For positive integers $k$ and $n$, let $S_k(n)$ be the sum of the $k$th powers of the first $n$ positive integers, i.e. $S_k(n) = \sum_{i=1}^{n} i^k$. When $n$ is understood, we may shorten $S_k(n)$ to $S_k$. Everybody knows the formula

$$S_1(n) = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

(discovered by Gauss as a school child). Also familiar to students learning mathematical induction are the formulae:

$$S_2(n) = \sum_{i}^{n} i^2 = \frac{n(n+1)(2n+1)}{6},$$

$$S_3(n) = \sum_{i}^{n} i^3 = \frac{n^2(n+1)^2}{4}.$$ 

These $S_k$ are interesting polynomials in $n$ and it is easy to prove (cf. Barnard and Child, 1936, ch. 8) that $S_k$ is of degree $k + 1$. In what follows, we give some historical background on the polynomials, and then examine a number of identities linking the $S_k$. They can all be considered generalizations of the observation that

$$S_3 = S_1^2.$$ 

Historical Background

The polynomials $S_1$, $S_2$ and $S_3$ were known to early Greek and Hindu mathematicians – in fact, Archimedes would have used $S_2$ to calculate the volume of the sphere and the cone. And apparently $S_4$ was known to Arab mathematicians prior to the year 1400. Rules equivalent to the polynomials $S_1$ and $S_2$ are found in the work of the Chinese mathematician Yang Hui around the year 1275.

If one assumes $S_k$ is a polynomial of degree $k + 1$, then by substituting 1, 2, ..., $k + 2$ successively one gets a system of $k + 2$ linear equations in the $k + 2$
coefficients of $S_k$. These may be solved to get $S_k$. In Table 1 are listed $S_k$ for $k \leq 7$ along with a general formula.

**Table 1**

\[
\begin{align*}
S_1 &= \frac{1}{2}n^2 + \frac{1}{2}n \\
S_2 &= -\frac{1}{4}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n \\
S_3 &= \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2 \\
S_4 &= \frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n \\
S_5 &= \frac{1}{6}n^6 + \frac{1}{2}n^5 + \frac{5}{12}n^4 - \frac{1}{12}n^2 \\
S_6 &= \frac{1}{7}n^7 + \frac{1}{2}n^6 + \frac{1}{2}n^5 - \frac{1}{6}n^3 + \frac{1}{42}n \\
S_7 &= \frac{1}{8}n^8 + \frac{1}{2}n^7 + \frac{7}{12}n^6 - \frac{7}{24}n^4 + \frac{1}{12}n^2 \\
S_k &= \frac{1}{k+1} \sum_{i=0}^{k} \binom{k+1}{i} b_i n^{k+1-i}
\end{align*}
\]

where $b_i$ are the Bernoulli numbers: $b_0 = 1$, $b_1 = \frac{1}{2}$, $b_3 = \frac{1}{6}$, $b_3 = 0$, $b_4 = \frac{1}{30}$, $b_5 = 0$, $b_6 = \frac{1}{42}$, $b_7 = 0$, $b_8 = -\frac{1}{30}$, \ldots. For odd $i > 4$, $b_i = 0$.

A modification of these polynomials yields the Bernoulli polynomials, about which much has been written.

A pattern becomes apparent when we write the polynomials in factored form as shown in Table 2. $S_k$ is divisible by $n^2(n+1)^2$ when $n$ is odd ($n > 1$), and divisible by $n(n+1)(2n+1)$ when $n$ is even.

