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Second order interaction of flexural gravity waves with bottom mounted vertical circular cylinder

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Introduction

Interaction of gravity waves with bottom mounted vertical circular cylinder received much attention in the past in the context of both the water waves and flexural gravity waves. Both the linear and higher order interactions were of concern. Unlike the problem of water waves where the semi-analytical solution for linear and second order problem is well mastered and agreed within the community, the solution for the linear problem of flexural gravity waves was proposed in different forms by the different authors and the second order problem seems has not been considered except for the 2D case [5]. In the present work, a general methodology is proposed both for the linear and the second order probem of the flexural gravity waves. The method relies on the use of the classical eigenfunction expansion principles through the definition of the relevant Green's function and the use of the Boundary Integral Equation technique.

Mathematical model

The second order theoretical model for flexural gravity waves was presented in [4] and will not be repeated here. We just mention that the generic Boundary Value Problem (BVP) for the velocity potential, can be defined as shown in Figure 1,

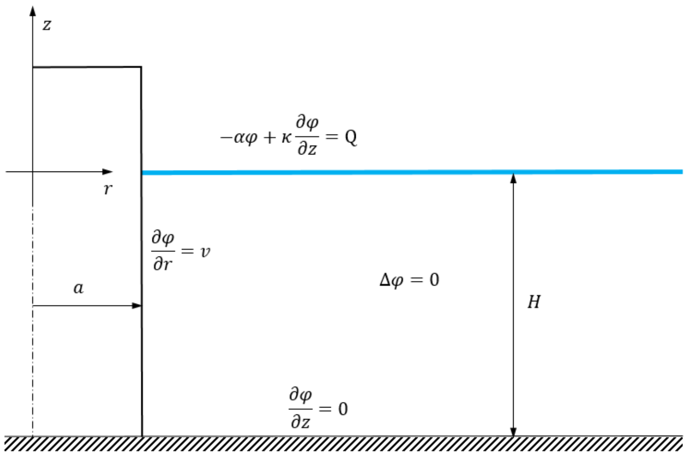

Figure 1: Basic configuration and definitions.

where the operator κ and the corresponding dispersion relation are given by:

$$\kappa = \mathcal{M} + \mathcal{D}\Delta_0^2 \qquad , \qquad \alpha = (\mathcal{M} + \mathcal{D}\mu^4)\mu \tanh \mu H \tag{1}$$

The coefficients $\mathcal{M}$ and $\mathcal{D}$ are given by:

$$\mathcal{M} = 1 - \alpha\frac{M}{\varrho} \qquad , \qquad \mathcal{D} = \frac{D}{\varrho g} \tag{2}$$

with M and D being the distributed mass and the stifness of the plate, repectively. The functions $v(z, \theta)$ and $Q(r, \theta)$ are specified for each particular problem and α is proportional to the square of the corresponding wave frequency.

In order to complete the above BVP the additional boundary conditions at the plate ends on the cylinder and at infinity should be introduced. These conditions describe the way in which the plate is attached to the cylinder i.e. clamped, free, simply supported ... For example, in the case of the plate clamped to the cylinder the conditions of the zero displacement and zero slope apply:

$$\frac{\partial\varphi}{\partial z}(a,\theta,0)=0 \quad , \quad \frac{\partial^2\varphi}{\partial r\partial z}(a,\theta,0)=0 \tag{3}$$

Solution methodology

The soultion procedure uses the eigenfunction expansion principles. In that respect, the method of separating variables for flows symmetric with respect to x axis provides:

$$\varphi(r,\theta,z)=\sum_{m=0}^{\infty}\epsilon_m\varphi_m(r,z)\cos m\theta=\sum_{m=0}^{\infty}\epsilon_m\sum_{n=-2}^{\infty}f_n(z)\varphi_{mn}(r)\cos m\theta \tag{4}$$

where $\epsilon_m=1$ for $m=0$ and $\epsilon_m=2$ for $m>0$, and the vertical eigenfunctions are defined by:

