Group actions on C^*-correspondences

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Outline

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1. C^*-correspondences
2. Group actions
3. Cuntz-Pimsner algebras

Right Hilbert C^*-modules

A Hilbert C^*-module is essentially a Hilbert space with the usual scalars (the complex numbers) replaced by an arbitrary C^*-algebra.

Definition

Let A be a C^*-algebra. A right Hilbert A-module is a Banach space X with pairing $\langle \cdot, \cdot \rangle : X \times X \to A$ (inner-product) and a right action $X \times A \to X$ (scalar multiplication) satisfying

- $\langle \cdot, \cdot \rangle$ $\mathbb{C}$*-linear in the second variable*
- $\langle x, y \cdot a \rangle = \langle x, y \rangle a$
- $\langle y, x \rangle = \langle x, y \rangle^*$
- $\langle x, x \rangle \geq 0$ *and* $\sqrt{\|\langle x, x \rangle\|_A} = \|x\|_X$

for all $x, y \in X$ and $a \in A$.

Adjointable and compact operators

Let X, Y be right Hilbert A-modules.

Definition

We say a linear operator $T : X \to Y$ is adjointable if there exists an operator $T^* : Y \to X$ such that

$$\langle Tx, y \rangle = \langle x, T^* y \rangle$$

for all $x \in X, y \in Y$.

We write $\mathcal{L}(X, Y)$ for the collection of all adjointable operators $T : X \to Y$.

$\mathcal{L}(X) := \mathcal{L}(X, X)$ is a C^*-algebra.

Adjointable and compact operators

For $x \in X, y \in Y$, define $\theta_{y,x} : X \to Y$ to be the operator satisfying

$$\theta_{y,x}(z) = y \cdot \langle x, z \rangle.$$

for all $z \in X$.

This is an adjointable operator with $(\theta_{y,x})^* = \theta_{x,y}$. We call

$$\mathcal{K}(X, Y) = \overline{\operatorname{span}}\{\theta_{y,x} : x \in X, y \in Y\}$$

the *compact* operators.

Then $\mathcal{K}(X) := \mathcal{K}(X, X)$ is a closed two-sided ideal in $\mathcal{L}(X)$ and $\mathcal{L}(X) = M(\mathcal{K}(X))$.

Definition

A C^*-correspondence is a right Hilbert A module X with a left action of A on X by adjointable operators, implemented by a homomorphism

$$\varphi_X : A \to \mathcal{L}(X).$$

We will write C^*-correspondences as pairs (X, A).

We write $a \cdot x$ for $\varphi_X(a)(x)$

Examples

Let D be a C^*-algebra. Then (D, D) is a C^*-correspondence with left and right actions given by multiplication and inner-product

$$\langle a, b \rangle = a^* b$$

Let $\alpha \in \mathrm{Aut}(D)$. There is a C^*-correspondence (D_α, D) with $D_\alpha = D$, right action and inner-product as above, and left action

$$a \cdot b = \alpha(a) b.$$

There are also examples arising from directed graphs and self-similar group actions.

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Morphisms between C^*-correspondences

Let (X, A) and (Y, B) be C^*-correspondences.

Definition

A morphism from (X, A) to (Y, B) is a pair of maps (ψ, π) where $\psi : X \to Y$ is linear and $\pi : A \to B$ is a C^*-homomorphism satisfying

- $\pi(\langle x, y\rangle) = \langle \psi(x), \psi(y)\rangle$
- $\pi(a) \cdot \psi(x) = \psi(a \cdot x)$ *and* $\psi(x) \cdot \pi(a) = \psi(x \cdot a)$
- $\varphi_Y(\pi(a)) = \psi^{(1)}(\varphi_X(a))$ *whenever* $a \in \varphi_X^{-1}(\mathcal{K}(X)) \cap \ker(\varphi_X)^{\perp}$, *where* $\psi^{(1)} : \mathcal{K}(X) \to \mathcal{K}(Y)$ *satisfies*

$$\psi^{(1)}(\theta_{x,y}) = \theta_{\psi(x),\psi(y)}$$

Let G be a locally compact group.

Definition

An action of G on a C^*-correspondence (X, A) is a pair (γ, α) where

- $\alpha : G \to \mathrm{Aut}(A)$ *is a continuous action of G on A*
- $\gamma : G \to \mathrm{Aut}(X)$ *is a continuous action of G on X; i.e. for any $s \in G, x \in X$ the map $s \mapsto \gamma_s(x)$ is continuous*
- *for each $s \in G$, the pair*

$$(\gamma_s, \alpha_s) : (X, A) \to (X, A)$$

is a C^*-correspondence morphism.

Crossed product C^*-algebras

Let G be a locally compact group and let

$$\alpha : G \to \mathrm{Aut}(A)$$

be a continuous action of G on A. We can define $*$-algebra structure on $C_c(G, A)$ as

$$(f * g)(s) = \int_G f(t)\alpha_t(g(t^{-1}s))d\mu(t)$$

$$f^*(s) = \Delta_G(s^{-1})\alpha_s(f(s^{-1})^*)$$

The (full) crossed product $A \rtimes_\alpha G$ is a C^*-completion of $C_c(G, A)$.

Crossed products are closely related to semi-direct products of groups: if a locally compact group H acts by automorphisms on another locally compact group N, then there is an induced action on the group C^*-algebra $C^*(N)$ and

$$C^*(N) \rtimes H \cong C^*(N \rtimes H).$$

Crossed product correspondence

Given $((X,A),G,(\gamma,\alpha))$ we can form the *crossed product C^*-correspondence* $(X \rtimes_\gamma G, A \rtimes_\alpha G)$ as follows:

Fix $f, g \in C_c(G,X)$, $a \in C_c(G,A)$ and $s \in G$.

$$\text{Inner-product}: \quad \langle f,g\rangle(s) = \int_G \alpha_{t^{-1}}\langle f(t), g(ts)\rangle d\mu(t)$$

$$\text{Right action}: \quad (f \cdot a)(s) = \int_G f(t)\alpha_t(a(t^{-1}s))d\mu(t)$$

$$\text{Left action}: \quad (a \cdot f)(s) = \int_G a(t)\gamma_t(f(t^{-1}s))d\mu(t)$$

Define $X \rtimes_\gamma G$ to be the completion of $C_c(G,X)$ with respect to the norm $\|f\| = \sqrt{\|\langle f,f\rangle\|}$.

Outline

Representations of C^*-correspondences

Definition

A representation of a C^*-correspondence (X, A) on a C^*-algebra D is a morphism $(\psi, \pi) : (X, A) \to (D, D)$.

Definition (Pimsner, 1997, Katsura, 2004)

The Cuntz-Pimsner algebra $\mathcal{O}_X$ is the universal C^*-algebra generated by a representation of (X, A). We denote the universal representation by $(k_X, k_A) : (X, A) \to \mathcal{O}_X$.

Example: $\mathcal{O}_{A_\alpha} = A \rtimes_\alpha \mathbb{Z}$

Crossed products

The Cuntz-Pimsner construction is functorial in the sense that given a morphism $(\psi, \pi) : (X, A) \to (Y, B)$ there is an induced C^*-homomorphism $\Psi : \mathcal{O}_X \to \mathcal{O}_Y$.

Therefore an action (γ, α) of G on (X, A) induces an action β of G on $\mathcal{O}_X$.

Theorem (Hao-Ng, 2008, Kaliszewski-Quigg-R, 2012)

Suppose G is amenable. Then there is an isomorphism

$$\mathcal{O}_{X \rtimes_\gamma G} \cong \mathcal{O}_X \rtimes_\beta G.$$

References

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