Fermat in 1636 discovered a recurrence relation for finding $S_k$ in terms of $S_{k-1}$, $S_{k-2}$, etc., but it was not convenient for use for large $k$. By 1654, Pascal had derived the recurrence relation

\[(n+1)^{k+1} - (n+1) = \binom{k+1}{1} S_1 + \binom{k+1}{2} S_2 + \cdots + \binom{k+1}{k} S_k\]

which is easier to use. For example, letting $k = 4$ we solve

\[(n+1)^4 - (n+1) = 5\frac{n(n+1)}{2} + 10\frac{n(n+1)(2n+1)}{6} + 10\frac{n^2(n+1)^2}{4} + 5S_4\]

to find $S_4$. This, however, is still a tedious process.
Both Fermat and Pascal were unaware of the work of a lesser known mathematician, Johann Faulhaber who prior to 1631 had published the formulae up to $S_{17}$, and, more importantly, had observed that $S_k$ is a polynomial in $n^2 + n$ if $k$ is odd or $(2n + 1) \times (a \text{ polynomial in } n^2 + n)$ if $k$ is even. See Table 3.

**Table 3**

<table>
<thead>
<tr>
<th>$S_k$</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_4$</td>
<td>$\frac{1}{30} (2n + 1)[3(n^2 + n)^2 - (n^2 + n)]$</td>
</tr>
<tr>
<td>$S_5$</td>
<td>$\frac{1}{12} [2(n^2 + n)^3 - (n^2 + n)^2]$</td>
</tr>
<tr>
<td>$S_6$</td>
<td>$\frac{1}{42} (2n + 1)[3(n^2 + n)^3 - 3(n^2 + n)^2 + (n^2 + n)]$</td>
</tr>
<tr>
<td>$S_7$</td>
<td>$\frac{1}{24} [3(n^2 + n)^4 - 4(n^2 + n)^3 + 2(n^2 + n)^2]$</td>
</tr>
<tr>
<td>$S_k$</td>
<td>$P(n^2 + n)$ if $k$ is odd, $2n + 1 P(n^2 + n)$ if $k$ is even</td>
</tr>
</tbody>
</table>

The $S_k$ were made well-known by Jacques Bernoulli through his book *Ars Conjectandi*, published in 1713, 8 years after his death. In this were included the formulae up to $S_{10}$ (including an error in $S_6$), but more importantly, he showed another way of calculating the numbers $b_k$ which appear in the coefficients. He gave what we now call the exponential generating function for the $b_k$, namely

$$\frac{x}{1 - e^{-x}} = \sum_{k=0}^{\infty} b_k \frac{x^k}{k!}.$$

These numbers were called by Euler the "Bernoulli numbers". Jacobi, in 1834, rediscovered the polynomials in the form given in Table 3.
Finally, it is sometimes more useful to have the $S_k$ expressed as polynomials in $n+1$. These can be found easily from the relation $S_k(n) = S_k(n+1) - (n+1)^k$. For example,

$$S_2 = \frac{1}{3}(n+1)^3 - \frac{1}{2}(n+1)^2 + \frac{1}{6}(n+1).$$

For more information, consult Edwards (1982).

**Identities among the $S_k$**

As a student I noticed that

$$S_1^2 = S_3$$

and wondered if there were other similar relationships among the $S_k$. There are many ways in which one may try to generalize (1):

(i) Can $S_2^2$, $S_3^2$, ... be expressed in terms of $S_k$, perhaps as linear combinations?

(ii) Turning (1) around, can $S_k$ for higher $k$ be simply expressed in terms of some $S_i$ with $i < k$?

(iii) Are there simple expressions for $S_1^3$, $S_1^4$, ...?

(iv) Instead of $S_1 \cdot S_1$, find a linear expression for $S_k \cdot S_m$ in terms of $S_i$'s.

(v) Other generalizations involving $S_k$.

(vi) Answer (i)-(v) for the sums of powers of the first $n$ odd numbers, or, indeed, any arithmetic progression.

