

Oblique scattering by a thick rectangular barrier in deep water

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1. Introduction

Water wave scattering problems involving thick rectangular barriers in deep water are yet to be investigated in the literature on water waves although for finite depth these have been studied. For example, Newman (1965) studied the problem of water wave scattering by a thick rectangular barrier in finite depth water employing a matching argument to obtain the reflection and transmission coefficients approximately. Shortly afterwards, Mei and Black (1969) applied variational formulation to investigate scattering problems involving surface-piercing or submerged rectangular barrier in finite depth water. Later Kanoria et al (1999) used single-term Galerkin technique involving ultrapherical Gegenbauer polynomials as basis functions to obtain reflection coefficient for water wave scattering problems involving thick vertical barriers in finite depth water. Here we have investigated the effect of thick partially immersed rectangular barrier (type 1) or thick submerged rectangular barrier (type 2) on surface waves in deep water by using multi-terms Galerkin technique involving simple polynomials as basis functions.

2. Mathematical formulation of the problem

Using the linearized theory of water waves, the mathematical problem is to solve the boundary value problem described by

$$\phi_{xx} + \phi_{yy} - \nu^2\phi = 0, y > 0, \quad (2.1)$$

$$K\phi + \phi_y = 0 \text{ on } y = 0, \begin{cases} |x| > b \text{ for type 1 barrier,} \\ -\infty < x < \infty \text{ for type 2 barrier,} \end{cases} \quad (2.2)$$

$$\phi_x = 0 \text{ on } x = \pm b, y \in L_j, \text{ for type } j \text{ (} j=1,2 \text{) barrier, (} L_1 = (0, a), L_2 = (c, \infty) \text{),} \quad (2.3)$$

$$\phi_y = 0 \text{ on } y = l_j, |x| < b \text{ for type } j \text{ (} j=1,2 \text{) barrier, (} l_1 = a, l_2 = c \text{),} \quad (2.4)$$

$$\nabla\phi \rightarrow 0 \text{ as } y \rightarrow \infty, \begin{cases} -\infty < x < \infty < \infty \text{ for type 1 barrier,} \\ |x| > b \text{ for type 2 barrier,} \end{cases} \quad (2.5)$$

$$r^{\frac{1}{3}}\nabla\phi = O(1) \text{ as } r \rightarrow 0, \text{ where } r \text{ is the distance from a submerged edge of the barrier,} \quad (2.6)$$

$$\phi(x, y) \rightarrow \begin{cases} T\phi^{inc}(x, y) \text{ as } x \rightarrow -\infty, \\ \phi^{inc}(x, y) + R\phi^{inc}(-x, y) \text{ as } x \rightarrow \infty, \end{cases} \quad (2.7)$$

where R and T denote the reflection and transmission coefficients respectively. The notations are explained below. Here the thick rectangular barrier is of breadth $2b$ along one horizontal direction (x -direction) and infinitely long along the other horizontal direction (z -direction) while it is either immersed upto a depth a below the mean free surface (xz -plane) or submerged from depth c and is extended infinitely downwards. A train of surface water waves represented by the velocity potential $Re \{ \phi^{inc}(x, y) e^{i\nu z - i\sigma t} \}$ is obliquely incident at an angle α on the thick barrier from a large distance in the positive x -direction, $\phi^{inc}(x, y)$ being given by $\phi^{inc}(x, y) = 2e^{-Ky - i\mu(x-b)}$. Here $K = \frac{\sigma^2}{g}$, σ being the circular frequency of the incoming wave train, g being the acceleration due to gravity, $\mu = K \cos \alpha$ and $\nu = K \sin \alpha$. Due to the geometry of the problem, the z -dependence can be eliminated altogether by assuming the velocity potential describing the resulting motion in the form $Re \{ \phi(x, y) e^{i\nu z - i\sigma t} \}$.

