Mysterious wavefront uncovered

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A wavemaker in harmonic motions generates progressive waves in a wave tank. Using the new formulation describing these waves given in Chen & Li (2018), three classes of waves in different regions including wavefront in the fore part, steady-state waves behind and transient waves in between, are then demystified by revealing their respective features. It is shown that the starting position x_F of wavefront propagates at the very group velocity associated with the frequency of wavemaker and has an amplitude value which is not the maximum of wave train and larger than that (1/2) predicted by Miles (1962). Wavelengths of wavefront increase at the quadratic rate with the distance $x > x_F$ while amplitudes decrease in order of $(x/x_F)^{-3/2}$. In addition to steady-state waves behind the wavefront $x < x_F$, there are transient waves whose amplitude decreases in order of $(x/x_F)^{5/2}$. Furthermore, we formulate the amplitude of the largest wave appearing behind the wavefront and its position relative to the wavefront receding backward with time.

1 Introduction

We consider a semi-infinite fluid of gravity $g = 1$ limited on the top by the free surface and use a Cartesian coordinate system (o, x, z) located at the mean surface with axis oz pointing upwards. The flexible vertical plate is located at $x = 0$ and oscillating with horizontal velocity $\mathcal{A}(z)\sin(\omega t)$ with amplitude $\mathcal{A}(z) = A\omega \exp(k_0 z)$ along the plate, frequency ω and wavenumber k_0. The wave elevation for $x > 0$ is obtained by the classical method based on Fourier transform in Dai & Duan (2008) and is written as :

$$\eta^I(x,t) = -A\sin(k_0 x)\cos(\omega t) - A\frac{2k_0}{\pi}⨍_0^\infty \frac{\cos(kx)\cos(\beta t)}{k^2 - k_0^2}\,dk \tag{1}$$

where $⨍$ stands for the principal value in the sense of Cauchy. The dispersion relation imposes :

$$\omega = \sqrt{k_0} \quad \text{and} \quad \beta = \sqrt{k}$$

The integral (1) can be rewritten in a compact form

$$\eta^I(x,t)/A = \Re\mathrm{e}\{\eta(x,t)\} \quad \text{with} \quad \eta(x,t) = e^{-i\tau}\int_0^\infty F(u)\frac{e^{i(1+u)^2\tau} + e^{i(1-u)^2\tau}}{u_0 - u}\,du \tag{2}$$

in which $\Re\mathrm{e}\{\cdot\}$ means to take the real part and $\int$ stands for the integration along the real axis but circumventing above the pole $u = u_0$. To obtain (2), we have made the change of variables :

$$k = (ut)^2/(2x)^2\ , \quad u_0 = 2\omega x/t \quad \text{and} \quad \tau = t^2/(4x) \tag{3}$$

and the function $F(u)$ is given by

$$F(u) = \frac{2u_0^2 u}{\pi(u_0^2 + u^2)(u_0 + u)} \tag{4}$$

associated with two oscillatory functions $\exp[i(1+u)^2\tau]$ and $\exp[i(1-u)^2\tau]$. The contour associated with the first along which the integrand is steepest descent is easy to find. The contour associated with the second is more complex due to the multiple values of $(1-u)^2$ for $u \in (0, 2)$ and the location of poles. The detail analysis is given in Chen & Li (2018) and the final result is a sum of four components

$$\eta(x,t) = \eta_S(x,t) + \eta_T(x,t) + \eta_L(x,t) + \eta_F(x,t) \tag{5}$$

in which the steady-state component present only for $x < x_F$ with $x_F = t/(2\omega)$ called the starting position of wavefront :

$$\eta_S(x,t) = \mathrm{i}\mathrm{e}^{\mathrm{i}(k_0 x - \omega t)} H(x_F - x) \tag{6}$$

with $H(\cdot)$ the Heaviside function. The initial component $\eta_T(x,t)$ given by

$$\eta_T(x,t) = -\mathrm{i}\mathrm{e}^{-\omega t - \mathrm{i}k_0 x} H(x_F - x) \tag{7}$$

decreases exponentially with t and only significant at the initial time and a position close to wave-maker. The local component $\eta_L(x,t)$ is smooth varying, of small value and approximated by

$$\eta_L(x,t) \approx \frac{1}{2\pi} \sum_{j=0}^{4} a_j \mathrm{e}^{b_j t^2/(4x)} E_1[b_j t^2/(4x)] \tag{8}$$

in which the coefficients (a_j, b_j) for $j = 0, 1, \cdots, 4$ are dependent on u_0 given in Tab.1 of Chen & Li (2018). In (8), $E_1(\cdot)$ represents the exponential integral function defined by (5.1.1) in Abramowitz & Stegun (1967). Finally, the wavefront component $\eta_F(x,t)$ is analysed below.