$$f_n(z)=\frac{\cosh\mu_n(z+H)}{\cosh\mu_n H} \tag{5}$$

The wave numbers μ_n are roots of the dispersion equation (1). The dispersion equation has two real roots ($\pm\mu_0$, $\mu_0>0$), infinite number of pure imaginary roots ($\pm\mu_n=\pm ik_n$, $k_n>0$, $n=1,\infty$) and four complex roots ($\mu_{-4}=-a_0-ib_0$, $\mu_{-3}=a_0-ib_0$, $\mu_{-2}=-a_0+ib_0$, $\mu_{-1}=a_0+ib_0$ with $a_0>0$, $b_0>0$). In the present analysis we follow the procedure from [6] and we restrict ourselves to the roots $\mu_{-2},\mu_{-1},\mu_0,\mu_n=ik_n$, $n=1,\infty$. It is important to mention that the eigenfunctions $f_n(z)$ are not orthogonal in a classical sense but they obey to the following orthogonal relation:

$$\int_{-H}^{0}f_m(z)f_n(z)dz+\frac{\mathcal{D}}{\alpha}\left(f_m^{'''}f_n^{'}+f_m^{'}f_n^{'''}\right)_{z=0}=\frac{\delta_{mn}}{2\mathcal{C}_n} \tag{6}$$

The Green's function associated with the above defined BVP, can be derived following the method presented in [2, 9]. Here we skip the details and we simply recall the final expression:

$$G(\boldsymbol{x};\boldsymbol{\xi}) = \sum_{m=0}^{\infty}\epsilon_m G_m(r,z;\rho,\zeta)\cos m(\theta-\vartheta) \tag{7}$$

$$G_m(r,z;\rho,\zeta) = -\frac{i}{2}\sum_{n=-2}^{\infty}\mathcal{C}_n\begin{pmatrix}H_m(\mu_n r)J_m(\mu_n\rho)\\ J_m(\mu_n r)H_m(\mu_n\rho)\end{pmatrix}f_n(\mu_n z)f_n(\mu_n\zeta) \quad , \quad \begin{pmatrix}r>\rho\\ r<\rho\end{pmatrix} \tag{8}$$

This expression is valid for two arbitrary points $\boldsymbol{x}=(r,\theta,z)$ and $\boldsymbol{\xi}=(\rho,\vartheta,\zeta)$ in the fluid domain $-H\le(z,\zeta)\le 0$. Now we recall the Green's theorem in its original form:

$$\begin{pmatrix}4\pi\varphi(\boldsymbol{x})\\ 0\end{pmatrix}-\iint_{S_a}\varphi(\boldsymbol{\xi})\frac{\partial G(\boldsymbol{x};\boldsymbol{\xi})}{\partial\rho}dS = \iint_{S_a}G(\boldsymbol{x};\boldsymbol{\xi})\frac{\partial\varphi(\boldsymbol{\xi})}{\partial\rho}dS+ \iint_{S_F}[\varphi(\boldsymbol{\xi})\frac{\partial G(\boldsymbol{x};\boldsymbol{\xi})}{\partial\zeta}-G(\boldsymbol{x};\boldsymbol{\xi})\frac{\partial\varphi(\boldsymbol{\xi})}{\partial\zeta}]dS \quad , \quad \begin{pmatrix}r>a\\ r<a\end{pmatrix} \tag{9}$$

where S_a denotes the surface of the cylinder $r=a$, S_F denotes the free surface $z=0$, and where it was accounted for the fact that the integrals at infinity and at the sea bottom ($z=-H$) dissapear. For convenience we denote the free surface integral in (9) by I_{S_F} and we rewrite it as follows:

$$I_{S_F} = \iint_{S_F}[\varphi\frac{\partial G}{\partial\zeta}-G\frac{\partial\varphi}{\partial\zeta}]dS=\frac{1}{\alpha}\iint_{S_F}\frac{\partial G}{\partial\zeta}QdS+\frac{\mathcal{D}}{\alpha}\iint_{S_F}[\frac{\partial G}{\partial\zeta}\Delta_0^2\frac{\partial\varphi}{\partial\zeta}-\frac{\partial\varphi}{\partial\zeta}\Delta_0^2\frac{\partial G}{\partial\zeta}]dS \tag{10}$$

