Clifford Fourier Transforms in Colour Image Analysis

Hardy Spaces and Paley-Wiener Spaces

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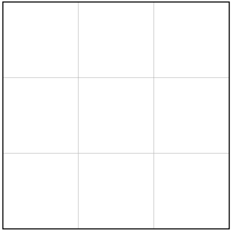
September 5, 2015

Introduction to Image Analysis



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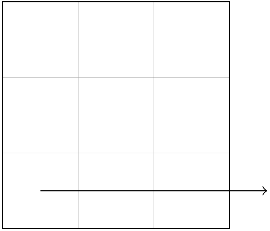
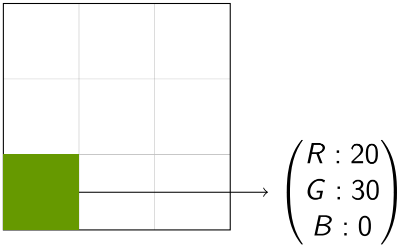


$$\begin{pmatrix} R : 20 \\ G : 30 \\ B : 0 \end{pmatrix}$$

Introduction to Image Analysis



Colour images $f : \mathbb{R}^2 \to \mathbb{R}^3$.

The Classical Fourier Transform

The Fourier Transform acts on $f : \mathbb{R} \to \mathbb{C}$ by

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On $f : \mathbb{R}^n \to \mathbb{C}$ by

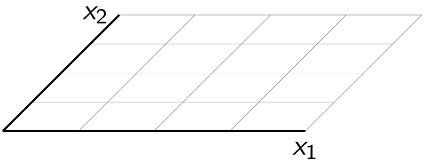
$$F\{f(x)\}(y) = \int e^{-i<x,y>} f(x) dx.$$

Properties of the Classical FT

For any $f, g : \mathbb{R}^n \to \mathbb{C}$, $\lambda, \mu \in \mathbb{R}$ and $x_0 \in \mathbb{R}^n$

Linearity	$F\{\lambda f + \mu g\}$	$= \lambda F\{f\} + \mu F\{g\}$
Translation	$F\{f(x - x_0)\}$	$= e^{-ix_0 y} F\{f\}$
Differentiation	$F\{\frac{d}{dx_j} f\}$	$= iy_j F\{f\}$
Scaling	$F\{f(\lambda x)\}$	$= \frac{1}{\lvert\lambda\rvert^n} F\{f\}(\frac{y}{\lambda})$
Plancherel	$\lVert F\{f\}\rVert^2$	$= \lVert f\rVert^2$
Convolution	$F\{f\}F\{g\}$	$= F\{\int f(x-y)g(y)dy\} = F\{f * g\}$

Problems



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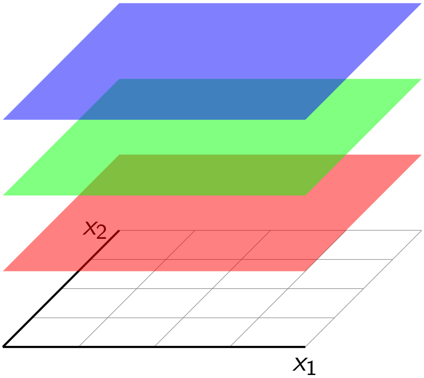

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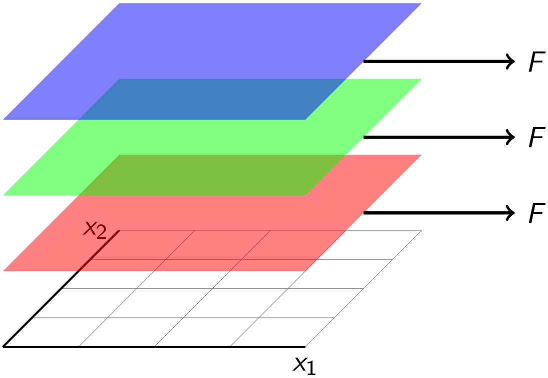

