The complete classification of unital graph C^*-algebras

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Graph C^*-algebras

Definition

A graph E is a 4-tuple (E^0, E^1, r, s), where E^0 is a countable set of vertices, E^1 is a countable set of edges, and $r, s\colon E^1 \to E^0$ are the range and source maps.

Definition

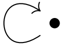
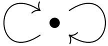
The graph C^*-algebra, $C^*(E)$, is the universal C^*-algebra generated by:

- pairwise orthogonal projections $\{p_u \mid u \in E^0\}$, and,
- partial isometries $\{s_e \mid e \in E^1\}$,

subject to the relations:

- $s_e^* s_f = 0$, if $e \neq f$,
- $s_e^* s_e = p_{r(e)}$,
- $s_e s_e^* \leq p_{s(e)}$, and,
- $p_u = \sum_{e \in s^{-1}(u)} s_e s_e^*$, if $0 < |s^{-1}(u)| < \infty$.

Examples of graph C^*-algebras

	$C(S^1)$
	$\mathcal{O}_2$
$\bullet \xrightarrow{(\infty)} \bullet$	$\widetilde{\mathbb{K}}$

$C^*(\mathbb{N}^2)$ is not a graph C^*-algebra

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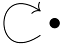

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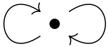

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$C^*(\mathbb{N}^2)$ is not a graph C^*-algebra

The first classification results

Theorem (Rørdam '95)

If E, F are strongly connected finite graphs, not a single cycle, then

$$C^*(E) \otimes \mathbb{K} \cong C^*(F) \otimes \mathbb{K} \iff K_0(C^*(E)) \cong K_0(C^*(F)).$$

Theorem (Restorff '04)

If E, F are finite graphs such that every vertex supports a loop and E, F satisfy condition (K), then

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Simplified proof strategy

Lemma

If $C^*(E)$ and $C^*(F)$ have the same K-theory and E and F both satisfy a positivity condition, then the shift spaces X_E and X_F are flow equivalent.

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If X_E and X_F are flow equivalent, then $C^*(E) \otimes \mathbb{K} \cong C^*(F) \otimes \mathbb{K}$.

Lemma

Given a graph G we can find a graph G' such that

- $C^*(G) \otimes \mathbb{K} \cong C^*(G') \otimes \mathbb{K}$.
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- G' satisfies the positivity condition.

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- $C^*(G)$ *and* $C^*(G')$ *have the same K-theory.*
- G' *satisfies the positivity condition.*

Rørdam's result revisited

Theorem (Rørdam)

Let E, F be strongly connected finite graphs, not a single cycle. The following are equivalent:

1. $C^*(E) \otimes \mathbb{K} \cong C^*(F) \otimes \mathbb{K}$.
2. $K_0(C^*(E)) \cong K_0(C^*(F))$.
3. *E can be transformed into F using flow moves and the Cuntz splice.*

Problems with extending the strategy

- Sinks and sources are a dynamical problem.
- Infinite emitters are a different dynamical problem.
- Graphs that do not satisfy condition (K) have infintely many ideals.
- Old positivity results need to be extended.
- We run into new positivity problems.

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Problematic graphs

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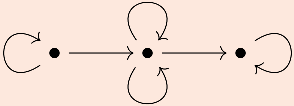

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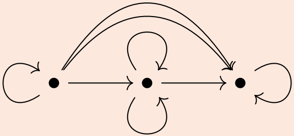

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- $C^*(E) \otimes \mathbb{K} \cong C^*(F) \otimes \mathbb{K}$.
- *We cannot transform E into F using the flow moves and the Cuntz splice.*

Positivity problems

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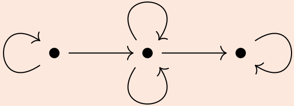

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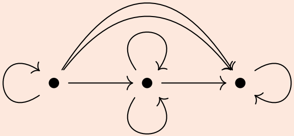

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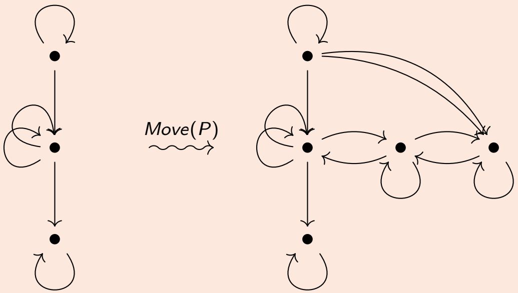
The solution

The Pulelehua move



Theorem

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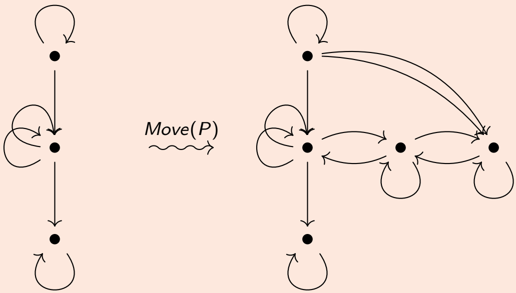

Theorem

The Pulelehua move preserves stable isomorphism.

The main result

Theorem

Let E, F be graphs with finitely many vertices. The following are equivalent:

1. $C^*(E) \otimes \mathbb{K} \cong C^*(F) \otimes \mathbb{K}$.
2. *E can be transformed into F using the flow moves, the Cuntz splice and the Pulelehua move.*
3. $FK^+_{R,\gamma}(C^*(E)) \cong FK^+_{R,\gamma}(C^*(F))$.