, (5)$^{(\cup)}$ - same as (4), (5) but using $\widetilde{\mathcal{D}^{(0)}}$

(6) $p_{E\cup F} = p_E \vee p_F,\ E, F \in \widetilde{\mathcal{D}^{(0)}}$

2. $C^{*\,(\cup)}(\Lambda)$: use (1) - (3), (4)$^{(\cup)}$, (5)$^{(\cup)}$, (6).

(7) If $\varphi_\zeta = \mathrm{id}_{A(\zeta)}$ then $v_{\alpha_1}^* v_{\beta_1} \cdots v_{\alpha_n}^* v_{\beta_n} = p_{A(\zeta)}$ (where $\zeta = (\alpha_1, \beta_1, \ldots, \alpha_n, \beta_n)$)

3. $C_s^*(\Lambda)$: use (1) - (3), (7).

4. $C_s^{*\,(\cup)}(\Lambda)$: use (1) - (3), (6), (7).

1. $C^*(\Lambda)$: use (1) - (5).

Let $\widetilde{\mathcal{D}^{(0)}} = \{\cup_{i=1}^n E_i : E_i \in \mathcal{D}^{(0)} \cup \{\varnothing\}\}$.

(4)$^{(\cup)}$, (5)$^{(\cup)}$ - same as (4), (5) but using $\widetilde{\mathcal{D}^{(0)}}$

(6) $p_{E\cup F} = p_E \vee p_F$, $E, F \in \widetilde{\mathcal{D}^{(0)}}$

2. $C^{*\,(\cup)}(\Lambda)$: use (1) - (3), (4)$^{(\cup)}$, (5)$^{(\cup)}$, (6).

(7) If $\varphi_\zeta = \mathrm{id}_{A(\zeta)}$ then $v_{\alpha_1}^* v_{\beta_1} \cdots v_{\alpha_n}^* v_{\beta_n} = p_{A(\zeta)}$ (where $\zeta = (\alpha_1, \beta_1, \ldots, \alpha_n, \beta_n)$)

3. $C_s^*(\Lambda)$: use (1) - (3), (7).

4. $C_s^{*\,(\cup)}(\Lambda)$: use (1) - (3), (6), (7).

5. $\mathcal{T}_\ell$ $(\subseteq B(\ell^2(\Lambda)))$

There is a commutative diagram:

$$
\begin{array}{ccccc}
C^*(\Lambda) & \xrightarrow{\pi_s} & C_s^*(\Lambda) & & \\
\Big\downarrow \pi^{(\cup)} & & \Big\downarrow \rho^{(\cup)} & & \\
C^{*\,(\cup)}(\Lambda) & \xrightarrow{\rho_s} & C_s^{*\,(\cup)}(\Lambda) & \xrightarrow{\pi_\ell} & \mathcal{T}_\ell(\Lambda)
\end{array}
$$

There is a commutative diagram:

$$\begin{array}{ccccc}
C^*(\Lambda) & \xrightarrow{\pi_s} & C^*_s(\Lambda) & & \\
\downarrow{\scriptstyle \pi^{(\cup)}} & & \downarrow{\scriptstyle \rho^{(\cup)}} & & \\
C^{*\,(\cup)}(\Lambda) & \xrightarrow{\rho_s} & C^{*\,(\cup)}_s(\Lambda) & \xrightarrow{\pi_\ell} & \mathcal{T}_\ell(\Lambda)
\end{array}$$

We will expand this diagram to include the Toeplitz algebras discussed earlier. First, one more algebra . . .

Definition. Let $ZM(\Lambda) := \{\varphi_\zeta : \zeta \in \mathcal{Z}(\Lambda)\} \cup \{\mathrm{id}_\varnothing\}$ (the set of all zigzag maps).

$ZM(\Lambda)$ is an inverse semigroup. We let $C^*(ZM(\Lambda))$ denote its universal C^*-algebra.

Theorem. There is a commutative diagram

$$
\begin{array}{ccccccccc}
C^*(\Lambda) & \xrightarrow{\pi_s} & C^*_s(\Lambda) & \xrightarrow{g} & C^*(ZM(\Lambda)) & & & & B(\ell^2(\Lambda)) \\
\Big\downarrow{\scriptstyle \pi^{(\cup)}} & & \Big\downarrow{\scriptstyle \rho^{(\cup)}} & & \Big\downarrow{\scriptstyle q} & & & & \Big\uparrow \\
C^{*\,(\cup)}(\Lambda) & \xrightarrow{\rho_s} & C_s^{*\,(\cup)}(\Lambda) & \xrightarrow{\mu} & \mathcal{T}(\Lambda) & \xrightarrow{\pi_r} & \mathcal{T}_r(\Lambda) & \xrightarrow{\pi_\ell} & \mathcal{T}_\ell(\Lambda)
\end{array}
$$

1. μ and g are isomorphisms.
2. π_s, ρ_s, $\pi^{(\cup)}$, $\rho^{(\cup)}$, q, π_r, π_ℓ are surjective.
3. ρ_s is an isomorphism if Λ is finitely aligned, but not in general (even if Λ is a submonoid of a group).
4. π_s, $\pi^{(\cup)}$, $\rho^{(\cup)}$, q are not isomorphisms in general (even if Λ is a finitely aligned submonoid of a group).