Common fixed point of generalized contraction in ordered metric spaces

H. Khandani[a], S.M. Vaezpour[b], B. Sims[c,*]

[a] *Dept. of Math., Science and Research Branch,*
Islamic Azad University(IAU), Tehran, Iran
khandani.hassan@yahoo.com

[b] *Dept. of Math., Amirkabir University of Technology, Tehran, Iran*
Vaez@aut.ac.ir

[b,c] *School of Math. and Phys. Sci., The University of Newcastle, NSW, Australia*
Brailey.Sims@newcastle.edu.au

Abstract

In this paper we give some theorems of generalized contractive mappings on ordered metric spaces and extend some results of Zhang Xian [Zhang. Xian, Common fixed point theorems for some new generalized contractive type mappings, Trans. Amer. Math. Soc. 266(1977)257-290] to ordered metric spaces and generalize a result of Agarwal, Ravi. P, El-gebeily, M. A. and D. O'Regan, donal [Agarwal, Ravi. P, El-gebeily, M. A. and D. O'Regan, donal(2008)Generalized contractions in partially ordered metric spaces, Applicable Analysis, 87:1, 109-116]. We also introduce some new type of contractive mappings on ordered metric spaces and prove some related results for them.

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1 Introduction

Banach contraction theorem has been extended in many directions. Ran and Reurings [4] extended Banach contraction theorem to ordered metric spaces

*Corresponding author. Fax: +61-249216898.

as follows.

Theorem 1.1. Let X be a partially ordered set, (X, d) be a complete metric spaces, and every $x, y \in X$ have upper and lower bound in X and the following conditions hold.

(i) $d(fx, fy) \leq ad(x, y)$ for each $x, y \in X$with $x \geq y$ where $0 \leq a < 1$,

(ii) there exists $x_0 \in X$ such that $x_0 \leq fx_0$ or $x_0 \geq fx_0$.

Then f has a unique fixed point.

Recall that if $(X, \leq)$ is a partially ordered set and $F : X \longrightarrow X$ is such that for each $x, y \in X$, $x \leq y$ implies $F(x) \leq F(y)$, then F is called non-decreasing. Very recently R. P. Agarwal, M. A. El-Gebeily and D. O'Regan [1] presented the following result for generalized contractive mappings.

Theorem 1.2. Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Assume there is a non decreasing function $\psi : [0, \infty) \longrightarrow [0, \infty)$ with $\lim_{n \longrightarrow \infty} \psi^n(t) = 0$ for each $t > 0$ and also suppose that f is a non-decreasing mapping with

(i) $d(fx, fy) \leq \psi(max\{d(x, y), d(x, fx), d(y, fy), \frac{d(fx,y)+d(x,fy)}{2}\}$, for each $x, y \in X$ with $x \geq y$. Also suppose that one of the following conditions hold:

(a) f is continuous,

(b) if $\{x_n\}$ is a non decreasing sequence with $x_n \longrightarrow x$ in X, then $x_n \leq x$ for all $n \in N$.
If there exists $x_0 \in X$ such that $x_0 \leq fx_0$, then f has a fixed point.

By removing the condition that ψ is non-decreasing R. P. Agarwal, M. A. El-Gebeily and D. O'Regan [1] presented the following result.

Theorem 1.3. Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Assume there is a continuous function $\psi : [0, \infty) \longrightarrow [0, \infty)$ with $\psi(t) < t$ for each $t > 0$ and also suppose that F is a non decreasing mapping and the following conditions hold:

(i) $d(fx, fy) \leq \psi(max\{d(x, y), d(x, fx), d(y, fy)\}$, for each $x, y \in X$ with $x \geq y$,

(ii) there exists $x_0 \in X$ such that $x_0 \leq fx_0$ and one of the following conditions hold,

(a) f is continuous,

(b) if $\{x_n\}$ is a non decreasing sequence with $x_n \longrightarrow x$ in X, then $x_n \leq x$ for all $n \in N$.

Then f has a fixed point.

Let $(X, \leq)$ be a partially ordered set and (X, d) be a metric space and $f, g : X \longrightarrow X$ two self mappings. Then we need the following notations in the sequel.