One can easily pursue method (i) by using mathematical induction – even without knowing the expression one is trying to prove! For example, in order to discover the expression for $S_2^2$, we go directly to the inductive step and look at $(S_2(n+1))^2$. Expressing $(S_2(n+1))^2$ as $(S_2(n) + (n + 1))^2$ and expanding, we get

$$(S_2(n+1))^2 = S_2(n)^2 + 2(n+1)^2S_2(n) + (n+1)^4$$

$$= S_2(n)^2 + 2(n+1)^2\left[\frac{1}{3}(n+1)^3 - \frac{1}{2}(n+1)^2 + \frac{1}{6}(n+1)\right]$$

$$+ (n+1)^4$$

$$= S_2(n)^2 + \frac{2}{3}(n+1)^5 + \frac{1}{3}(n+1)^3.$$ 

Now we can see what the inductive hypothesis must be – assume that

$$S_2(n)^2 = \frac{2}{3}S_5(n) + \frac{1}{3}S_3(n),$$

which is easily seen to hold at $n = 1$. We conclude
Notice that we did not need to know the formulae for $S_5$ and $S_3$ in order to find this expression for $S_2^2$. Similarly we get the pretty expression

$$S_3^2 = \frac{1}{2} S_7 + \frac{1}{2} S_6.$$  

This method can be applied equally well to find, for any $i$ and $j$, the product $S_i S_j$ expressed as a linear combination of the $S_k$. Thus we are able to carry out the generalizations (i) and (iv).

The relation $S_1^2 = S_3$ was known to the Arabs in the Middle Ages, but is apparently first found in the work of the Hindu mathematician Alkarkhi around the year A.D. 1000. The formula for $S_3^2$ was known to Jacobi before 1851. However, it seems likely that at least $S_2^2$ and $S_3^2$ were known earlier than this. The method used by the Hindus to discover the expression for $S_1^2$ is worth studying because it can be used equally well to calculate any product $S_i S_j$. The method, called the "Indian Method" by French number theorist Édouard Lucas, depends on grouping the $n^2$ products in the expansion of $(1 + 2 + \ldots + n)(1 + 2 + \ldots + n)$ as illustrated in Figure 1.

\begin{figure}
\begin{tabular}{cccc}
1.1 & 1.2 & 1.3 & 1.4 & \ldots & 1, n \\
2.1 & 2.2 & 2.3 & 2.4 & \ldots & 2, n \\
3.1 & 3.2 & 3.3 & 3.4 & \ldots & 3, n \\
4.1 & 4.2 & 4.3 & 4.4 & \ldots & 4, n \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
n.1 & n.2 & n.3 & n.4 & \ldots & n, n \\
\end{tabular}
\caption{Figure 1}
\end{figure}

It is easily seen that the sum of the $k$th group is $2kS_1(k) - k^2$. Summing this expression from $k = 1$ to $k = n$, being careful not to count the diagonal
terms twice, gives:

\[ S_1^2 = 2 \left( \sum_{k=1}^{n} kS_1(k) \right) - S_2 \]
\[ = 2 \left( \sum_{k=1}^{n} \frac{k^2(k+1)}{2} \right) - S_2 \]
\[ = (S_3 + S_2) - S_2 \]
\[ = S_3. \]

Notice again that this method needs only the formula for \( S_1 \), not those for \( S_2 \) and \( S_3 \). Listed in Table 4 are results for \( S_i^2 \) for \( i \) up to 6, and also the general formula. The resemblance of the general case to the expression for \( S_k \) given in Table 1 is surprising.

**Table 4**

<table>
<thead>
<tr>
<th>( S_i^2 )</th>
<th>( S_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( S_1^2 )</td>
<td>( S_3 )</td>
</tr>
<tr>
<td>( S_2^2 )</td>
<td>( \frac{2}{3} S_6 + \frac{2}{6} S_3 )</td>
</tr>
<tr>
<td>( S_3^2 )</td>
<td>( \frac{2}{4} S_7 + \frac{3}{6} S_5 )</td>
</tr>
<tr>
<td>( S_4^2 )</td>
<td>( \frac{2}{5} S_9 + \frac{4}{6} S_7 - \frac{1}{15} S_5 )</td>
</tr>
<tr>
<td>( S_5^2 )</td>
<td>( \frac{2}{6} S_{11} + \frac{5}{6} S_9 - \frac{1}{6} S_7 )</td>
</tr>
<tr>
<td>( S_6^2 )</td>
<td>( \frac{2}{7} S_{13} + S_{11} - \frac{1}{3} S_9 + \frac{1}{21} S_7 )</td>
</tr>
<tr>
<td>( S_k^2 )</td>
<td>( \frac{2}{k+1} \sum_{0 \leq 2i \leq k+1} b_{2i} \binom{k+1}{2i} S_{2k+1-2i} )</td>
</tr>
</tbody>
</table>