3. Method of solution

Due to the geometrical symmetry of the barrier about $x = 0$, it is convenient to split $\phi(x, y)$ into its symmetric and antisymmetric parts $\phi^s(x, y)$ and $\phi^a(x, y)$ respectively so that $\phi(x, y) = \phi^s(x, y) + \phi^a(x, y)$, where

$$\phi^s(-x, y) = \phi^s(x, y), \phi^a(-x, y) = -\phi^a(x, y) \text{ so that } \phi_x^s(0, y) = 0, \phi_x^a(0, y) = 0, 0 < y < \infty. \quad (3.1)$$

Now $\phi^{s,a}(x, y)$ satisfy (2.1) to (2.6) together with (3.1). Thus, one may restrict the analysis to the region $x \geq 0$ only. Let the behavior of $\phi^{s,a}(x, y)$ for large x be represented by

$$\phi^{s,a}(x, y) \approx e^{-Ky} \left\{ e^{-i\mu(x-b)} + R^{s,a} e^{i\mu(x-b)} \right\} \text{ as } x \rightarrow \infty, \quad (3.2)$$

where $R^{s,a}$ are unknown constants. By using (2.7), we find that these constants are related to R and T by

$$R, T = \frac{1}{2}(R^s \pm R^a) e^{-2i\mu b}. \quad (3.3)$$

Now the expansions of $\phi^{s,a}(x, y)$ satisfying the equations (2.1) to (2.5) for $x \geq 0$ and (3.1) in the different regions for each configuration of the thick barriers are given below.

Region I ($x > b, 0 < y < \infty$) :

$$\phi^{s,a}(x, y) = e^{-Ky} \left\{ e^{-i\mu(x-b)} + R^{s,a} e^{i\mu(x-b)} \right\} + \int_0^\infty A^{s,a}(k) S(k, y) e^{-k_1(x-b)} dk \quad (3.4)$$

where $S(k, y) = k \cos(ky) - K \sin(ky)$ and $k_1 = \sqrt{\nu^2 + k^2}$

Region II ($0 < x < b, y \in \bar{L} = \bar{L}_j = (0, \infty) - L_j, j = 1, 2$) :

For $y \in \bar{L}_1 = (a, \infty)$, $\phi^{s,a}(x, y)$ are given by

$$\begin{pmatrix} \phi^s(x, y) \\ \phi^a(x, y) \end{pmatrix} = \begin{pmatrix} B_0^s \cosh \nu x \\ B_0^a \sinh \nu x \end{pmatrix} + \begin{pmatrix} \int_0^\infty B^s(k) \cosh(k_1 x) \cos k(y-a) dk \\ \int_0^\infty B^a(k) \sinh(k_1 x) \cos k(y-a) dk \end{pmatrix}. \quad (3.5)$$

For $y \in \bar{L}_2 = (0, c)$, $\phi^{s,a}(x, y)$ are given by

$$\begin{pmatrix} \phi^s(x, y) \\ \phi^a(x, y) \end{pmatrix} = \begin{pmatrix} C_0^s \cos \sqrt{(k_0^2 - \nu^2)x} \\ C_0^a \sin \sqrt{(k_0^2 - \nu^2)x} \end{pmatrix} \frac{\cosh k_0(c-y)}{\cosh(k_0 c)} + \sum_{n=1}^\infty \begin{pmatrix} C_n^s \cosh \sqrt{(k_n^2 + \nu^2)x} \\ C_n^a \sinh \sqrt{(k_n^2 + \nu^2)x} \end{pmatrix} \cos k_n(c-y) \quad (3.6)$$

where $\pm k_0, \pm i k_n (n = 1, 2, \dots)$ are the roots of the equation $k \tanh(kc) = K$. $B_0^{s,a}, C_0^{s,a} (n = 0, 1, \dots)$ are unknown constants and $A^{s,a}(k), B^{s,a}(k)$ are unknown functions such that the integrals in (3.4), (3.5), (3.6) and in the mathematical analysis below in which they appear, are convergent. Let us now define

$$\phi_x^{s,a}(b+, y) = f^{s,a}(y), 0 < y < \infty. \quad (3.7)$$

Then

$$f^{s,a}(y) = 0 \text{ for } y \in L = L_j (j = 1, 2) \text{ and } \phi_x^{s,a}(b-, y) = f^{s,a}(y) \text{ for } y \in \bar{L} = (0, \infty) - L. \quad (3.8)$$