(i) by R we denote the real numbers, $R^+ = [0, +\infty)$, and N will denote natural numbers,

(ii) $m(x, y) = max\{d(x, y), d(x, fx), d(y, gy), \frac{d(fx,y)+d(x,gy)}{2}\}$with $x, y \in X$,

(iii) $X_{\leq} = \{(x, y) : x \leq y \quad or \quad y \leq x\}$,

(iv) the orbit of f at $x \in X$ is denoted by $O_f(x)$ and is defined by $O_f(x) := \{x, fx, f^2x, ...\}$.

(v) f is called a banach operator if there exists a nonnegative number $\alpha < 1$ such that $d(fx, f^2x) \leq \alpha d(x, fx)$ for each $x \in X$(see[2]).

(vi) if $U \subseteq X \times X$, then f is called orbitally U-continuous (see[3]) if:[$x \in X$ and $f^{n_i}x \longrightarrow a \in X$, as $i \longrightarrow \infty$ and $(f^{n_i}x, a) \in U$ for any $i \in N$] imply [$f^{n_i+1}x \longrightarrow fa \in X$, as $i \longrightarrow \infty$].

Let $\lambda \in [0, \infty)$ and $F : [0, \lambda) \longrightarrow R^+$ then $F \in \mathcal{F}[0, \lambda)$ iff F satisfy the following conditions.

(i) $F(0) = 0$ and $F(t) > 0$ for each $t \in (0, A)$,

(ii) F is non decreasing,

(iii) F is continuous.

Lemma 1.4. Let $\lambda \in [0, \infty)$ and $F \in \mathcal{F}[0, \lambda)$. If $\{\epsilon_n\}_{n=0}^{\infty} \subseteq [0, \lambda)$ and $limF(\epsilon_n) = 0$, then $lim_{n \longrightarrow \infty} \epsilon_n = 0$.
Proof. It is straightforward, see [5].
Let $\lambda \in [0, \infty)$ and $\varphi : [0, \lambda) \longrightarrow R^+$ then $g \in \mathcal{G}[0, \lambda)$ if and only if φ satisfy the following conditions.

(i) $\varphi(t) < t$ for each $t \in (0, \lambda)$,

(ii) φ is non decreasing and upper semi-continuous,

(iii) $\lim_{n \longrightarrow \infty} \varphi^n(t) = 0$ for each $t \in [0, \lambda)$.

Xian Zhang [5] proved the following common fixed point theorem for generalized contractive mappings and in this paper we extend it to ordered metric spaces.
Theorem 1.5. Let X be a complete metric space and let $d = sup\{d(x, y) :$

$x, y \in X\}$. Set $\lambda = d$ if $d = \infty$ and $\lambda > d$ if $d < \infty$. Suppose that $f, g : X \longrightarrow X$, $F \in \mathcal{F}[0, \lambda)$ and $\varphi \in \mathcal{G}[0, F(\lambda - 0))$ satisfy

$$F(d(x,y)) \leq \varphi(F(m(x,y))) \quad for \quad each \quad x, y \in X$$

Then f, g have a common fixed point in X. Moreover for each $x_0 \in X$, the iterated sequence $\{x_n\}$ with $x_{2n+1} = gx_{2n}$ and $x_{2n+2} = fx_{2n+1}$ converges to the common fixed point of f and g.

2 Main Results

Lemma 2.1. Let (X, d) be a metric space, $\lambda \in [0, +\infty)$, $F \in \mathcal{F}[0, \lambda)$, $\varphi : R^+ \longrightarrow R^+$ is non-decreasing function and $f, g : X \longrightarrow X$ such that $F(d(y, gy)) \leq \varphi(F(m(x, y)))$ where $x, y \in X$with $y = fx$. Then $F(d(y, gy)) \leq \varphi(F(d(x, y)))$ provided y is not a fixed point of g. A similar statement hold if g is replaced by f.
proof. Let $y = fx$ then

$$F(d(y,gy)) \leq \varphi(F(max\{d(x,y), d(x,y), d(y,gy), \frac{d(y,y)+d(x,gy)}{2}\}))$$

$$\leq \varphi(F(max\{d(x,y), d(y,gy), \frac{d(x,y)+d(y,gy)}{2}\}))$$
$$= \varphi(max\{d(x,y), d(y,gy)\}),$$

since y is not a fixed point of g hence, $max\{d(x,y), d(y,gy)\} = d(x,y)$ and the proof is follows. ∎