The following identity, valid for two arbitrary harmonic functions ϕ and ψ :

$$\iint_S(\psi\Delta^2\phi-\phi\Delta^2\psi)dS=\int_C[\Delta\psi\frac{\partial\phi}{\partial n}-\Delta\phi\frac{\partial\psi}{\partial n}+\psi\frac{\partial}{\partial n}\Delta\phi-\phi\frac{\partial}{\partial n}\Delta\psi]dC \tag{11}$$

is used and the integral I_{S_F} is rewritten as:

$$I_{S_F} \quad = \quad \frac{1}{\alpha}\iint_{S_F}\frac{\partial G}{\partial\zeta}QdS - \frac{\mathcal{D}}{\alpha}\int_{r=a}[\frac{\partial^3 G}{\partial\zeta^3}\frac{\partial^2\varphi}{\partial\rho\partial\zeta} - \frac{\partial^3\varphi}{\partial\zeta^3}\frac{\partial^2 G}{\partial\rho\partial\zeta} + \frac{\partial G}{\partial\zeta}\frac{\partial^4\varphi}{\partial\rho\partial\zeta^3} - \frac{\partial\varphi}{\partial\zeta}\frac{\partial^4 G}{\partial\rho\partial\zeta^3}]dC \qquad (12)$$

After using the orthogonality of the Fourier series we can write the following expression, for a point inside the cylinder [$r = a - \delta$ ($\delta > 0$)] :

$$\int_{-H}^{0}\varphi_m(a,\zeta)\frac{\partial G_m(r,z;a,\zeta)}{\partial\rho}d\zeta \quad = \quad \int_{-H}^{0}G_m(r,z;a,\zeta)v_m(a,\zeta)d\zeta - \frac{1}{a\alpha}\int_a^\infty\frac{\partial G_m(r,z;\rho,0)}{\partial\zeta}Q_m(\varrho)d\rho$$

$$+\frac{\mathcal{D}}{\alpha}\left[\frac{\partial^3 G_m}{\partial\zeta^3}\frac{\partial^2\varphi_m}{\partial\rho\partial\zeta} - \frac{\partial^3\varphi_m}{\partial\zeta^3}\frac{\partial^2 G_m}{\partial\rho\partial\zeta} + \frac{\partial G_m}{\partial\zeta}\frac{\partial^4\varphi_m}{\partial\rho\partial\zeta^3} - \frac{\partial\varphi_m}{\partial\zeta}\frac{\partial^4 G_m}{\partial\rho\partial\zeta^3}\right]_{\substack{\rho=a\\ \zeta=0}} \qquad (13)$$

After developing the above expression and using the fact that $f_k^{'} = -\mu_k\tanh\mu_k H$ and $f_k^{'''} = \mu_k^3\tanh\mu_k H$ at $z = 0$ we obtain for $\varphi_{mn}(a)$ the following expression:

$$\varphi_{mn}(a) \quad = \quad \frac{2\mathcal{C}_n}{\mu_n H_m^{'}(\mu_n a)}\left\{H_m(\mu_n a)\int_{-H}^{0}f_n(z)v_m(a,z)dz - \right. \qquad (14)$$

$$\left.\frac{\mu_n\tanh\mu_n H}{\alpha}\left[\frac{1}{a}\int_a^\infty H_m(\mu_n\rho)Q_m(\varrho)d\rho - \mathcal{D}H_m(\mu_n a)\left(\mu_n^2\frac{\partial^2\varphi_m}{\partial r\partial z} - \frac{\partial^4\varphi_m}{\partial r\partial z^3}\right)_{\substack{r=a\\ z=0}}\right]\right\}$$

The quantities $\partial^2\varphi_m/\partial r\partial z$ and $\partial^4\varphi_m/\partial r\partial z^3$ at the connecting line $(a, 0)$, are denoted by γ_m and σ_m respectively, and their values can be deduced from the appropriate edge conditions. Indeed, when applying the edge conditions, and whatever the type of these conditions, the system of equations for γ_m and σ_m can be deduced in the following form:

$$J_1\gamma_m + J_2\sigma_m = J_3 \qquad , \qquad J_4\gamma_m + J_5\sigma_m = J_6 \qquad (15)$$

where J_n , $n = 1, 6$ are the known coefficients.
Solution of this system gives the coefficients γ_m and σ_m so that the problem is formally solved.