This accomplishes the generalization (i), but going to the more general case \( S_i S_j \), we get the following examples, among others:

\[ S_1 S_2 = \frac{5}{6} S_4 + \frac{1}{6} S_2 \]
\[ S_2 S_4 = \frac{8}{15} S_7 + \frac{1}{2} S_5 - \frac{1}{30} S_3 \]
\[ S_3 S_5 = \frac{5}{12} S_9 + \frac{2}{3} S_7 - \frac{1}{12} S_5 \]

The general situation can be stated as a theorem whose proof is really just an application of the Indian method.
Theorem 1

\[ S_k S_m = \frac{1}{m+1} \sum_i \binom{m+1}{2i} b_{2i} S_{k+m+1-2i} + \frac{1}{k+1} \sum_i \binom{k+1}{2i} b_{2i} S_{k+m+1-2i} \]

The special case in which \( k = m \) produced the results in Table 4. Theorem 1 was known to Lucas by 1891 (see Lucas, 1891, Ch. xiv).

We look next at generalization (iii). Computing \( S_1^2, S_1^3, S_1^4, \) etc. with the help of Theorem 1 allows us to easily produce the results listed in Table 5. A very pretty pattern can be observed.

**Table 5**

\[
\begin{align*}
S_1^2 &= \frac{1}{2} (2S_3) \\
S_1^3 &= \frac{1}{4} (3S_6 + S_3) \\
S_1^4 &= \frac{1}{8} (4S_9 + 4S_3) \\
S_1^5 &= \frac{1}{16} (5S_9 + 10S_7 + S_3) \\
S_1^6 &= \frac{1}{32} (6S_{11} + 20S_9 + 6S_7) \\
S_1^7 &= \frac{1}{64} (7S_{13} + 35S_{11} + 21S_9 + S_7)
\end{align*}
\]

These are instances of the following theorem which has appeared in a paper by Piza in 1952, but which was known as far back as 1877 (Lampe) and 1878 (Stern).

**Theorem 2**

\[ S_1^k = \frac{1}{2^{k-1}} \sum_i \binom{k}{2i-1} S_{2k+1-2i} \]

the sum being over those \( i \) for which \( 0 \leq 2i \leq k+1 \). This is another good example of the many facts concerning the sums of powers that have been rediscovered again and again by people unfamiliar with the literature (and accepted for publication by referees or editors equally unfamiliar with the literature!).

Having successfully achieved generalization (iii), we are now able to generalize in the direction (ii) simply by solving the equations in Table 5.
recursively. This, of course, only works for \(S_k\) when \(k\) is odd. For example:

\[
S_6 = \frac{1}{3}(4S_1^3 - S_1^2) \\
S_7 = \frac{1}{3}(2S_1^4 - 4S_1^3 + S_1^2) \\
S_9 = \frac{1}{5}(16S_1^5 - 20S_1^4 + 52S_1^3 - 3S_1^2)
\]

Actually, these can be obtained directly from Table 3, although the general formula is not known. This, in fact, is precisely what Faulhaber observed in 1631.