Now we extend the definition of orbitally U-continuous for two self mappings on an ordered metric space in the following way.
Definition 2.2. Let $(X, \leq)$ be a partially ordered space and (X, d) be a metric space and $U \subseteq X \times X$ and $x \in X$. Let $O_{f,g}(x) = \{x, gx, fgx, gfgx, ...\}$ and $O_{\{f,g\}}(x) = O_{f,g}(x) \cup O_{g,f}(x)$. Two function $f, g : X \longrightarrow X$ are called co-orbitally U-continuous if for each $x, a \in X$ and each subsequence $\{x_{n_i}\}$ of $O_{\{f,g\}}(x)$ such that $x_{n_i} \longrightarrow a \in X$, as $i \longrightarrow \infty$ and $(x_{n_i}, a) \in U$ then $fx_{n_i} \longrightarrow fa$ and $gx_{n_i} \longrightarrow ga$, as $i \longrightarrow \infty$. If $U = X$ then f and g are called co-orbitally continuous. f is orbitally $U-continuous$ if (f, f) are co-orbitally $U-continuous$.
Now we extend theorem 1 of [5] as follows.
Theorem 2.3. Let $(X, \leq)$ be a partially ordered space, (X, d) be a complete metric space and $d = sup\{d(x,y) : x, y \in X\}$. Set $\lambda = d$ if $d = \infty$ and $\lambda > d$ if $d < \infty$. Suppose that $F \in \mathcal{F}[0, \lambda)$ and $\varphi \in \mathcal{G}[0, F(\lambda - 0))$. Also let $f, g : X \longrightarrow X$ and $A = \{x \in X : (x, gx) \in X_{\leq}\}$, $B = \{x \in X : (x, fx) \in X_{\leq}\}$ satisfy the following conditions.

(i) $F(d(fx,gy)) \leq \varphi(F(m(x,y))$ for each $(x,y) \in X_{\leq}$,

(ii) $g(A) \subseteq B$ and $f(B) \subseteq A$,

(iii) $(x_0, fx_0) \in X_{\leq}$ or $(x_0, gx_0) \in X_{\leq}$,

(iv) f and g are $X_{\leq}$ co-orbitally continuous or,

(v) if $\{x_n\}_{n=0}^{\infty} \subseteq O_{\{f,g\}}(x)$ and $a \in X$ such that $x_n \longrightarrow a$ then $\{n : (x_n, a) \in X_{\leq}\}$ is infinite,

(vi) if $(x,y) \in X_{\leq}$ and $(y,z) \in X_{\leq}$ then $(x,z) \in X_{\leq}$.

Then f and g have a unique common fixed point.
Proof. It is evident that every fixed point of f is also a fixed point of g and conversely, let $n \in N$ then by lemma 2.1 we have,

$$F(d(x_{2n+1}, x_{2n})) = F(d(gx_{2n}, fx_{2n-1})) \leq \varphi(F(d(x_{2n}, x_{2n-1}))),$$

and

$$F(d(x_{2n+2}, x_{2n+1})) = F(d(fx_{2n+1}, gx_{2n})) \leq \varphi(F(d(x_{2n}, x_{2n+1}))).$$

So $F(d(x_{n+1}, x_n)) \leq \varphi(F(d(x_n, x_{n-1})))$ for each $n \geq 1$, and by induction we get $F(d(x_{n+1}, x_n)) \leq \varphi^n(F(d(x_0, x_1)))$. Hence $\lim_{n \longrightarrow \infty} F(d(x_{n+1}, x_n)) = 0$, and by lemma 1.5 we get

$$\lim_{n \longrightarrow \infty} d(x_{n+1}, x_n) = 0.$$

Suppose that $\{x_n\}$ is not Cauchy, then we can choose the subsequence $\{n_i\}_{i=0}^{\infty}$, $\{m_i\}_{i=0}^{\infty}$ and $\epsilon > 0$ such that $d(x_{n_i}, x_{m_i}) > 2\epsilon$. When i is large enough $d(x_{n_i}, x_{n_i+1}) < \frac{\epsilon}{2}$ and $d(x_{m_i}, x_{m_i+1}) < \frac{\epsilon}{2}$ and by the following inequalities

$$d(x_{n_i+1}, x_{m_i}) \geq d(x_{n_i}, x_{m_i}) - d(x_{n_i+1}, x_{n_i}) > \epsilon,$$

