Mathematicians at Morning Coffee?

The recent article entitled *Mathematicians at Lunch* by Neville de Mestre and Sean McElwain described an interesting problem about Australian Rules Football scores. The problem is suitable for high school students, and the solution presented by de Mestre and McElwain illustrates the benefit of curve-sketching in problem solving. When I first heard the problem from Sean McElwain at morning coffee I set about trying to solve it. As it turned out my solution (after a longer coffee break than usual!) was completely different from theirs, but equally straightforward. Since it involves some ideas not emphasized in high school mathematics, I thought that readers of *ASMJ* might be interested in it.

The Problem

The problem as posed by de Mestre and McElwain (1989) is:

*What score in Australian Rules Football is such that the product of the number of goals and the number of behinds equals the actual score?*

and, as stated there, amounts to solving the equation

$$6X + Y = XY$$

(1)

in non-negative integers, where $X$ is the number of goals and $Y$ is the number of behinds. Clearly $(0,0)$ is a solution, so from now on let us assume that $X$ and $Y$ are positive integers.

Rewriting (1) as

$$X(Y - 6) = Y,$$

we can see that $X$ is a divisor of $Y$, so we can write $Y = kX$. Substituting this into the original equation and cancelling $X$ gives

$$6 + k = kX,$$

or, equivalently,

$$k(X - 1) = 6.$$
Thus $k$ must be a divisor of 6. There are of course, only 4 possibilities for $k$, and the corresponding values for $X$ and $Y$ are listed in the following table:

<table>
<thead>
<tr>
<th>$k$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X$</td>
<td>7</td>
<td>4</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>$Y$</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>12</td>
</tr>
<tr>
<td>Score</td>
<td>49</td>
<td>32</td>
<td>27</td>
<td>24</td>
</tr>
</tbody>
</table>

**Generalization**

Of course, as soon as one solves a problem, one should look around for other problems for which the same method of solution might be expected to work. (This is one of the secrets of research mathematicians – find a successful method for solving one problem and exploit it in as many different contexts as possible.) If we have a game in which there are two kinds of scores, one worth $m$ points and another worth $n$ points, we are led to the equation:

$$mX + nY = XY$$

and it can be solved by exactly the same line of reasoning.

Rewriting (2) as $X(Y - m) = nY$ we see that $X$ must be a divisor of $nY$. Letting $nY = kX$ and substituting in (2) gives

$$mX + kX = \frac{kX^2}{n}$$

which simplifies to

$$k(X - n) = mn.$$  (3)

So $k$ must be a divisor of $mn$. It is easy now to determine how many non-trivial solutions there will be. If we write $mn$ is its prime factorization $p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_r^{\alpha_r}$, then the number of divisors of $mn$ is

$$\tau(mn) = (\alpha_1 + 1)(\alpha_2 + 1)\cdots(\alpha_r + 1).$$

For our original example of $6X + Y = XY$, the number of solutions will be $\tau(6)$, and $\tau(6) = \tau(2^1.3^1) = (1 + 1)(1 + 1) = 4$, as we have seen. If the scoring plays in our game are worth 8 points and 6 points, say, then we are looking for the positive integral solutions of

$$8X + 6Y = XY.$$  

By the above reasoning, there will be $\tau(48)$ solutions, and

$$\tau(48) = \tau(2^4.3^1) = (4 + 1)(1 + 1) = 10.$$
Thus there are 10 nontrivial solutions.

By solving (3) for $X$, and using $nY = kX$, we can write down all the solutions of (2):

$$X = n + \frac{mn}{k}, \quad Y = k + m,$$

where $k$ runs through the set of divisors of $mn$. These solutions are clearly all distinct. In our example with $m = 8$, $n = 6$, the solutions are of the form $(6 + 48/k, 8 + k)$, where $k$ is a divisor of 48; these are shown in the table:

<table>
<thead>
<tr>
<th>$k$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>12</th>
<th>16</th>
<th>24</th>
<th>48</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X$</td>
<td>54</td>
<td>30</td>
<td>22</td>
<td>18</td>
<td>14</td>
<td>12</td>
<td>10</td>
<td>9</td>
<td>8</td>
<td>7</td>
</tr>
<tr>
<td>$Y$</td>
<td>9</td>
<td>10</td>
<td>11</td>
<td>12</td>
<td>14</td>
<td>16</td>
<td>20</td>
<td>24</td>
<td>32</td>
<td>56</td>
</tr>
</tbody>
</table>