2 Wavefront component

The wavefront component in (5) is defined by

$$\eta_F(x,t) = \mathrm{e}^{-\mathrm{i}t^2/(4x)} \mathcal{F}(u_0, \tau) \tag{9}$$

involving the wavefront function given in closed form :

$$\mathcal{F}(u_0,\tau) = -\frac{\mathrm{i}}{2}\left[\mathbf{Cef}(\alpha_1\sqrt{\tau/2}) - \mathbf{Cef}(\alpha_2\sqrt{\tau/2}) - \mathbf{Cef}(\alpha_3\sqrt{\tau/2}) + \mathbf{Cef}(\alpha_4\sqrt{\tau/2})\right] \tag{10a}$$

for $u_0 < 1$ or $x < x_F$ and

$$\mathcal{F}(u_0,\tau) = -\frac{\mathrm{i}}{2}\left[-\mathbf{Cef}(-\alpha_1\sqrt{\tau/2}) + \mathbf{Cef}(-\alpha_2\sqrt{\tau/2}) - \mathbf{Cef}(\alpha_3\sqrt{\tau/2}) + \mathbf{Cef}(\alpha_4\sqrt{\tau/2})\right] \tag{10b}$$

for $u_0 > 1$ or $x > x_F$. The coefficients α_j for $j = 1, 2, 3, 4$ are given by

$$\begin{aligned} \alpha_1 &= (1-u_0) + \mathrm{i}(1-u_0)\ ; \quad \alpha_2 = (1-u_0) + \mathrm{i}(1+u_0) \\ \alpha_3 &= (1+u_0) + \mathrm{i}(1-u_0)\ ; \quad \alpha_4 = (1+u_0) + \mathrm{i}(1+u_0) \end{aligned} \tag{11}$$

The special function $\mathbf{Cef}(z)$ in (10) is defined by

$$\mathbf{Cef}(z) = \mathrm{e}^{z^2}\mathrm{erfc}(z) \approx \begin{cases} 1 - 2z/\sqrt{\pi} + z^2 - O(z^3) & |z| \to 0 \\ z^{-1}/\sqrt{\pi} - O(z^{-3}) & |z| \to \infty \end{cases} \tag{12}$$

involving the complex complementary error function $\mathrm{erfc}(z)$ defined by (7.1.2) in Abramowitz & Stegun (1967). The wavefront function $\mathcal{F}(u_0,\tau)$ is a smooth function for $x \lessgtr x_F$ and discontinuous at $x = x_F$ due to the first term on the right hand side of (10).

The work in Miles (1962) gives expressions equivalent to the first term on the right hand side of (10). His prediction of wave envelope at $x = x_F^+$ is then 1/2 which is very approximative. Following the same expressions, the magnitude of waves in wavefront decreases in order of $(x/x_F)^{-1/2}$ which is not true. In fact, asymptotic analyses of (10) at $x \approx x_F$ for large values of x_F give

$$\mathcal{F}(u_0,\tau) \approx \mp\mathrm{i}/2 + \mathrm{e}^{\mathrm{i}\pi/4}/(4\omega\sqrt{\pi x_F}) \tag{13}$$

for $x \lesseqgtr x_F$, which means that the wave magnitude at $x = x_F^+$ is larger than 1/2. Furthermore, asymptotic analysis of (10) gives the amplitude of waves in wavefront

$$\mathcal{F}(u_0, \tau) \approx 2\mathrm{e}^{\mathrm{i}\pi/4}/(\omega\sqrt{\pi x_F})(x/x_F)^{-3/2} \tag{14}$$

for $x/x_F \gg 1$ and that of transient waves behind the wavefront

$$\mathcal{F}(u_0, \tau) \approx -2\mathrm{e}^{\mathrm{i}\pi/4}/(\omega\sqrt{\pi x_F})(x/x_F)^{5/2} \tag{15}$$

for $x/x_F \ll 1$. According to the oscillator $\mathrm{e}^{-\mathrm{i}t^2/(4x)} = \mathrm{e}^{-\mathrm{i}(2\pi/\lambda_F)x}$ in (9), the wavelengths of waves in wavefront

$$\lambda_F = 8\pi x^2/t^2 = (2\pi/\omega^2)(x/x_F)^2 \tag{16}$$

are proportional to squared distance from x_F, i.e., waves are stretching out quadratically with distance. Waves are longer and longer in the far wavefront.