Intro to Clifford Algebras

Create new units $e_1, \ldots, e_n$ such that $e_j^2 = -1$ and $e_j e_k = -e_k e_j$.
A number is

$$a = a_0 + \sum_{j=1}^{n} a_j e_j + \sum_{j<k} a_{jk} e_j e_k + \ldots + a_{1\ldots n} e_1 e_2 \ldots e_n$$

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This 2^n dimensional space is denoted $\mathbb{R}_{(n)}$.
Note $\mathbb{R}_{(0)} = \mathbb{R}$, $\mathbb{R}_{(1)} = \mathbb{C}$,$\mathbb{R}_{(2)} = \mathbb{H}$.

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Consider a function $f : \mathbb{R}^n \to \mathbb{R}_{(n)}$ by $a_\beta : \mathbb{R}^n \to \mathbb{R}$.

Some Clifford Operators

For functions $f : \mathbb{R}^n \to \mathbb{R}_{(n)}$, we have

- Position $x = \sum_i e_i x_i$
- Dirac $D = \sum_i e_i \frac{\partial}{\partial x_i}$

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If $Df = 0$ then f is monogenic.

Some Clifford Operators

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- Position $x = \sum_i e_i x_i$
- Dirac $D = \sum_i e_i \frac{\partial}{\partial x_i}$
- Gamma $\Gamma = -\sum_{i<j} e_i e_j (x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i})$

Parity Matrices

Split $\mathbb{R}_{(n)} = \Lambda_e \oplus \Lambda_o$, the odd and even parts of the algebra.
Hence the parity matrix of $f : \mathbb{R}^n \to \mathbb{R}_{(n)}$ is

$$[f] = \begin{bmatrix} f_e(x) & f_o(x) \\ f_o(-x) & f_e(-x) \end{bmatrix}.$$

Clifford Fourier Transform

The Classical Fourier Transform can be written

$$F\{f\} = \exp(-i\frac{\pi}{2}H)\{f\}$$

where the Hermite operator is $H = \frac{1}{2}(-\Delta_n + \|x\|^2 - n)$ and Δ_n is the Laplacian on $\mathbb{R}^n$.

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So we define for $f : \mathbb{R}^n \to \mathbb{R}_{(n)}$,

$$\mathcal{F}\{f\} = \exp(-i\frac{\pi}{2}H)\{f\}$$

where the Hermite operator is $H = \frac{1}{2}(-\Delta_n + \|x\|^2 - n)$ and Δ_n is the Laplacian on $\mathbb{R}^n$.

Alternate Representations

Equivalently, for $f : \mathbb{R}^n \to \mathbb{R}_{(n)}$

$$\mathcal{F}\{f(x)\}(y) = \int e^{-i\frac{\pi}{2}\Gamma} e^{-i<x,y>} f(x)dx$$

or equivalently

$$\mathcal{F}\{f(x)\}(y) = e^{-i\frac{\pi}{2}\Gamma} F\{f(x)\}.$$

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In 2D specifically,

$$\mathcal{F}\{f(x)\}(y) = \frac{1}{2\pi} \int e^{-x\wedge y} f(x)dx.$$

Properties of the Clifford FT

For any $f, g : \mathbb{R}^n \to \mathbb{R}_{(n)}$, $\lambda, \mu \in \mathbb{R}_{(n)}$ and $x_0 \in \mathbb{R}^n$