Comments

There are several points worth making about this problem. The first is that elementary problems in number theory can be easily understood by students at a very early stage in their mathematical lives. These are the sorts of questions that can capture the attention of bright students and get them hooked on mathematical problem-solving. Even with very little experience, students can begin to explore on their own, discovering facts new to them and learning to pose their own problems. Teachers, at least at the secondary level, ought to have a store of problems and elementary properties of numbers at their disposal for use as bait. It is for this reason I have long believed that an introduction to number theory should be a required topic in the tertiary studies of any prospective mathematics teacher. (One could argue on the same grounds for discrete/combinatorial mathematics.)

A second point concerns computing. Many students will be learning some BASIC programming skills along with their mathematics. It should be within their abilities to write a program which searches for solutions to a problem such as the one described here. This is a useful check that the solution you have reasoned out is correct. On the other hand, if you can't solve the problem, a computer search for solutions might show what you ought to be trying to prove. This is one aspect of the 'laboratory subject' approach Brailey Sims (1989) is referring to in his article in the last ASMJ.

Finally, in the spirit of generalization, let me suggest a more difficult problem which naturally comes to mind. Suppose we have a game in which there are three different scoring plays. In basketball, a foul shot (F) is worth 1 point, an ordinary field goal (G) 2 points, and a long range field goal (L) 3
points. We could ask the same question as before, which amounts to asking for positive integer solutions to the equation

\[ F + 2G + 3L = FGL. \]

The reader is invited to prove there are 8 possible scores.

References


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Many Solutions

From time to time one encounters in a high school Mathematics examination paper a question such as the following:

Find a and b given that

\[ \frac{\sqrt{3} - 1}{\sqrt{3} + 1} = a + b\sqrt{3}. \]  

(1)

Such a question is defective in that there will be an infinite number of correct answers, unless the replacement set for a and b is explicitly stated. Presumably, what is intended in such a question is that the replacement set for a and b is the set of rational numbers. Then the following theorem may be invoked:

Theorem. If a and b are rational numbers and \( a + b\sqrt{3} = 0 \) then

\[ a = 0 \quad \text{and} \quad b = 0. \]

Proof: If a and b are rational, each may be expressed as the ratio of two integers.

\[ \therefore \quad a + b\sqrt{3} = \frac{p_1}{q_1} + \frac{p_2}{q_2}\sqrt{3} = 0, \]  

(2)

where \( p_1, q_1, p_2, q_2 \) are integers and \( q_1 \neq 0, q_2 \neq 0 \). Clearing the fractions in (2) gives

\[ p_1 q_2 + p_2 q_1 \sqrt{3} = 0. \]  

(3)
\[
\therefore \sqrt{3} = \frac{-p_1 q_2}{p_2 q_1} \quad \text{or} \quad p_2 q_1 = 0.
\]

The first is not possible, since \(\sqrt{3}\) is irrational.

\[\therefore p_2 q_1 = 0, \text{ and since } q_1 \neq 0, \text{ then } p_2 = 0.\]

Hence, from (3), \(p_1\) is also zero.

i.e. \(a = 0\) and \(b = 0\). \[\blacksquare\]

**Corollary:** The above result obviously holds if \(\sqrt{3}\) is replaced by any irrational number.

Rationalising the denominator of the LHS of (1) gives

\[2 - \sqrt{3} = a + b\sqrt{3}.\]

\[\therefore (a - 2) + (b + 1)\sqrt{3} = 0, \text{ where } a \text{ and } b \text{ (and hence } a - 2 \text{ and } b + 1) \text{ are rational}\]

\[\therefore a = 2 \text{ and } b = -1, \text{ by the above theorem.}\]

On the other hand, if, as in (1), the replacement set for \(a\) and \(b\) is not given, and one assumes the widest possible set at the typical high school level, viz. the set of real numbers, then there will be an infinite number of solutions. In fact, if \(b\) is any real number, then

\[a = 2 - (b + 1)\sqrt{3} \quad \text{will always satisfy (1).}\]

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