Pythagoras' Theorem and an Exercise in Calculus

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Probably the most widely known "theorem" of mathematics is that of Pythagoras - it is one of the few results from Euclidean geometry that a significant number of people remember from their school days. There are many ways to extend or generalise this theorem and in this article we show how to prove a wide-ranging generalisation using elementary calculus.

Pythagoras' theorem is also arguably the oldest theorem of mathematics - it predates Pythagoras by at least 1000 years. Indeed, it seems clear that the Babylonians as early as 1600 BC knew how to find all integer solutions to $a^2 + b^2 = h^2$ and used these to generate tables of secants or tangents [1]. By the time of the publication of Euclid's Elements in 300 BC, about 200 years after Pythagoras, the theorem was an important basic result in geometry. It appears as proposition 47 in Book I. More interesting, however, is the fact that by this time Euclid (or some predecessor) had come up with generalisations of the theorem. The first of these generalisations is to remove the restriction that the triangle contain a right angle. The corresponding statement we now know as the cosine law and this is found in Propositions 12 and 13 of Book II of the Elements. I found this quite remarkable, although of course the statements do not mention cosines, but are instead expressed in terms of projections.

The second generalisation is to replace the square with some other polygon. This occurs as proposition 31 in Book VI and follows a series of propositions dealing with similar triangles, parallelograms and polygons. In the language of Heath's translation [2]:

In right-angled triangles the figure on the side subtending the right angle is equal to the similar and similarly described figures on the sides containing the right angle.

In this context, "figure" means polygon, and "equal" means equal in area.

In the 2300 years since Euclid, mathematicians have found a remarkable number of different ways to extend this famous theorem. I want to prove the following generalisation which is in the spirit of the second one mentioned above. The idea is to replace Euclid's polygonal figure with any region whose boundary can be described by a continuous function, as illustrated below.
Theorem

The area of the figure on the hypotenuse AB is the sum of the areas of the similar figures on sides AC and BC.

This result is part of the folklore of elementary mathematics but I have not seen a proof written down. The proof is simple and provides a good exercise in elementary calculus, focusing on an understanding of how to scale up or down the domain and range of a function. It illustrates some basic properties of the definite integral while at the same time proving something the students will likely find interesting.

Proof

Suppose that the hypotenuse were placed on the x-axis of a rectangular coordinate system with vertex A coinciding with the origin and vertex B falling on the point (c,0). Let the curve which forms the other boundary of the region be denoted by f(x). Then the area of the region is

\[ A_b = \int_c f(x) \, dx \]

To calculate the area of the region on side BC, let BC lie on the x-axis with B at the origin. Then vertex C falls on the point (0,a). We now need to describe the function g(x) which forms the boundary. First shrink the curve f(x) horizontally so that its domain is [0,a], that is, shrink by a factor of \( \frac{a}{c} \). The function \( f \left( \frac{c}{a} x \right) \) does this. (In general, to shrink by factor k, use \( f \left( \frac{1}{k} x \right) \)).

We also need to shrink the curve vertically by the same factor \( \frac{a}{c} \). This is easier - just multiply by \( \frac{a}{c} \). Thus we get

\[ g(x) = \frac{a}{c} f \left( \frac{c}{a} x \right) \]

Now to find the area under g(x), we integrate as before:

\[ A_s = \int_c^a g(x) \, dx \]

By the same reasoning, we find the area of the region on side AC to be

\[ A_s = \int_0^c h(x) \, dx \quad \text{where} \quad h(x) = \frac{b}{c} f \left( \frac{c}{b} x \right) \]

Finally evaluate the two integrals, \( A_s \) and \( A_b \):
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\[ A_a = \int_a^c \left[ \frac{c}{a} x \right] dx \]

\[ = \frac{c}{a} \int_a^c x dx \]

We make the simple substitution \( t = \frac{c}{a} x \) so that \( dt = \frac{c}{a} dx \) and then \( dx = \frac{a}{c} dt \).

We need to change the limits of integration, also. Corresponding to \( x=a \) is \( t=a \). Thus

\[ A_a = \frac{a}{c} \int_a^c f(t) \frac{a}{c} dt \]

\[ = \frac{a^2}{c^2} \int_a^c f(t) dt \]

\[ = \frac{a^2}{c^2} A_c \]

We can rephrase the last equation as "the ratio of \( A_a \) to \( A_c \) is the same as the ratio of \( a^2 \) to \( c^2 \)." This is a generalisation of the fact that the ratio of the areas of similar polygons is equal to the ratio of the squares of the corresponding sides.

By a similar calculation, we get

\[ A_b = \frac{b^2}{c^2} A_c \]

so that, finally, we get by adding:

\[ A_a + A_b = \frac{a^2}{c^2} A_c + \frac{b^2}{c^2} A_c \]

\[ = \frac{a^2 + b^2}{c^2} A_c \]

\[ = A_c \]

since \( a^2 + b^2 = c^2 \) by Pythagoras' original theorem.

This theorem can be seen to be an extension of Pythagoras' proposition 31 in a very natural way if we interpret the definite integral as a limiting sum of areas of rectangles or trapezoids. For example, if we approximate the area of our region on side AC with

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trapezoids as illustrated, then we have a polygonal region with area $A_i$. Proposition 31 can be applied directly and we get

$$A_i = A_2 + A_3$$

(1)

where $A_2$ and $A_3$ are the areas of the similar polygons on the other two sides of the triangle. Taking the limit of both sides of (1) as the widths of the trapezoids decrease yields our theorem.

It's worth noticing some special cases of the theorem that can be found by taking the boundary $y=f(x)$ to be some interesting curve. As one example, if the regions are chosen

![Diagram](Image)

semi-circular as illustrated, then the areas are easy to calculate. We find

$$A_3 + A_4 = \frac{1}{2} \pi \left[ \frac{a}{2} \right]^2 + \frac{1}{2} \pi \left[ \frac{b}{2} \right]^2$$

$$= \frac{1}{2} \pi \left( \frac{a^2 + b^2}{4} \right) = \frac{1}{2} \pi \left[ \frac{c^2}{2} \right] = \frac{1}{2} \pi \left[ \frac{c}{2} \right]^2 = A_4$$

without recourse to the integral.

References