Also, due to the edge condition described by (2.6), we must have $f^{s,a}(y) = O(|y - l_j|^{-\frac{1}{3}})$ as $y \rightarrow l_j (j = 1, 2)$. Use of appropriate expansions for $\phi^{s,a}(x, y)$ in Eqns.(3.4) followed by Havelock's inversion formula and after using the conditions (3.8), we obtain the relations between $R^{s,a}$ and $f^{s,a}(y)$ as given by

$$1 - R^{s,a} = \frac{2iK}{\mu} \int_{\bar{L}} f^{s,a}(u) e^{-Ku} du. \quad (3.9)$$

Also $B_0^{s,a}, C_n^{s,a} (n = 0, 1, \dots), A^{s,a}(k), B^{s,a}(k)$ can be obtained in terms of integrals involving $f^{s,a}(y)$. Now matching of $\phi^{s,a}(x, y)$ across the line $x = b$ through the right corner points of the gap gives rise to the relations $\phi^{s,a}(b+, y) = \phi^{s,a}(b-, y), y \in \bar{L}$ which ultimately produce the integral equations

$$\int_{\bar{L}} F^{s,a}(u) \mathcal{M}^{s,a}(y, u) du = e^{-Ky} + \lambda, \text{ where } \lambda \text{ is a real constant and } y \in \bar{L} \quad (3.10)$$

$$\text{where } F^{s,a}(y) = \frac{2}{\pi(1 + R^{s,a})} f^{s,a}(y), y \in \bar{L}. \quad (3.11)$$

For type 1 barrier, $y, u \in \bar{L} = \bar{L}_1 = (a, \infty)$, $\lambda = -\frac{1}{2}e^{-Ka}$ and $\mathcal{M}^s(y, u)$ is given by

$$\begin{aligned} \mathcal{M}^s(y, u) = & \int_0^\infty \frac{1}{k_1(k^2 + K^2)} (S(k, u)S(k, y) + K S(k, u) \sin(ka) - K^2 \coth(k_1 b) \cos k(u-a)) dk \\ & + \int_0^\infty \frac{\coth(k_1 b) \cos k(y-a) \cos k(u-a)}{k_1} dk. \end{aligned} \quad (3.12)$$

The expression of $\mathcal{M}^a(y, u)$ is obtained by replacing 'coth' by 'tanh' in the expression of $\mathcal{M}^s(y, u)$. For type 2 barrier, $y, u \in \bar{L} = \bar{L}_2 = (0, c)$, $\lambda = 0$ and $\mathcal{M}^s(y, u)$ is given by

$$\begin{aligned} \mathcal{M}^s(y, u) = & \int_0^\infty \frac{S(k, u)S(k, y)}{k_1(k^2 + K^2)} dk - \sum_{n=1}^\infty \frac{2\pi k_n \coth \sqrt{(k_n^2 + \nu^2)b}}{(2k_n c + \sin(2k_n c))\sqrt{k_n^2 + \nu^2}} \cos k_n(c - y) \cos k_n(c - u) \\ & - \frac{2\pi k_0 \cot \sqrt{(k_0^2 + \nu^2)b}}{(2k_0 c + \sin(2k_0 c))\sqrt{k_0^2 + \nu^2}} \cosh k_0(c - y) \cosh k_0(c - u). \end{aligned} \quad (3.13)$$

The expression of $\mathcal{M}^a(y, u)$ is obtained by replacing 'cot' and 'coth' by '-tan' and '-tanh' respectively in the expression of $\mathcal{M}^s(y, u)$. Now by using (3.11) in Eqns.(3.9), we find that

$$\int_{\bar{L}} F^{s,a}(y) e^{-Ky} dy = C^{s,a} = -\frac{i \cos \alpha}{\pi} \cdot \frac{1 - R^{s,a}}{1 + R^{s,a}}. \quad (3.15)$$

