Effect of a submerged plate on flexural-gravity wave blocking

Santu Das*[1], Trilochan Sahoo[2] and Michael H. Meylan[3]

[1]Department of Mathematics, Indian Institute of Information Technology Bhagalpur, Bhagalpur - 813210, India

[2]Department of Ocean Engineering and Naval Architecture, Indian Institute of Technology Kharagpur, Kharagpur - 721302, India

[3]School of Mathematical and Physical Sciences, University of Newcastle, NSW 2308, Australia

* E-mail: *santudas20072@gmail.com*

Highlights

- Flexural-gravity wave blocking in the presence of a submerged plate is studied.
- Effect of plate properties on wave blocking is analyzed.
- Dependency between ocean current and wave-period during wave blocking is demonstrated.

1. Introduction

The study demonstrates the propagation of flexural gravity waves in the case of submerged and floating plate system. It is the continuation of the previous works by [1, 2, 3, 4]. The dominant roles of the submerged and floating plates completely depends on the plate position, structural rigidity and compressive forces acting on the plate. The effects of both the plates' properties as well as compressive force on wave blocking of both surface (along floating plate) and internal (along submerged plate) modes are analyzed in details for a specific case of a submerged plate being placed at shallow water depth with infinite-depth water underneath. Further, the dependency between blocking frequency and critical opposing current is analytically derived and presented graphically.

2. Mathematical formulation

In the present study, the floating ice sheet/structure and the submerged structure are modeled as thin elastic plates which are assumed to have a small amplitude structural responses. The physical problem is considered in the two-dimensional Cartesian coordinate system with x-axis being horizontal and z-axis being in the vertically downward direction (see Fig. (1)). The infinitely stretched ice-sheet/structure is assumed to be floating on the mean free surface $z = 0$ of water of finite depth H and another horizontal thin structure is submerged at a depth $z = h$, both of which generate flexural gravity waves due to interaction with ocean water. The fluid domain in between the two structures occupies the region $-\infty < x, y < \infty$, $0 \leq z \leq h$ and the fluid underneath the submerged structure occupies the region $-\infty < x, y < \infty$, $h \leq z \leq H$. Moreover, it is assumed that there is a uniform flow of constant velocity U. The fluid is assumed to be inviscid, incompressible and both fluids motion are irrotational which ensures the existence of the velocity potentials $\Phi_j(x, z, t)$ for $j = 1, 2$ which satisfy the two-dimensional Laplace equation is given by

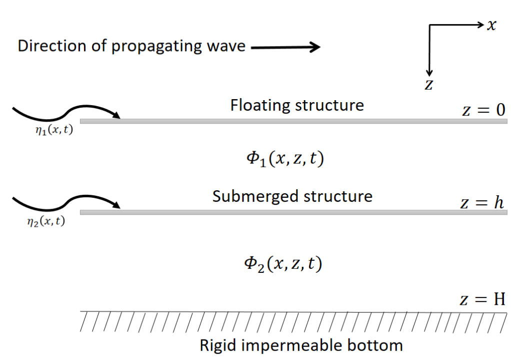


Figure 1: Schematic of the problem

$$\nabla^2 \Phi_j = 0, \quad j = 1, 2, \tag{1}$$

in the upper and lower layer fluid respectively, and floating and submerged plates are respectively referred to as $j\,(j = 1, 2)$-th plate. The linearized kinematic boundary conditions on the plate covered surface and submerged plate are given by

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\eta_1 = \frac{\partial \Phi_1}{\partial z} \quad \text{on} \quad z = 0, \tag{2}$$

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\eta_2 = \frac{\partial\Phi_2}{\partial z}=\frac{\partial\Phi_1}{\partial z} \quad \text{on} \quad z=h, \tag{3}$$

where $\eta_1(x,t)$ and $\eta_2(x,t)$ are the respective deflections of the floating and submerged elastic plates. Bernoulli equation yields the linearized hydrodynamics pressure $p_j(x,z,t)$ in the fluid regions ($j=1,2$ refer to upper and lower layer fluid regions respectively) as

$$p_j(x,y,t)=-\rho\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\Phi_j+\rho g z, \tag{4}$$

with $p_0(x,z,t)$ being the atmospheric pressure acting on the floating elastic plate. Further, in the presence of in-plane compressive forces N_j acting along x-direction on the homogeneous floating and submerged plates respectively, the plate deflections are given by (as in [5] and [6])