Particular cases

In order to demonstrate the generality of the above solution, here below we consider some particular situations for which the solutions have been presented in the past by different authors.

Linear case $Q = 0$

For the linear case the free surface condition is homogeneous ($Q = 0$) and the solution valid not only at the cylinder but in the whole fluid domain $r \geq a$ is given by:

$$\varphi_{mn}(r) = \frac{2\mathcal{C}_n}{\mu_n H_m^{'}(\mu_n a)}\left[\int_{-H}^{0}f_n(z)v_m(a,z)dz - \frac{\mu_n\tanh\mu_n H}{\alpha}\mathcal{D}\Big(\mu_n^2\gamma_m - \sigma_m\Big)_{\substack{r=a\\ z=0}}\right]H_m(\mu_n r) \qquad (16)$$

Water waves

In the case of the water waves we have $M = 0$ and $D = 0$ and, at the same time, the edge conditions (3) are no more relevant. With this in mind, the classical dispersion relation for water waves is easily recovered:

$$\alpha = \mu\tanh\mu H \qquad (17)$$

The solution of this equation gives one positive root (μ_0) and an infinite number of purely imaginary roots ($i\mu_n$, $n = 1, \infty$). The corresponding eigenfunctions in vertical direction keep the same form (5) and the orthogonality relation simplifies to:

$$\int_{-H}^{0}f_m(z)f_n(z)dz = \frac{\delta_{mn}}{2C_n} \qquad (18)$$

and the expression for φ_{mn} becomes:

$$\varphi_{mn}(a) = \frac{2\mathcal{C}_n}{\mu_n H'_m(\mu_n a)} \left[H_m(\mu_n a) \int_{-H}^{0} f_n(z) v_m(a,z) dz - \frac{1}{a} \int_{a}^{\infty} H_m(\mu_n \rho) Q_m(\varrho) d\rho \right] \tag{19}$$

This expression is fully equivalent to the solution proposed in [7].

Linear diffraction of monochromatic waves

Since the above solution is valid for any type of the external disturbance $v(a,z,\theta)$ and $Q(r,\theta)$, we can apply it directly to the case of wave diffraction. For that purpose, the incident wave is defined as the progressive sinusoidal wave with the folowing well known expression for the corresponding velocity potential φ_I:

$$\varphi_I = \mathcal{A} f_0(z) \sum_{m=0}^{\infty} \epsilon_m i^m J_m(\mu_0 r) \cos m\theta \tag{20}$$

where $\mathcal{A} = -i\zeta_A/\omega$ and ζ_A is the wave amplitude.
After replacing $v(a,z,\theta)$ by $-\partial\varphi_I/\partial r(a)$, we can deduce the $\varphi_{mn}(r)$ in the form:

$$\frac{\varphi_{mn}(r)}{H_m(\mu_n r)} = -\mathcal{A} i^m \frac{J'_m(\mu_0 a)}{H'_m(\mu_0 a)} \delta_{n0} + \frac{2\mathcal{C}_n \tanh \mu_n H}{H'_m(\mu_n a)} \frac{\mathcal{D}}{\alpha} \left[\mathcal{A} i^m \mu_0 J'_m(\mu_0 a) \tanh \mu_0 H \left(\mu_n^2 + \mu_0^2\right) + \left(\mu_n^2 \gamma_m - \sigma_m\right) \right] \tag{21}$$

It can be shown that this solution is identical to the one proposed in [6]. Furthermore, for the case of water waves, the well known McCamy and Fuchs solution is recovered:

$$\varphi(r,z,\theta) \quad = \quad -\mathcal{A} f_0(z) \sum_{m=0}^{\infty} i^m \frac{J'_m(\mu_0 a)}{H'_m(\mu_0 a)} H_m(\mu_0 r) \cos m\theta \tag{22}$$

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