Embedding Baumslag-Solitar groups into totally disconnected locally compact groups

Murray Elder, George Willis

THE UNIVERSITY OF
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AUSTRALIA

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Scale

van Dantzig: every totally disconnected locally compact (tdlc) group has a compact open subgroup.

For each $x \in$ G and each compact open subgroup V of G, $x^{-1}\mathsf{V}x \cap \mathsf{V}$ is open, so its cosets form an open cover of V.

Since V is compact, this means that $[\mathsf{V} : x^{-1}\mathsf{V}x \cap \mathsf{V}]$ is finite.

Define the **scale** of x to be $s(x) = \min\, [\mathsf{V} : x^{-1}\mathsf{V}x \cap \mathsf{V}]$. A subgroup V realising this minimum is called **minimising** for x.

Scale

The scale function $s : \mathsf{G} \to \mathbb{Z}^+$ enjoys the following properties:

- s is continuous
- $s(gxg^{-1}) = s(x)$
- if V is minimising for x then it is minimising for x^{-1}.

Commensurated subgroups

If H is a subgroup of G, and $x\mathsf{H}x^{-1} \cap \mathsf{H}$ is finite index in both $x\mathsf{H}x^{-1}$ and H, we say H is **commensurated by** G.

Eg

- SL$(n, \mathbb{Z})$ is commensurated by SL$(n, \mathbb{Q})$.
- $\langle a \rangle$ is commensurated by BS$(m, n) = \langle a, t \mid ta^m t^{-1} = a^n \rangle$

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Building a tdlc group

Let G be an abstract group with (commensurated) subgroup H.

Then G acts on G/H by permuting cosets, so G $\leq$ Sym(G/H).

For each $x \in$ Sym(G/H) and each finite subset F of G/H, put

$$\mathsf{N}(x, \mathsf{F}) = \{y \in \mathrm{Sym}(\mathsf{G}/\mathsf{H}) \mid y.(g\mathsf{H}) = x.(g\mathsf{H})\ \forall\ (g\mathsf{H}) \in \mathsf{F}\}.$$

These sets form a basis for a topology on Sym(G/H).

Building a tdlc group

If H has no subgroup that is normal in G, this topology is Hausdorf (and totally disconnected).

Take the **closure** of G in Sym(G/H) we obtain a tdlc group in which G embeds as a **dense** subgroup

(it is locally compact since H is commensurated).

Embedding BS(m,n) in a tdlc group

Applying this to BS(m, n) for $|m| \neq |n|$, with H $= \langle a \rangle$, we obtain a tdlc in which BS(m, n) is dense, which we call $\mathrm{G}_{m,n}$.

To see that we are getting (new) (interesting) (different) groups, we can try to compute the **scales** of elements.

Scales of $\mathsf{G}_{m,n}$

Thm (E, Willis): The set of scales for $\mathsf{G}_{m,n}$ for $m, n \neq 0, |m| \neq |n|$ is

$$\left\{ \left(\frac{\mathrm{lcm}(m,n)}{m} \right)^{\rho}, \left(\frac{\mathrm{lcm}(m,n)}{n} \right)^{\rho} \ : \ \rho \in \mathbb{N} \right\}$$

Thus, for every pair of relatively prime integers m, n we get a distinct tdlc group.

Computing scales

Since $s : \mathsf{G}_{m,n} \to \mathbb{Z}^+$ is continuous and $\mathsf{BS}(m,n)$ is dense in $\mathsf{G}_{m,n}$, scales of limit points cannot take different values to scales of elements in $\mathsf{BS}(m,n)$.

If V is a compact open subgroup of $\mathsf{G}_{m,n}$, put $\mathsf{U} = \mathsf{V} \cap \mathsf{BS}(m,n)$. The orbit of $g\mathsf{H}$ under V is the same as the its orbit under U, so $[\mathsf{U} : x^{-1}\mathsf{U}x \cap \mathsf{U}] = [\mathsf{V} : x^{-1}\mathsf{V}x \cap \mathsf{V}]$.

It follows that to compute scale we can work completely in $\mathsf{BS}(m,n)$ rather than $\mathsf{G}_{m,n}$.

Useful facts about BS(m,n)

A **pinch** is a subword of the form $ta^{mp}t^{-1}$ or $t^{-1}a^{np}t$.

Lemma X If $w = a^q t^{\pm 1} u$ is freely reduced and contains no pinches, then

$$w^{-1}\left\langle a^i \right\rangle w \cap \langle a \rangle = u^{-1}\left(t^{\mp 1}\left\langle a^i \right\rangle t^{\pm 1} \cap \langle a \rangle\right) u \cap \langle a \rangle\,.$$

BS(1,n)

Since $ta \to a^n t$ and $at^{-1} \to t^{-1}a^n$, any $x \in \mathrm{BS}(1,n)$ equals a word of the form $t^{-p}a^s t^q$ $(p, q \geq 0)$.

Since scale is invariant under conjugation, $s(x) = s(a^s t^{q-p})$.

If $q \geq p$, put $\rho = q - p$ (we call this the t-*exponent sum*).

Then $x^{-1}\langle a\rangle x = t^{-\rho}a^{-s}\langle a\rangle a^s t^\rho = t^{-\rho}\langle a\rangle t^\rho$ and $t^{-\rho}\langle a\rangle t^\rho \cap \langle a\rangle = \langle a\rangle$, which means $s(x) = 1$ and $\langle a\rangle$ is minimising for x.

BS(1,n)

Now suppose $x = a^s t^{q-p}$ with $q < p$. Put $\tau = p - q$.