It is important to note that $F^{s,a}(y)$ and $C^{s,a}$ are all real quantities. Thus after solving the integral equation (3.10), $C^{s,a}$ are obtain from the relation (3.15) and these produce the actual reflection and transmission coefficients $|R|$ and $|T|$ given respectively by

$$|R| = \frac{|1 + (\pi \sec \alpha)^2 C^s C^a|}{\sqrt{\{1 + (\pi \sec \alpha C^s)^2\} \{1 + (\pi \sec \alpha C^a)^2\}}} \quad \text{and} \quad |T| = \frac{\pi \sec \alpha |C^s - C^a|}{\sqrt{\{1 + (\pi \sec \alpha C^s)^2\} \{1 + (\pi \sec \alpha C^a)^2\}}} \quad (3.17)$$

so that $|R|^2 + |T|^2 = 1$ which is the energy identity.

4. Multi-term Galerkin approximation

For the solution of Eqns.(3.10), we choose multi-term Galerkin approximation as

$$F^{s,a}(y) = \sum_{l=0}^N \alpha_l^{s,a} f_l(y), y \in \bar{L} \quad (4.1)$$

where $f_l(y)$ ($l = 0, 1, \dots, N$) are suitable basis functions. To find the unknown constants $\alpha_l^{s,a}$ ($l = 0, 1, \dots, N$) we substitute Eqns.(4.1) into the integral Eqns.(3.10) and multiply by $f_m(y)$ and integrate over \bar{L} to obtain the linear systems $\sum_{l=0}^N \alpha_l^{s,a} L_{lm}^{s,a} = G_m$, ($m = 0, 1, \dots, N$) where $L_{lm}^{s,a}, G_m$ ($m = 0, 1, \dots, N$) can be computed numerically and $C^{s,a}$ are obtained as $C^{s,a} = \sum_{l=0}^N \alpha_l^{s,a} G_l$.

For thick rectangular partially immersed barrier case $L = (0, a)$ so that $\bar{L} = (a, \infty)$ and $F^{s,a}(y)$ are chosen as

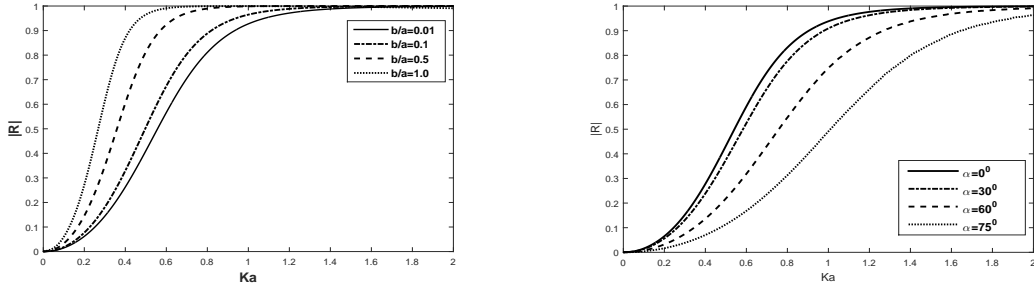
$$F^{s,a}(y) = \left(\frac{y}{a} - 1\right)^{-\frac{1}{3}} e^{-Ky} \sum_{l=0}^N \alpha_l^{s,a} \left(\frac{y}{a}\right)^l, a < y < \infty \quad (4.2)$$

while for thick rectangular submerged barrier $L = (c, \infty)$ so that $\bar{L} = (0, c)$ and $F^{s,a}(y)$ are chosen as

$$F^{s,a}(y) = \left(1 - \frac{y}{c}\right)^{-\frac{1}{3}} \sum_{l=0}^N \alpha_l^{s,a} \left(\frac{y}{c}\right)^l, 0 < y < c. \quad (4.3)$$