$$\left(E_1I_1\frac{\partial^4}{\partial x^4}+N_1\frac{\partial^2}{\partial x^2}+\rho_{p_1}d_1\frac{\partial^2}{\partial t^2}\right)\eta_1=-(p_1|_{z=0}-p_0|_{z=0}), \tag{5}$$

$$\left(E_2I_2\frac{\partial^4}{\partial x^4}+N_2\frac{\partial^2}{\partial x^2}+\rho_{p_2}d_2\frac{\partial^2}{\partial t^2}\right)\eta_2=-(p_2|_{z=h}-p_1|_{z=h}). \tag{6}$$

with E_j being the Young's modulus, $I_j=d_j^3/(12(1-\nu_j^2))$, ν_j being the Poisson's ratio, d_j being the thickness of the elastic plate and ρ_{p_j} being the density of the plate.
Hence, the linearized conditions on the floating and submerged plates are obtained from Eqs. (4) - (6) as

$$\left(E_1I_1\frac{\partial^4}{\partial x^4}+N_1\frac{\partial^2}{\partial x^2}+\rho_{p_1}d_1\frac{\partial^2}{\partial t^2}\right)\frac{\partial\Phi_1}{\partial z}=\rho\left\{\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)^2\Phi_1-g\frac{\partial\Phi_1}{\partial z}\right\}, \tag{7}$$

$$\left(E_2I_2\frac{\partial^4}{\partial x^4}+N_2\frac{\partial^2}{\partial x^2}+\rho_{p_2}d_2\frac{\partial^2}{\partial t^2}\right)\frac{\partial\Phi_2}{\partial z}=\rho\left\{\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)^2(\Phi_2-\Phi_1)\right\}. \tag{8}$$

Further, the linearized kinematic condition Eq. (3) on submerged plate yields,

$$\left.\frac{\partial\Phi_2}{\partial z}\right|_{z=h^+}=\left.\frac{\partial\Phi_1}{\partial z}\right|_{z=h^-}. \tag{9}$$

Finally, at the rigid bottom, the boundary condition is given by

$$\frac{\partial\Phi_2}{\partial z}=0, \quad z=H. \tag{10}$$

3. Characteristics of wave motion

Here, the characteristics of monochromatic flexural gravity waves are analyzed under the assumptions

$$\eta_j(x,t)=\eta_{j,0}\cos(kx-\omega t), \quad j=1,2,$$

where k is the wavenumber of the plane progressive wave with angular frequency ω and $j=1,2$ correspond to wave amplitudes in surface and internal mode, respectively. Thus, the velocity potentials $\Phi_j(x,z,t)$, $(j=1,2)$ in water of finite depth is obtained as

$$\Phi_1(x,z,t)=Ux-\frac{\mathrm{i}(\omega-Uk)\eta_{1,0}}{k}(a\cosh kz+\sinh kz)\sin(kx-\omega t), \tag{11}$$

$$\Phi_2(x,z,t)=Ux-\frac{\mathrm{i}(\omega-Uk)\eta_{2,0}\cosh k(H-z)}{k\sinh k(H-h)}\sin(kx-\omega t). \tag{12}$$

where

$$a=-\frac{(\omega-Uk)^2\{1+\coth kh\coth k(H-h)\}-gkA_2\coth kh}{(\omega-Uk)^2\{\coth kh+\coth k(H-h)\}-gkA_2}, \tag{13}$$

U is the uniform current speed along x-direction, and the wave number k satisfies the dispersion relation

$$R(\omega-Uk)^4-S(\omega-Uk)^2+T=0, \tag{14}$$

where

$$
\begin{aligned}
R =& 1 + \coth kh \coth k(H-h), \\
S =& gk[A_2 \coth kh + A_1\{\coth kh + \coth k(H-h)\}], \\
T =& g^2k^2A_1A_2, \quad A_1 = D_1k^4 - Q_1k^2 + 1, \quad A_2 = D_2k^4 - Q_2k^2, \\
D_i =& E_iI_i/(\rho g), \quad Q_i = N_i/(\rho g).
\end{aligned}
$$

with $E_1I_1,\ E_2I_2$ being the flexural rigidities of floating and submerged plates respectively with $N_1,\ N_2$ being the corresponding compressive forces, ρ being the density of the fluid medium, H being the total water depth and h being the distance of the submerged elastic plate from the floating elastic plate. Further, the amplitude ratio of floating and submerged plates are given as

$$
\left|\frac{\eta_{2,0}}{\eta_{1,0}}\right| = \frac{(\omega - Uk)^2}{\sinh kh[(\omega-Uk)^2\{\coth kh + \coth k(H-h)\} - gkA_2]}. \tag{15}
$$