$$d(x_{n_i}, x_{m_i-1}) \geq d(x_{n_i}, x_{m_i}) - d(x_{m_i-1}, x_{m_i}) > \epsilon,$$

$$d(x_{n_i+1}, x_{m_i-1}) \geq d(x_{n_i}, x_{m_i}) - d(x_{m_i-1}, x_{m_i}) - d(x_{n_i+1}, x_{n_i}) > \epsilon,$$

we can assume that $n_i's$ are even numbers and $m_i's$ are odd numbers and $d(x_{n_i}, x_{m_i}) > \epsilon$ for all $i \in N$. Let k_i be the smallest positive number such that $d(x_{n_i}, x_{k_i}) > \epsilon$, so

$$\epsilon \leq d(x_{n_i}, x_{k_i}) \leq d(x_{n_i}, x_{k_i-2}) + d(x_{k_i-2}, x_{k_i}) \leq \epsilon + d(x_{k_i-2}, x_{k_i}),$$

and we get

$$\lim_{n \longrightarrow \infty} d(x_{n_i}, x_{k_i}) = \epsilon.$$

Also by

$$d(x_{n_i}, x_{k_i}) \leq d(x_{n_i}, x_{n_i-1}) + d(x_{n_i-1}, x_{k_i-1}) + d(x_{k_i-1}, x_{k_i}),$$

and

$$d(x_{k_i-1}, x_{n_i-1}) \leq d(x_{k_i-1}, x_{k_i}) + d(x_{k_i}, x_{n_i}) + d(x_{n_i}, x_{n_i-1}),$$

we get

$$\lim_{n \longrightarrow \infty} d(x_{n_i-1}, x_{k_i-1}) = \epsilon,$$

$$m(x_{n_i}, x_{k_i}) = max\{d(x_{n_i}, x_{k_i}), d(x_{n_i}, x_{n_i+1}), d(x_{k_i}, x_{k_i+1}), \frac{d(x_{n_i+1}, x_{k_i}) + d(x_{n_i}, x_{k_i+1})}{2}\}.$$

Since $d(x_{n_i}, x_{n_i+1}) \longrightarrow \epsilon$, so when i is large enough we have

$$m(x_{n_i}, x_{k_i}) \leq d(x_{n_i}, x_{k_i}) + \delta_i,$$

where $\delta_i \longrightarrow 0$ as $i \longrightarrow \infty$. But

$$F(d(x_{n_i}, x_{k_i})) \leq \varphi(F(d(fx_{n_i-1}, gx_{k_i-1}))), \tag{1}$$

and φ is upper semi-continuous from right and by letting $i \longrightarrow \infty$ in (1) we get

$$F(\epsilon) \leq \varphi(F(\epsilon)) < F(\epsilon).$$

This contradiction shows that $\{x_n\}$ is Cauchy and by completeness of (X, d) it converges to some $x \in X$. At first we assume that (iv) hold. Since $x_n \longrightarrow x$ and $gx_{2n} = x_{2n+1} \longrightarrow gx$ then $x = gx$ and by the first part of the proof, x is also a fixed point of f. Now assume that (v) hold, without loose of generality we can suppose that there is a subsequence $\{n_k\}$ of positive odd number such that $(x_{n_k}, x) \in X_{\leq}$ for each $k \in N$, so we have $F(d(fx_{n_k}, gx)) \leq \varphi(F(d(x_{n_k}, x))) \longrightarrow 0 \quad as \quad k \longrightarrow \infty$, and so $F(d(x_{n_k+1}, gx)) \longrightarrow 0 \quad as \quad k \longrightarrow \infty$, which implies $x = gx \quad and \quad hence \quad x = fx$. It is easy to see that the common fixed point of f and g must be unique. ■

Remark 2.4. In previous theorem if A, B is defined as $A = \{x \in X : x \leq gx\}$ and $B = \{x \in X : x \leq fx\}$ and instead of (iii) we assume that there is $x_0 \in X$ such that $x_0 \leq fx_0$ or $x_0 \leq gx_0$ the theorem is again hold, and in this setting the condition (vi) is no longer needed. The same statement is true when in this remark every $\leq$ is replaced by $\geq$.

Remark 2.5. It is just because of the term $\frac{d(fx,y)+d(x,gy)}{2}$ in $m(x, y)$ that we have to suppose that ψ is non-decreasing in theorem 2.4. So theorem 1.3 is an immediate consequence of 2.3.

