The boundary-path space of a directed graph

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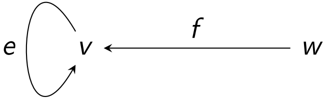
Definition

A *directed graph* E is a set E^0 of vertices and a set E^1 of directed edges, with direction determined by range and source maps $r, s : E^1 \to E^0$.

Example



$$E^0 = \{v, w\} \quad E^1 = \{e, f\}$$
$$s(e) = r(e) = r(f) = v \quad s(f) = w$$

Paths

- A sequence $\mu_1\mu_2\mu_3\ldots$ of edges is a *path* if $s(\mu_i) = r(\mu_{i+1})$ for all i.

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- $E^n = \{\mu : \mu \text{ is a path with } n \text{ (possibly } = \infty) \text{ edges}\}$
- $E^* = \{\mu : \mu \text{ has finitely many edges}\}$.
- For $V \subset E^0$ and $F \subset E^*$, define $VF := F \cap r^{-1}(V)$.
- In particular, for $v \in E^0$, $vF = F \cap r^{-1}(v)$.

Graph C^*-algebras

- $E^{\leq n} := \{\mu \in E^* : |\mu| = n, \text{ or } |\mu| < n \text{ and } s(\mu)E^1 = \emptyset\}$.
- The graph C^*-algebra $C^*(E)$ is universal for C^*-algebras containing a *Cuntz-Krieger E-family*: a family consisting of mutually orthogonal projections $\{s_v : v \in E^0\}$ and partial isometries $\{s_\mu : \mu \in E^*\}$ such that $\{s_\mu : \mu \in E^{\leq n}\}$ have mutually orthogonal ranges for each $n \in \mathbb{N}$, and such that
  1. $s_\mu^* s_\mu = s_{s(\mu)}$;
  2. $s_\mu s_\nu = s_{\mu\nu}$ when $s(\mu) = r(\nu)$;
  3. $s_\mu s_\mu^* \leq s_{r(\mu)}$; and
  4. $s_v = \sum_{\mu \in vE^{\leq n}} s_\mu s_\mu^*$ for every $v \in E^0$ and $n \in \mathbb{N}$ such that $|vE^{\leq n}| < \infty$.

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- Hence for each $\phi \in \Delta_D$, the elements of $\{\lambda : \phi(s_\lambda s_\lambda^*) = 1\}$ determine a path.

Boundary Paths

- The paths we get turn out to be all infinite paths, and all finite paths whose source is a *singular vertex*: elements $v \in E^0$ satisfying either
  - $vE^1 = \emptyset$, in which case we call v a *source*; or
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- The formula

$$h_E(x)(s_\mu s_\mu^*) = \begin{cases} 1 & \text{if } \mu \preceq x \\ 0 & \text{otherwise.} \end{cases}$$

uniquely determines a bijection from ∂E onto Δ_D [W].

Topology

- Following the approach of [PW], define $\alpha : E^* \cup E^\infty \to \{0,1\}^{E^*}$ by
$$\alpha(x)(\mu) = \begin{cases} 1 & \text{if } x = \mu\mu' \\ 0 & \textit{otherwise}. \end{cases}$$
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- For $\mu \in E^*$, define $\mathcal{Z}(\mu) := \{\mu\mu' \in E^* \cup E^\infty\}$.
- For $G \subset E^*$, we write $\mathcal{Z}(\mu \setminus G) := \mathcal{Z}(\mu) \setminus \bigcup_{\nu \in G} \mathcal{Z}(\nu)$.
- The *cylinder sets* $\{\mathcal{Z}(\mu \setminus G) : \mu \in E^*, G \subset s(\mu)E^1 \text{ is finite}\}$ are a basis for our topology. [W].

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- With this topology, $E^* \cup E^\infty$ is locally compact and Hausdorff [W].

Boundary Paths

- Fix a path $\mu \in E^*$ with $0 < |s(\mu)E^1| < \infty$.
- Then $\{\mu\} = \mathcal{Z}\big(\mu \setminus \{s(\mu)E^1\}\big)$ an open set.
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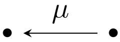
- So $\partial E = U^c$ is closed in $E^* \cup E^\infty$, and hence locally compact and Hausdorff.
- The map $h_E : \partial E \to \Delta_D$ is a homeomorphism [W].

Desingularisation

Drinen and Tomforde developed a construction they called *desingularisation* [DT]:

- Suppose E has some singular vertices. Fix $\mu \in \partial E \cap E^*$.
- If $|s(\mu)E^1| = 0$, then append on an infinite path:



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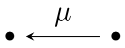

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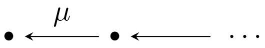

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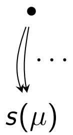
- If $|s(\mu)| = \infty$, then append an infinite path, and distribute the incoming edges along it:



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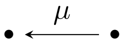

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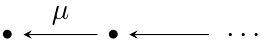

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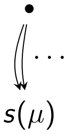

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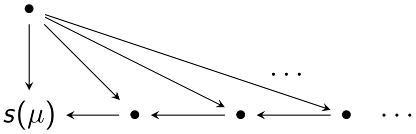

Desingularisation

- Let E be a directed graph, and F be a Drinen-Tomforde desingularisation of E.
- This gives a homeomorphism $\phi_\infty : E^0F^\infty \to \partial E$ [DT,W].
- Then there exists a full projection p and an isomorphism $\pi : C^*(E) \to pC^*(F)p$ [DT].

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- π induces a homeomorphism $\pi^* : \Delta_{pD_F p} \to \Delta_{D_E}$ [W].

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- These maps commute [W]:

$$\begin{array}{ccc} E^0 F^\infty & \xrightarrow{\ \phi_\infty\ } & \partial E \\ {\scriptstyle \eta}\big\downarrow & & \big\downarrow{\scriptstyle h_E} \\ \Delta_{pD_F p} & \xrightarrow[\ \pi^*\ ]{} & \Delta_{D_E} \end{array}$$

Where η is essentially the restriction of h_F to paths with ranges in E^0.

References

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[PW] A.L.T. Paterson and A.E. Welch, *Tychonoff's theorem for locally compact spaces and an elementary approach to the topology of path spaces*, Proc. Amer. Math. Soc. **133** (2005), 2761–2770.

[W] S.B.G. Webster, *The path space of a directed graph*, Proc. Amer. Math. Soc., to appear.

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