Next we describe some other ways to generalize the identity (1) which do not seem to have been published before (although this is a risky claim to make). These are much less obvious, but come about by regarding (1) as

\[
\binom{2}{1} S_1 S_1 = 2S_3.
\]

We consider all products \(S_k S_m\) of the same degree, i.e. where \(k + m\) is some constant, for example \(S_2^2\), \(S_2 S_4\) and \(S_1 S_5\) (see Tables 5 and 6). Multiplying by the corresponding binomial coefficient and taking the alternating sum, we find

\[
\binom{6}{1} S_1 S_5 = \binom{6}{2} S_2 S_4 + \binom{6}{3} S_3^2 - \binom{6}{4} S_4 S_2 + \binom{6}{5} S_5 S_1 = 2S_7.
\]

This is an instance of the following identity,

**Theorem 3**

If \(k\) is even, then

\[
\sum_{i=1}^{k-1} (-1)^{i-1} \binom{k}{i} S_i S_{k-i} = 2S_{k+1}.
\]

The proof of this depends on the standard binomial coefficient identity

\[
\sum_{k=m}^{n} (-1)^k \binom{n}{k} \binom{k}{m} = 0, \quad m \neq n.
\]

If \(k\) is odd, then the terms in the sum in Theorem 3 all cancel out and no result is obtained. Another identity of this kind is obtained by including the coefficient \(i\) in each term. This result is easily derived from Theorem 3.
Theorem 4

If $k$ is even, then
\[ \sum_{i=1}^{k-1} (-1)^{i-1} \binom{k}{i} S_i S_{k-i} = k S_{k+1}. \]

As mentioned above, these last two identities are apparently new, and there may well be other identities of this kind, using different coefficients.

Finally, turning to generalization (vi), we consider other arithmetic progressions starting at 1, such as 1, 3, 5, 7, \ldots or 1, 4, 7, 10, \ldots. Quite clearly, the Indian method can still be used, so that an analogue of Theorem 1 exists. In the case of sums of powers of odd numbers, let

\[ T_k(n) = \sum_{i=1}^{n} (2i - 1)^k. \]

We find, for example:

\[ T_1 T_2 = \frac{5}{12} T_4 + \frac{7}{12} T_2 \]

\[ T_2 T_3 = \frac{7}{24} T_6 + \frac{5}{6} T_4 - \frac{1}{8} T_2 \]

\[ T_4^2 = \frac{1}{5} T_9 + \frac{4}{3} T_7 - \frac{8}{15} T_5 \]

The analogue of Theorem 2 is particularly nice:

Theorem 5

\[ T_1^k = \frac{1}{2^k - 1} \sum_{i=1}^{k} \binom{2k}{2i} T_{2^{k+1-2i}}. \]

Of course, the similarity between the results of Theorem 5 and Theorem 2 suggest there should be similar results for other arithmetic progressions. The reader is encouraged to look for them.

We conclude with a final generalization in yet another direction. Let $P_k(n)$ represent the sum of the $k$th powers of the first $n$ terms of the arithmetic progression $1, 1 + d, 1 + 2d, \ldots$, so that when $d = 1$, $P_1$ is $S_1$, when $d = 2$, $P_1$ is $T_1$, etc. In Table 6, we display expressions obtained for $P_1^2$ for various values of $d$. Here $P_3(n)$ means $1^3 + (1 + d)^3 + (1 + 2d)^3 + \cdots$ to $n$ terms.
Table 6
\[ d = 1 : \quad P_1^2 = P_3 \]
\[ d = 2 : \quad P_1^2 = \frac{1}{2}(P_3 + P_1) \]
\[ d = 3 : \quad P_1^2 = \frac{1}{3}(P_3 + 2P_1) \]
\[ d = 4 : \quad P_1^2 = \frac{1}{4}(P_3 + 3P_1) \]

The pattern is clear, and the general case is easy to prove by induction, so we have

\[ P_1^2 = \frac{1}{d}(P_3 + (d-1)P_1) \]

The search for other generalizations of \( S_1^2 = S_3 \) seems to be limited only by one's imagination.

References