Since $\langle a \rangle$ is minimising for x^{-1} it is minimising for x, so we compute $x^{-1}\langle a \rangle x \cap \langle a \rangle = t^{\tau}\langle a \rangle t^{-\tau} \cap \langle a \rangle = \langle a^{n^{\tau}} \rangle$

so the scale is $[\langle a \rangle : \langle a^{n^{\tau}} \rangle] = n^{\tau}$.

BS(m, n), $|m|, |n| \geq 2$

In this case we make use of an asymptotic formula of **Möller**: for *any* compact open subgroup V,

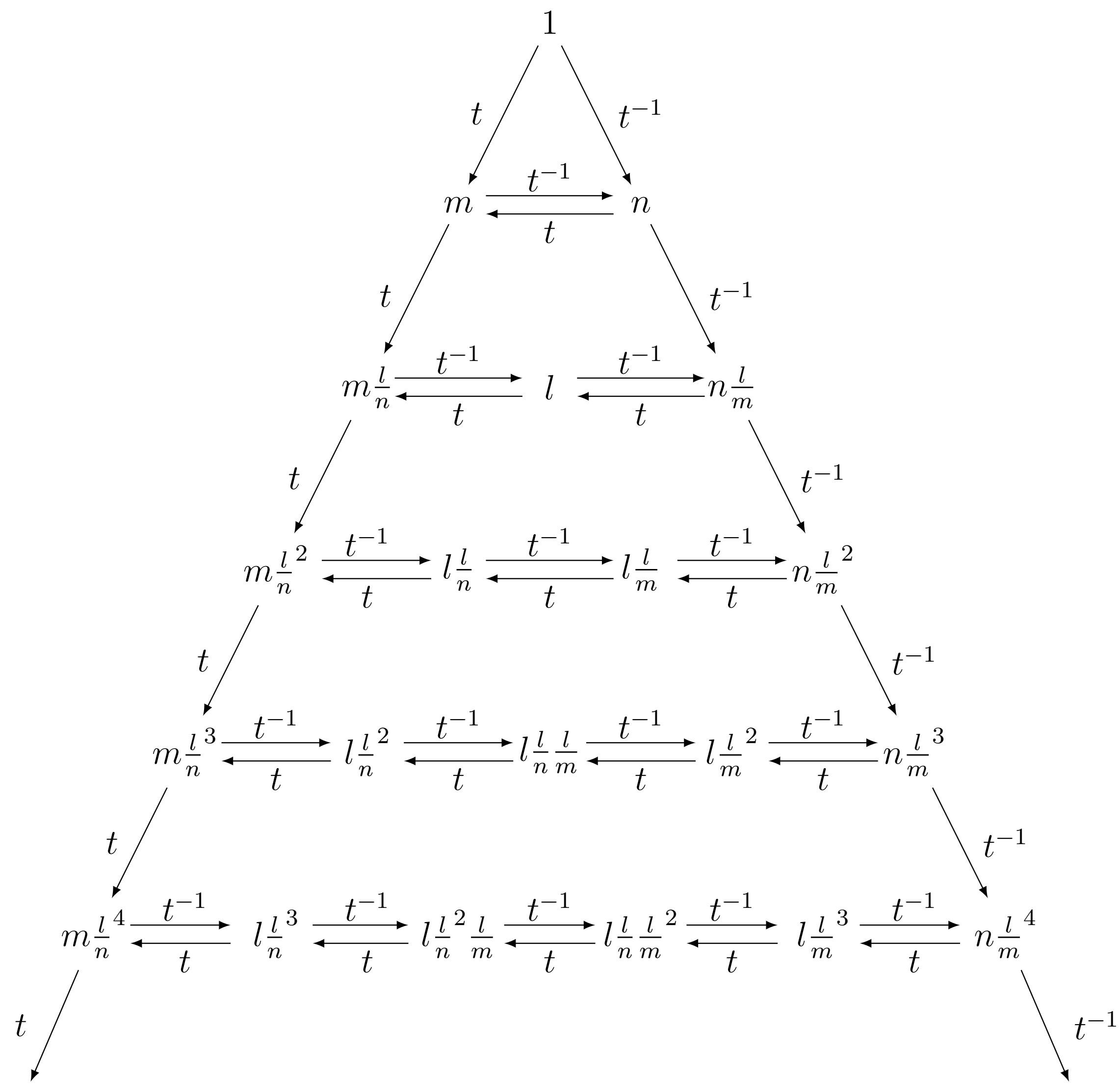
$$s(x) = \lim_{k \to \infty} \left[\mathsf{V} : x^{-k}\mathsf{V}x^k \cap \mathsf{V}\right]^{1/k}.$$

We might as well choose V to be the closure of $\langle a \rangle$.

BS(m, n), $|m|, |n| \geq 2$

To compute $x^{-k}\langle a\rangle x^k \cap \langle a\rangle$ we use Lemma X, and draw a **graph** of the computation as follows.

Put $p(x)$ = the path (or word in the free monoid over t, t^{-1}) tracing the computation for x, ρ = the t-exponent sum of x. and assume xx is freely reduced and contains no pinches (this can be arranged).



Facts about the graph

– Level i has i horizontal edges.

– Say $p(x)$ ends at position i on level L:

- if $\rho = 0$ then x^k stays in level L and ends at position i.

- if $\rho > 0$ and i is distance d from the left side, then x^k is distance d from the left side and on level $\mathsf{L} + k\rho$.

- if $\rho < 0$ and i is distance d from the right side, then x^k is distance d from the right side and on level $\mathsf{L} + k|\rho|$.

Computing scale

Using these facts and the formula of Möller we can compute the scale for any x:

(on board)

Thanks and References

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