Linearity	$\mathcal{F}\{f\lambda + g\mu\}$	$=\mathcal{F}\{f\}\lambda + \mathcal{F}\{g\}\mu$
Translation(2D)	$\mathcal{F}\{f(x - x_0)\}$	$=e^{y\wedge x_0}\mathcal{F}\{f\}$
Differentiation	$\mathcal{F}\{Df\}$	$=-y\mathcal{F}_{-}\{f\}$
Scaling	$\mathcal{F}\{f(\lambda x)\}$	$=\frac{1}{\lvert\lambda\rvert^n}\mathcal{F}\{f\}(\frac{y}{\lambda})$
Plancherel	$\lVert\mathcal{F}\{f\}\rVert^2$	$=\lVert f\rVert^2$
Convolution	$\mathcal{F}\{f\}\mathcal{F}\{g\}$	$=\mathcal{F}\{e^{i\frac{\pi}{2}\Gamma}(e^{-i\frac{\pi}{2}\Gamma}f) * (e^{-i\frac{\pi}{2}\Gamma}g)\}$
Convolution(2D)	$[\mathcal{F}f][\mathcal{F}g]$	$=[\mathcal{F}\{f * g\}]$

Classical Hardy Theorem

If a function $f : \mathbb{R}^n \to \mathbb{R}$ has a Fourier Transform supported on the positive half line,

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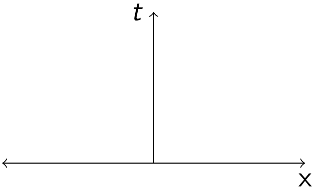

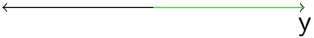

If a function $f : \mathbb{R}^n \to \mathbb{R}$ has a Fourier Transform supported on the positive half line, then it has an analytic extension $f(x, t)$ for $t > 0$.

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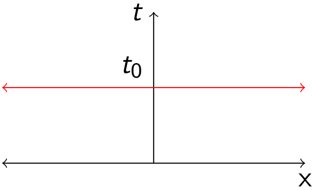

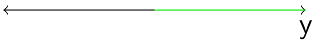

If a function $f : \mathbb{R}^n \to \mathbb{R}$ has a Fourier Transform supported on the positive half line, then it has an analytic extension $f(x,t)$ for $t > 0$. Further $\int |f(x,t_0)|^2 dx \leq \|f\|^2$.

Clifford Hardy Spaces

Algebraically, $f \in H^2_{\pm}$ if

$$\frac{1}{2}(1 \pm \frac{y}{|y|})Ff = Ff.$$

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Note $[\frac{1}{2}(1 + \frac{y}{|y|})]^2 = [\frac{1}{2}(1 + \frac{y}{|y|})]$ and $[\frac{1}{2}(1 + \frac{y}{|y|})][\frac{1}{2}(1 - \frac{y}{|y|})] = 0$.

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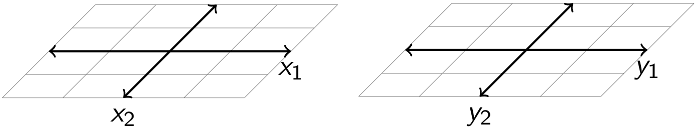

Theorem (Franklin, 2015)

If a function $f : \mathbb{R}^n \to \mathbb{R}_{(n)}$ *has a Clifford Fourier Transform such that* $[\frac{1}{2}(1 - \frac{y}{|y|})][\mathcal{F}f] = [\mathcal{F}f]$,

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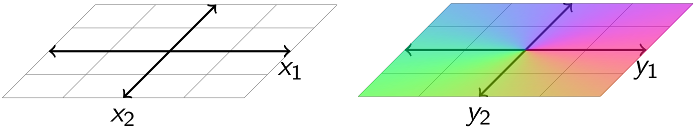

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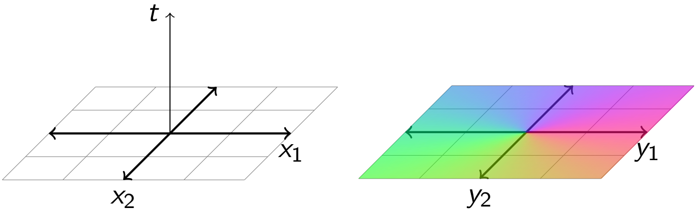

