The Maximum Dimension of a Subspace of Nilpotent Matrices of Index 2

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Abstract
A matrix $M$ is nilpotent of index 2 if $M^2 = 0$. Let $V$ be a space of nilpotent $n \times n$ matrices of index 2 over a field $k$ where $\text{card } k > n$ and suppose that $r$ is the maximum rank of any matrix in $V$. The object of this paper is to give an elementary proof of the fact that $\dim V \leq r(n-r)$. We show that the inequality is sharp and construct all such subspaces of maximum dimension. We use the result to find the maximum dimension of spaces of anti-commuting matrices and zero subalgebras of special Jordan Algebras.

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1 Introduction
An $n \times n$ matrix $M$ is nilpotent if $M^t = 0$ for some $t > 0$. We are concerned with linear spaces of nilpotent matrices over a field $k$. As far back as 1959, Gerstenhaber [4] showed that the maximum dimension of a space of nilpotent matrices was $\frac{n(n-1)}{2}$. In this paper we are interested in matrices nilpotent of index 2. Naturally such a space will have smaller dimension. We are able to show that the maximum dimension of such a space depends on the maximum $r$ of the ranks of matrices in the space: $r(n-r)$. This bound is sharp and we
characterize those spaces attaining this maximum dimension. While this might seem to be a very specialized result, it has some important consequences. It gives an immediate proof that \( r(n-r) \) is the maximum possible dimension of a space of anti-commuting matrices over any field of \( \text{card} \ k > n/2 \) (and \( \text{char} \ k \neq 2 \)). It also shows that \( r(n-r) \) is the maximum dimension of a zero subalgebra of a special Jordan Algebra. All of the proofs involve only elementary linear algebra.

Related work has been done by Brualdi and Chavey [2]. They have investigated the more general problem of finding the maximal dimension of a space of nilpotent matrices of bounded index \( k \). Their arguments are combinatorial in nature and do not imply our result. Atkinson and Lloyd [1] and others have also studied spaces of matrices of bounded rank, but their results do not overlap ours.

2 Preliminary Theorems

**Theorem 1.** Let \( V \) be a space of \( n \times n \) matrices over a field \( k \) where \( \text{card} k \geq n \). Let \( A \in V \) have the property that \( r = \text{rank} A \geq \text{rank} X \) for every \( X \in V \). If \( a \in \ker A \) then \( Ba \in \text{Im} A \) for all \( B \in V \).

**Proof.** The result is obvious when \( r = n \) so assume \( r < n \).

Let \( S = \{a_1, a_2, ..., a_k\} \) be a basis of \( \ker A \) and extend \( S \) to a basis \( B_1 = \{a_1, a_2, ..., a_k, a_{k+1}, ..., a_n\} \) of \( k^n \). Then \( T = \{Aa_{k+1}, Aa_{k+2}, ..., Aa_n\} \) is a basis of \( \text{Im} A \) and we extend \( T \) to a basis \( B_2 = \{c_1, c_2, ..., c_k, Aa_{k+1}, ..., Aa_n\} \) of \( k^n \).

Now let the vectors in \( B_1 \) form the columns of a matrix \( Q \) and the vectors in \( B_2 \) form the columns of a matrix \( P \). Then

\[
P^{-1}AQ = \begin{pmatrix} 0 & 0 \\ 0 & I_r \end{pmatrix}
\]

where \( I_r \) is an \( r \times r \) identity matrix. Let \( B \) be any matrix in \( V \) and assume

\[
P^{-1}BQ = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}
\]

where \( B_4 \) is an \( r \times r \) matrix. Then for any \( x \in k \) we have

\[
P^{-1}(B + xA)Q = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 + xI_r \end{pmatrix}.
\]