Behind the wavefront for $x < x_F$, transient waves are composed essentially of the steady-state component $\eta_S(x,t)$ (6) and wavefront component $\eta_F(x,t)$ (9) :

$$\eta(x,t) = \eta_S(x,t) + \eta_F(x,t) \approx \mathrm{i}\mathrm{e}^{\mathrm{i}(k_0 x - \omega t)} - (\mathrm{i}/2)\mathrm{e}^{-\mathrm{i}t^2/(4x)}\mathbf{Cef}(\mathrm{e}^{\mathrm{i}\pi/4}\sqrt{\pi/2}Z) \tag{17}$$

with $Z = (t - 2\omega x)/\sqrt{2\pi x}$. In (17), we have used the first term on the right hand side of (10a) to represent the wavefront function. By using (7.3.22) in Abramowitz & Stegun (1967) for transforming the error function into Fresnel integrals, we have

$$\eta(x,t) \approx \mathrm{i}\mathrm{e}^{\mathrm{i}(k_0 x - \omega t)}\left\{1 + C(Z) + S(Z) + \mathrm{i}\big[C(Z) - S(Z)\big]\right\}/2 \tag{18}$$

in which $C(Z)$ and $S(Z)$ are Fresnel integrals. The envelope of transient waves (18) behind wavefront is then

$$E(x,t) = \sqrt{2C(Z)\big[C(Z)+1\big] + 2S(Z)\big[S(Z)+1\big] + 1}/2 \tag{19}$$

which has a maximum value of $E = E_m = 1.1707$ at $Z = Z_m = 1.2172$. The position $x = x_m < x_F$ at which the maximum wave appears is then given by

$$x_F - x_m = (2\pi/\omega^2)(Z_m^2/8)\left(\sqrt{1 + (8/Z_m^2)\omega t/(2\pi)} - 1\right) \tag{20}$$

increasing with time t. This can be understood that transient waves behind the wavefront propagate at lower speed than the starting position $x = x_F$ of wavefront which move at the group velocity, and the maximum wave recedes backward from the wavefront when the time increases. Therefore more transient waves appear between the maximum wave and the wavefront.

3 Numerical results

The wavefront component $\eta_F(x,t)$ defined by (9) is first analysed and depicted on Fig.1 along x/x_F varying from 0 to 2 for $k = 3/2 = \omega^2$ and at $\bar{t} = t/T = 15$ with the period $T = 2\pi/\omega = 2\pi/\sqrt{3/2}$. The wavefront component is oscillatory and has a discontinuous at $x/x_F = 1$. Wavelengths in wavefront increase with distance and decrease behind wavefront as shown by the oscillator $\mathrm{e}^{-\mathrm{i}t^2/(4x)}$. The envelope (magnitude) determined by the wavefront function $\mathcal{F}(u_0, \tau)$ given in closed form (10) and represented by the dashed red lines. The wavefront function is a smoothly varying for $x \lesseqgtr x_F$ and discontinuous at $x = x_F$. Unlike the prediction in Miles (1962), the envelope value depending on the time and smaller (larger) than 1/2 for $x = x_F^-$ ($x = x_F^+$), respectively. Behind the wavefront $x < x_F$, the amplitude increases asymptotically at the rate $c(x/x_F)^{5/2}$ with the coefficient $c = 2/\sqrt{\pi\omega^2 x_F} = (2/\pi)\sqrt{T/t}$ according to (15). In wavefront $x > x_F$, the amplitude decreases asymptotically at the rate $c(x/x_F)^{-3/2}$ according to (14).