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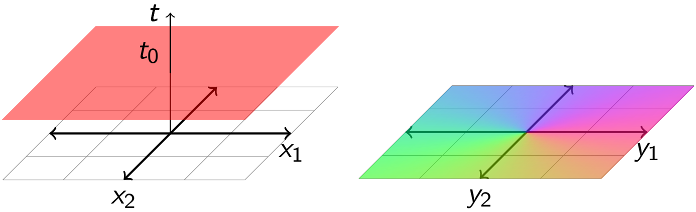

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Classical Paley-Wiener Theorem

If a function $f : \mathbb{R}^n \to \mathbb{R}$ has a Fourier Transform supported on a interval of length R,

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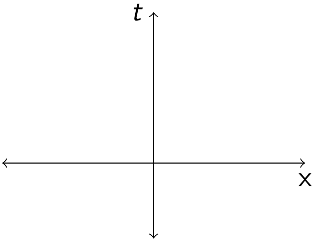

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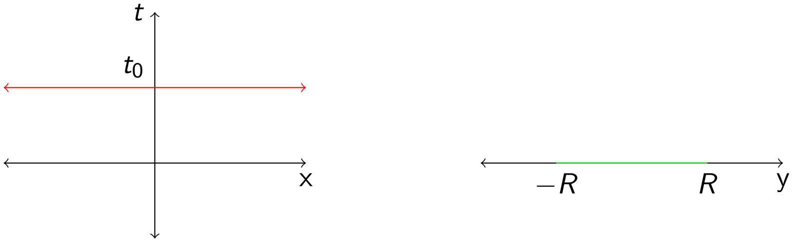
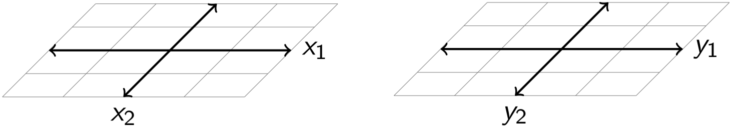


If a function $f : \mathbb{R}^n \to \mathbb{R}$ has a Fourier Transform supported on a interval of length R, then it has an analytic extension $f(x,t)$, such that $\int |f(x,t_0)|^2 dx \leq e^{2|t|R} \|f\|^2$.

Clifford Paley-Wiener Theorem



Theorem (Franklin, 2015)

If a function $f : \mathbb{R}^n \to \mathbb{R}_{(n)}$ *has a Clifford Fourier Transform supported on the ball of radius* R*,*

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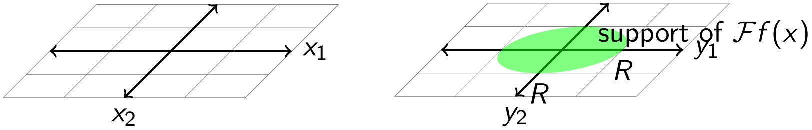

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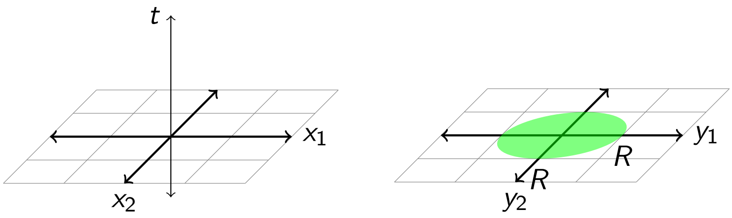

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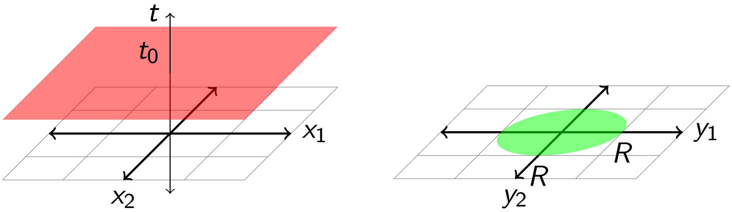

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Conclusion and Questions

Thanks to Jeff Hogan and Kieran Larkin for their valuable support and advice.

Thanks for listening.