Assuming the elastic restoring force to be much larger than the inertial effect (as in [7]), the term $\gamma\omega^2 << 1$ is neglected. Thus, the dispersion relation associated with flexural gravity wave motion in the presence of uniform current and compressive force in finite water depth satisfies

$$
(\omega - Uk)^2_{\pm} = \frac{S \pm \sqrt{S^2 - 4RT}}{2R}. \tag{16}
$$

where $\pm$ sign corresponds to the wave in the surface mode that propagates along the floating ice-sheet/structure and the wave in internal mode that propagates along the submerged plate, respectively.

Now, considering a specific case of infinite depth water underneath submerged structure which is placed at a shallow depth, the above dispersion relations can be modified into

$$
(\omega - Uk)^2_{+} = \left(S_1 - \frac{T_1}{S_1}\right) \text{ and } (\omega - Uk)^2_{-} = \left(\frac{T_1}{S_1}\right). \tag{17}
$$

The amplitude ratio $\left|\frac{\eta_{2,0}}{\eta_{1,0}}\right|$ is approximated as

$$
\left|\frac{\eta_{1,0}}{\eta_{2,0}}\right| = \frac{[(\omega-Uk)^2(1+kh) - gkA_2kh]}{(\omega-Uk)^2}. \tag{18}
$$

The condition of wave blocking is obtained as $\frac{d\omega_{\pm}}{dk} = 0$ which, consequently, provides the opposing current speed $U^b_{\pm}$ at the time of blocking as

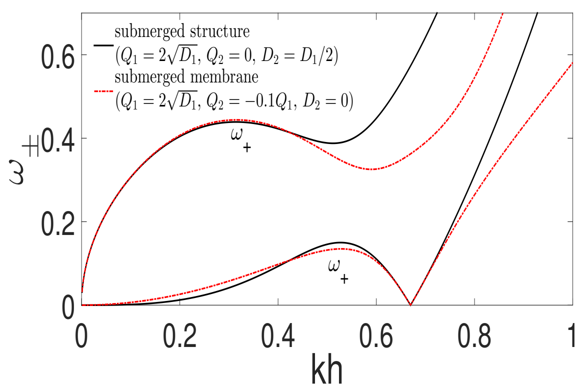
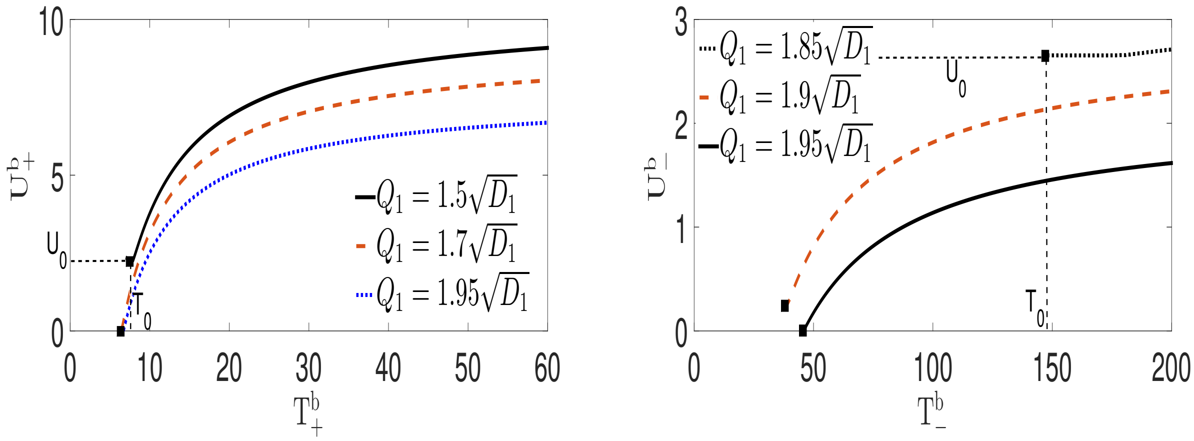


