Simple Groups of Automorphisms of Trees

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Groups acting on trees are well-studied examples of tdlc groups.

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Local actions and universal groups

Definition

For $G < \mathrm{Aut}(\mathcal{T})$, the *local action* of G at any vertex $v \in V(\mathcal{T})$ is the permutation group formed by restricting $\mathrm{Stab}_G(v)$ to $E(v)$; the set of edges incident on v.

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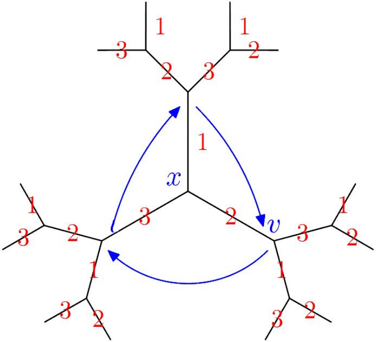

For $F < S_d$ the *universal group* $U(F)$ acts vertex-transitively on the d-regular tree with local action F.

The k-closure of G acting on $\mathcal{T}$

Definition

Suppose $G < \mathrm{Aut}(\mathcal{T})$ and $k \in \mathbb{N}$. Let d be the distance metric on $\mathcal{T}$ and let $B = B(v, k)$ be the closed ball of radius k centred at $v \in V(\mathcal{T})$.

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Proposition

$G^{(k)} = H^{(k)}$ iff G, H act with the same orbits and $\mathrm{Stab}_G v|_{B(v,k)} = \mathrm{Stab}_H v|_{B(v,k)}$ for vertices in each orbit.

Characterising and Calculating $G^{(k)}$

Let Γ be a finite graph with universal covering tree $\mathcal{T}$. Then a discrete group $G < \mathrm{Aut}(\mathcal{T})$ exists such that

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Example

The graphs $\Gamma = C(p, r, 1)$ introduced in [Gardiner & Praeger, 1994], with vertex set $\{(i, k) : i \in \mathbb{Z}_r, 1 \leq k \leq p\}$ and (i, k) adjacent to (j, l) iff $j = i \pm 1$. If $r > 4$ then $\mathrm{Stab}(v) \cong (S_{p-1} \times S_p^{r-1}) \rtimes \mathbb{Z}_2$.

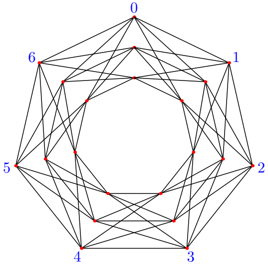

Figure: $C(3, 7, 1)$

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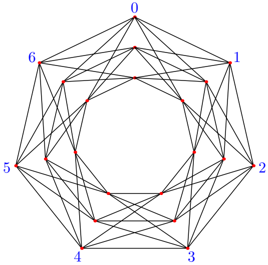
Γ is covered by $\mathcal{T}_{2p}$; G (and $G^{(k)}$) is vertex-transitive with local action $\cong S_p^2 \rtimes \mathbb{Z}_2$.



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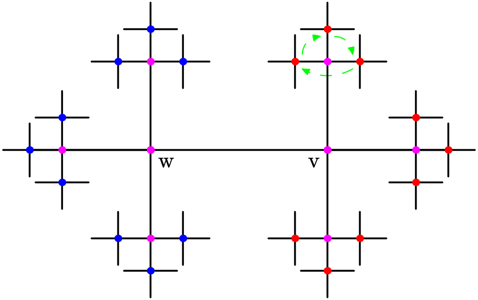
Independence Property (P^k)

Definition

Suppose $G < \mathrm{Aut}(\mathcal{T})$ and fix $k \in \mathbb{N}$. For any edge $\{v, w\}$, let $\mathcal{T}_{(v,w)}$ denote the semitree of $\mathcal{T}$ containing v but not $\{v, w\}$. Let $\mathcal{B} = B(v, k) \cap B(w, k)$ and denote $F := \mathrm{Fix}_G(\mathcal{B})$.



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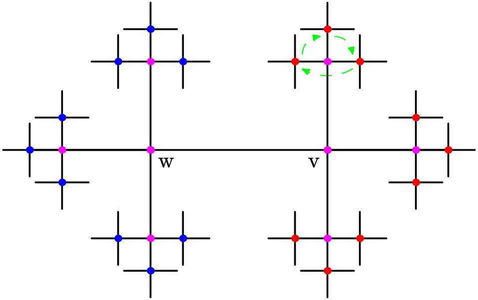

Properties of Property (P^k)

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- Property (P^k) implies Property (P^j) for all $j > k$.
- The sequence $\{G^{(k)} : k \in \mathbb{N}\}$ terminates at $G^{(k)} = \overline{G}$ iff G has Property (P^k).

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Theorem

Let $G < \mathrm{Aut}(\mathcal{T})$ be closed, fix $k \in \mathbb{N}$ and let $G^{(k)+}$ denote the group generated by automorphisms in $\mathrm{Fix}_G(\mathcal{B})$ for any edge of $\mathcal{T}$. Suppose that G satisfies Property (P^k) and does not stabilise a proper non-empty subtree or an end of $\mathcal{T}$.

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References and Further Information

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