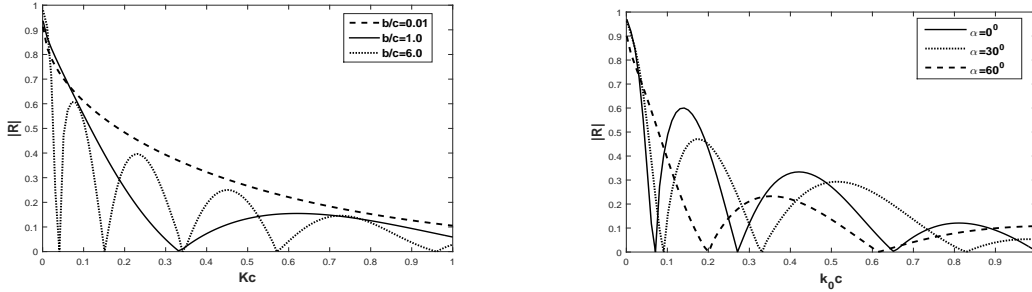
5. Numerical results

Due to the energy identity $|R|^2 + |T|^2 = 1$, we calculate only $|R|$ numerically. Here the numerical results are obtained by choosing $N = 2$ (three-term expansion) and these are displayed graphically against the wavenumber for different parameters. In Figure 1(a), $|R|$ is plotted against Ka for different values of $b/a = 0.01, 0.1, 0.5, 1.0$ and $\alpha = 0^\circ$. From this figure it is seen that for fixed values of α and b/a , the reflection coefficient increases as the wavenumber increases and asymptotically becomes unity for large wavenumber. This is because for large wavenumber the waves are confined within a thin layer below the surface and are totally reflected by the barrier. Also, it is observed from this figure that for a fixed value of the wavenumber Ka , $|R|$ increases as thickness of the barrier increases. Hence it is clear that $|R|$ crucially depends on both wavenumber and thickness of the barrier. The curve of $|R|$ in this figure corresponding to $b/a = 0.01$ almost coincides with the curve of $|R|$ in figure 2 of Ursell (1947) for thin partially immersed barrier. In this case the barrier can be assumed to be thin. Taking $\alpha = 0^\circ, 30^\circ, 60^\circ, 75^\circ$ and $b/a = 0.1$, $|R|$ is depicted against Ka in figure 1(b). It is observed from this figure that for fixed b/a , $|R|$ decreases as α increases.



(a) $|R|$ vs Ka for different values of b/a and $\alpha = 0^\circ$. (b) $|R|$ vs Ka for different values of α , and $b/a = 0.1$.

Figure 1: $|R|$ for thick partially immersed rectangular barrier



(a) $|R|$ vs Kc for different values of b/a , and $\alpha = 0^\circ$. (b) $|R|$ vs k_0c for different values of α , and $b/c = 4.43$.

Figure 2: $|R|$ for thick submerged rectangular barrier

In figure 2(a), $|R|$ is plotted against wavenumber Kc corresponding to $\alpha = 0^\circ$ and $b/c = 0.01, 1.0, 6.0$. It may be noted that, when barrier is comparatively thin ($b/c = 0.01$) and $\alpha = 0^\circ$, the curve of $|R|$ coincides with corresponding curve given by Dean (1945) for the case of deep water. It is clear from this figure that as the thickness increases, $|R|$ starts fluctuating and the fluctuations becomes rapid as the thickness of the barrier further increases. Hence $|R|$ crucially depends on wavenumber and thickness of the barrier. For large Kc , $|R|$ becomes zero asymptotically which is plausible.

The curve of $|R|$ display an oscillatory nature for both normal and oblique incidence of the surface wave which is shown in figure 2(b). In this figure, $|R|$ is plotted against k_0c for $b/c = 4.43$ and $\alpha = 0^\circ, 30^\circ, 60^\circ$. For $\alpha = 0^\circ$ and $b/c = 4.43$ (this is same as $l/H = 4.43$ in the notation of Mei and Black(1969)), almost the same graphical results as in Mei and Black (1969) are recovered. From this figure it is seen that the number of oscillations of $|R|$ decreases as the angle of incidence increases while oscillation of $|R|$ increases as the thickness of the barrier increases. Hence, thickness of the barrier and angle of incidence play opposite role in the context of oscillations(i.e, number of zeros) of the reflection coefficient.

6. Conclusion

The Galerkin technique employed here involves only three terms ($N = 2$) in the expansion, the basis functions being chosen as simple polynomials multiplied by appropriate weight functions. This may be regarded as some sort of novelty in this work.

Acknowledgement: This work is supported by UGC and SERB, India, through the research project no. EMR/2016/005315.

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