The whole profiles $\eta(x,t)$ defined by (5) of waves generated by harmonic motions of wavemaker with frequency $\omega = \sqrt{3/2}$ are shown by dashed lines on Fig.2 along the distance $\bar{x} = x/\lambda$ from 0 to 40, while the envelopes are depicted by solid lines. Results for different instants $\bar{t} = t/T$ equal

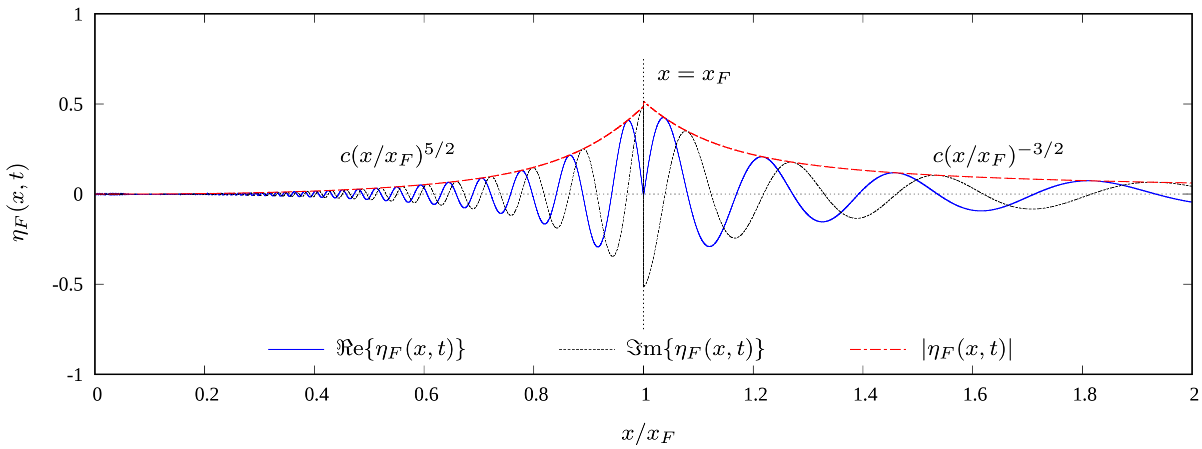

Figure 1: Wavefront component along x/x_F for $k = 3/2$ at $\bar{t} = t/T = 15$

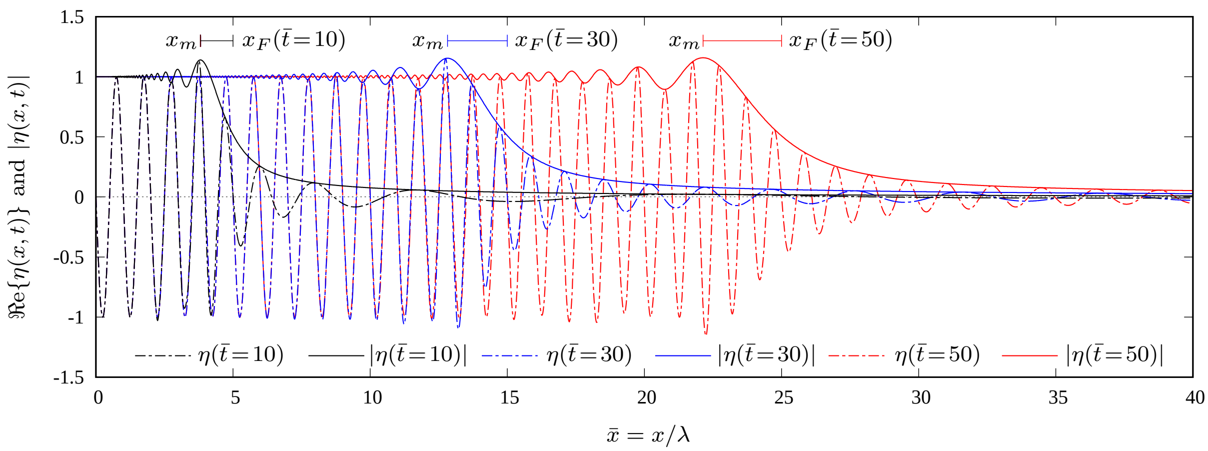

Figure 2: Transient waves and their envelopes for three instants $\bar{t} = t/T = 10, 30$ and 50.

to 10, 30 and 50 are represented by the black, blue and red lines, respectively. In addition, the position of the maximum wave at each instant defined by (20) is marked together with associated x_F. Wave profiles confirm the three regions of waves: the wave front starting from the position x_F, the steady-state waves from the wavemaker to a distance $x_S < x_m < x_F$ and transient waves in between. The distance $x_F - x_m$ between the maximum wave x_m and wavefront x_F increases with time as predicted by (20). It is interesting to remark that the position of maximum wave behind wavefront recedes backward from the wavefront so that more transient waves exist between the maximum wave and wavefront.

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