Theorem 2.6. Let $(X, \leq)$ be a partially ordered space and (X, d) is complete metric space. Also let $f, g : X \longrightarrow X$, $\alpha : [0, \infty) \longrightarrow [0, 1)$ with $\limsup_{t \longrightarrow +r} \alpha(t) < 1$ for all $r \in R^+$, $A = \{x : (x, gx) \in X_{\leq}\}$, $B = \{x : (x, fx) \in X_{\leq}\}$ satisfy the following conditions:

(i) $d(gx, fgx) \leq \alpha(d(x, gx))d(x, gx)$ for each $(x, gx) \in X_{\leq}$,

(ii) $d(fx, gfx) \leq \alpha(d(x, fx))d(x, fx)$ for each $(x, fx) \in X_{\leq}$,

(iii) $g(A) \subseteq B$ and $f(B) \subseteq A$,

(iv) there is $x_0 \in X$ such that $(x_0 \leq fx_0)$ or $(x_0 \leq gx_0)$,

(v) f and g are co-orbitally continuous, or f and g are co-orbitally $X_{\leq}$ continuous and for each $x \in X$ and any sequence $\{x_n\} \subseteq O_{\{f,g\}}(x)$ such that $x_n \longrightarrow a$ there is a subsequence $\{x_{n_k}\}_{k\geq 0}$ of $\{x_n\}$ such that $(x_{n_k}, a) \in X_{\leq}$ for all $k \in N$,

(vi) if $(x, y) \in X_{\leq}$ and $(y, z) \in X_{\leq}$ then $(x, z) \in X_{\leq}$.

Then f and g have a common fixed point.
Proof. Let $x_0 \leq gx_0$, define $x_{2n+1} := gx_{2n}$ and $x_{2n+2} := fx_{2n+1}$ for each $n \in N$, by our assumption $x_{2n} \in A$, $x_{2n+1} \in B$ for all $n \in N$ and from $(iii), (vi)$ we have;

$$d(x_{2n}, gx_{2n}) \leq \alpha(d(x_{2n-1}, x_{2n}))d(x_{2n-1}, x_{2n}),$$

$$d(x_{2n+1}, fx_{2n+1}) \leq \alpha(d(x_{2n}, x_{2n+1}))d(x_{2n}, x_{2n+1}).$$

So for every $n \in N$,

$$d(x_n, x_{n+1}) \leq \alpha(d(x_{n-1}, x_n))d(x_{n-1}, x_n).$$

If we define $d_n := d(x_n, x_{n+1})$ for each $n \in N$ then $\{d_n\}_{n\geq 0}$ is a decreasing sequence in R^+ and it must converges to some point $d \in R^+$. If $x_n = x_{n+1}$ for some $n \in N$ then the proof is follows. Suppose that $x_n \neq x_{n+1}$ for all $n \in N$, Since $\limsup_{t\longrightarrow +d} \alpha(t) = r_0 < 1$ hence, we can choose $r_0 < r < 1$ and $m \in N$ such that $\alpha(d_n) < r$ for each $n \geq m$. We have $d_{m+1} \leq \alpha(d_m)d_m < rd_m$ and by induction we get $d_{m+k} \leq r^k d_m$ for each $k \in N$. Now we have

$$\sum_{n=0}^{+\infty} d(x_n, x_{n+1}) = \sum_{n=0}^{m} d(x_n, x_{n+1}) + \sum_{k=1}^{+\infty} d(x_{m+k}, x_{m+k+1})$$

$$\leq \sum_{n=0}^{m} d(x_n, x_{n+1}) + \sum_{k=1}^{\infty} r^k d_m < +\infty.$$

So $\{x_n\}_{n=0}^{\infty}$ is Cauchy and converges to some point $a \in X$. If f and g are co-orbitally continuous, the result is follows. Let f and g are co-orbitally $X_{\leq}$ continuous, from by (v) without loose of generality there is a subsequence $\{n_k\}$ of positive even number such that $(x_{n_k}, a) \in X_{\leq}$ for each $n \in N$, but $x_{n_k+1} = gx_{n_k} \longrightarrow ga$ and we get $a = ga$. It is easy to see that every fixed point of g is also a fixed point f and conversely, and the proof is complete. ■