Figure 2: Dispersion graphs for both the modes are plotted for two different cases. Wave blocking can be observed from the optima of dispersion curves.

$$
U^b_{\pm} = \begin{cases} \dfrac{1}{2}\left[\dfrac{1}{\sqrt{S_1(S_1^2 - T_1)}}\dfrac{dT_1}{dk} - \dfrac{S_1 + \frac{T_1}{S_1}}{\sqrt{S_1(S_1^2-T_1)}}\dfrac{dS_1}{dk}\right] & \text{for surface mode,} \\ \dfrac{1}{2}\left[\dfrac{1}{S_1}\sqrt{\dfrac{T_1}{S_1}}\dfrac{dS_1}{dk} - \dfrac{1}{\sqrt{S_1T_1}}\dfrac{dT_1}{dk}\right] & \text{for internal mode.} \end{cases}
$$

The corresponding blocking frequencies $\omega^b_{\pm}$ can be obtained from Eq. (17) as (taking the positive branch of the dispersion relation)

$$
\omega_b^{\pm} = \begin{cases} \left(\sqrt{S_1 - \dfrac{T_1}{S_1}} + U_b^+ k\right) & \text{for surface mode,} \\ \left(\sqrt{\dfrac{T_1}{S_1}} + U_b^- k\right) & \text{for internal mode,} \end{cases}
$$



(a) Surface mode (b) Internal mode

Figure 3: Dependency between the opposing ocean current and the time period of incoming blocked waves for different values of compressive force acting on the floating plate. Figure 3(a) illustrates the dependency for surface waves for three different values of compressive force, namely $Q_1 = 1.5\sqrt{D_1}$, $1.7\sqrt{D_1}$, $1.95\sqrt{D_1}$. The occurrence of blocking is observed even without current for $Q_1 = 1.7\sqrt{D_1}$, $1.95\sqrt{D_1}$ at almost the same time period (blue-dot and red-dash curves). Any incoming wave having less time-period cannot be blocked. On the other hand, when $Q_1 = 1.5\sqrt{D_1}$, a minimum opposing current (U_0) is required to block the incoming wave of any time-period (rigid black curve). Similarly, any incoming wave having time-periodicity less than T_0 cannot be blocked by any opposing current. Figure 3(b) depicts the same for waves in internal mode, but for three different values of compressive force, namely $Q_1 = 1.85\sqrt{D_1}$, $1.9\sqrt{D_1}$, $1.95\sqrt{D_1}$. The choices of such high values of Q_1 is chosen because of very high minimum values of (T_0, U_0) (the case with $Q_1 = 1.85\sqrt{D_1}$) to initiate blocking in internal mode (black-dot curve). The pattern shows that for relatively smaller values of Q_1, (T_0, U_0) values will be very large.

The above equations will be solved for unknown wavenumber k for fixed incoming wave frequency and then the corresponding $U^b_\pm$ will be obtained.

References

[1] S. Das, T. Sahoo, and M. H. Meylan. Dynamics of flexural gravity waves: from sea ice to hawkings radiation and analogue gravity. *Proc. R. Soc. A*, 474(2209):20170223, 2018.

[2] S. Das, T. Sahoo, and M. H. Meylan. Flexural-gravity wave dynamics in two–layer fluid: blocking and dead water analogue. *J. Fluid Mech.*, 854:121–145, 2018.

[3] S. Das, P. Kar, T. Sahoo, and M. H. Meylan. Flexural-gravity wave motion in the presence of shear current: wave blocking and negative energy waves. *Phy. Fluid*, 30(10):106606, 2018.

[4] M. H. Meylan and M. A. Peter. Water-wave scattering by submerged elastic plates. *The Quarterly Journal of Mechanics & Applied Mathematics*, 62(3):321–344, 2009.

[5] S. R. Manam, J. Bhattacharjee, and T. Sahoo. Expansion formulae in wave structure interaction problems. In *P. Roy. Soc. Lond. A Mat.*, volume 462, pages 263–287. The Royal Society, 2006.

[6] C. Fox and V. A. Squire. On the oblique reflexion and transmission of ocean waves at shore fast sea ice. *Philos. T. Roy. Soc. A*, 347(1682):185–218, 1994.

[7] R. M. S. M. Schulkes, R. J. Hosking, and A. D. Sneyd. Waves due to a steadily moving source on a floating ice plate. part 2. *J. Fluid Mech.*, 180:297–318, 7 1987.