Remark 2.7. In previous theorem if A, B is defined as $A = \{x \in X : x \leq gx\}$and $B = \{x \in X : x \leq fx\}$ and instead of (iv) we assume that there is $x_0 \in X$ such that $x_0 \leq fx_0$ or $x_0 \leq gx_0$ the theorem is again hold, and in this way the condition (vi) is no longer needed. The same statement is true when in this remark every $\leq$ is replaced by $\geq$.
We extend the definition of Banach operators to ordered metric spaces as the following.
Definition 2.8. Let $(X, \leq)$ be a partially ordered space and (X, d) be metric space, $f : X \longrightarrow X$ is called an ordered Banach operator if satisfies the following contraction,

$$d(fx, f^2x) \leq \alpha d(x, fx)$$

for each $(x, fx) \in X_{\leq}$.

Take $f = g$ in theorem 2.6, the following corollary is resulted.
Corollary 2.9. Let $(X, \leq)$ be a partially orderede space and (X, d) be complete metric space. Also let $f : X \longrightarrow X$, $\alpha : [0, \infty) \longrightarrow [0, 1)$ with $\limsup_{t \longrightarrow +r} \alpha(t) < 1$ for all $r \in R^+$, $A = \{x \leq fx\}$ satisfy the following conditions.

(i) $d(fx, f^2x) \leq \alpha(d(x, fx))d(x, fx)$ for each $(x, fx) \in X_{\leq}$,

(ii) $f(A) \subseteq A$,

(iii) there is $x_0 \in X$ such that $x_0 \leq fx_0$,

(iv) f is orbitally continuous, or f is orbitally $X_{\leq}$ continuous and for each $x \in X$ and any sequence $\{x_n\} \subseteq O_f(x)$ such that $x_n \longrightarrow a$ there is a subsequence $\{x_{n_k}\}_{k \geq 0}$ of $\{x_n\}$ such that $(x_{n_k}, a) \in X_{\leq}$ for all $k \in N$.

Then f has a fixed point.

The following examples shows that some contractive functions that have been discussed in this paper are not the same.

Let $(X, \leq)$ be a partially ordered set, (X, d) a metric space, $k \in (0, 1)$, and consider the following conditions:

(i) $d(fx, fy) \leq kd(x, y)$ for each $(x, y) \in X_{\leq}$,

(ii) $d(fx, f^2x) \leq kd(x, fx)$ for each $(x, fx) \in X_{\leq}$,

(iii) $d(y, fy) \leq kd(x, y)$ for each $(x, y) \in X_{\leq}$ with $x \neq y$.

It is easy to see that every contraction of type (i) is also a contraction of type (ii) and every contraction of type (iii) is a contraction of type (ii) . We show that there is some contraction of type (ii) such that it is not a contraction of type (i) or (ii).

Example 2.10. Let $(X = [0, +\infty), \leq)$ be ordered metric space with the usual order of R, define:

$$fx = 0 \quad if \quad x = 0 \quad otherwise \quad fx = -1.$$

$x \leq fx$ iff $x = 0$. $d(fx, f^2x) \leq (1/2)d(x, fx)$ for each $(x, fx) \in X_{\leq}$. But with $x = 0$ and $y = 1$, f does not satisfy in (i) and (ii). Notice that $A = \{x : x \leq fx\} = \{0\}$ and f is orbitally continuous and $f(A) \subseteq A$ and f satisfies all conditions of remark 2.7 and its fixed point is $x = 0$. So corollary 2.9 is a genuine generalization of theorem 1.1.

References

[1] R. P. Agarwal, M. A. El-gebeily, D. O'Regan *Generalized contractions in partially ordered metric spaces*, Applicable Analysis,87(1)(2008) 109-116.

[2] U. C. Gairola and A. S. Rawat, *A fixed point theorem for integral type inequality*, Int. Journal of Math. Analysis, 2(15)(2008) 709-712.

[3] D. O'Regan and A. Petrusel, *fixed point theorem for generalized contractions in ordered metric spaces*, J. Math. Anal. Appl. 341(2008) 1241-1252.

[4] A.C.M. Ran and M.C. Reurings, *A fixed point theorem in ordered sets and some application to matrixs equations*, Proc Am. Math. Soc. 132(2004) 1435-1443.

[5] Zhang Xian, *Common fixed point theorems for some new generalized contractive type mappings*, Trans. Amer. Math. Soc. 266(1977) 257-290.