Let \( S \) be any \((r+1) \times (r+1)\) submatrix of \( P^{-1}(B + xA)Q \) containing \( B_4 + xI_r \). Then \( \text{det} S = 0 \). Since \( \text{card} k \geq n > r \), each term of this polynomial must be identically 0. The fact that the coefficient of \( x^n \) is 0 implies that each element of \( B_1 \) must be 0. So

\[
P^{-1}BQ = \begin{pmatrix} 0 & B_2 \\ B_3 & B_4 \end{pmatrix}.
\]

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Now suppose \( a_0 \in \ker A \). Then

\[
    a_0 = \sum_{i=1}^{k} x_i a_i = Q \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \\ 0 \\ \vdots \\ 0 \end{pmatrix}.
\]

So we have

\[
    B a_0 = B Q \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \\ 0 \\ \vdots \\ 0 \end{pmatrix} = P \begin{pmatrix} 0 & B_2 \\ 0 & B_4 \\ \vdots & \vdots \\ y_{k+1} & \vdots & \vdots & y_n \end{pmatrix} = P \sum_{j=1}^{n-k} y_{k+j} A a_{k+j} \in \text{Im } A
\]

We note here that the proof actually only required the cardinality of the field \( k \) to be more than \( r \), the maximum rank of a matrix in \( V \).

**Lemma 1.** Let \( V \) be a space of \( m \times n \) matrices over any field \( k \) and partition \( V \) as

\[
    V = \{ (A_1 A_2) \}.
\]

Let \( W = \{ A_1 : A \in V \} \) and \( U = \{ (0 A_2) : A \in V \} \).

Then \( \dim W + \dim U = \dim V \).

**Proof.** Let \( \dim W = s \), \( \dim U = t \), and \( \dim V = k \). There must exist independent matrices \( B_1, B_2, \cdots B_s \in V \) so that if \( B_1 = \left( B_{i,1} \left( B_{i,3} B_{i,4} \right) \right) \), then \( \{ B_{1,1}, B_{2,1}, \cdots, B_{s,1} \} \) is a basis of \( W \). Extend \( B_1, B_2, \cdots B_s \) to a basis

\[
    B = \{ B_1, B_2, \cdots B_s, \cdots, B_k \}
\]

of \( V \).

For \( 1 \leq j \leq k-s \) let

\[
    B_{s+j} = \begin{pmatrix} B_{s+j,1} & B_{s+j,2} \\ B_{s+j,3} & B_{s+j,4} \end{pmatrix}.
\]

Then

\[
    B_{s+j,1} = c_{s+j,1} B_{1,1} + c_{s+j,2} B_{2,1} + \cdots + c_{s+j,s} B_{s,1}
\]
for suitable scalars \(c_{s+j,1}, c_{s+j,2}, \ldots, c_{s+j,s}\). Replace \(B_{s+j}\) by

\[
B'_{s+j} = B_{s+j} - (c_{s+j,1}B_1 + c_{s+j,2}B_2 + \cdots + c_{s+j,s}B_s)
\]

and let \(B' = \{B_1, B_2, \ldots, B_s, B'_{s+1}, B'_{s+2}, \ldots, B'_k\}\). It is easy to show that \(B'\) is a basis of \(V\) and

\[
U = \text{span} \{B'_{s+1}, B'_{s+2}, \ldots, B'_k\}.
\]

Note: Lemma 1 is more significant than it first appears. For our proof we chose \(A_1\) as the space \(W\), but in principle there is nothing special about that choice. In fact a result similar to the statement of the Lemma holds if \(A_1\) is replaced by any set of \(r\) fixed positions in the matrices found in \(V\), so long as \(U\) is chosen as the set of complementary positions. We will use this principle repeatedly in Section 3.

We need the following lemma in Section 3. It is equivalent to a known result, but we include a short and simple proof.

**Lemma 2.** Let \(V\) be a subspace of \(M_{m,n}(k)\) where \(k\) is any field and let \(V^R = \{A \in M_{n,m}(k) \midXA = 0\text{ for all }X \in V\}\). Then \(\dim V + \dim V^R \leq mn\).

**Proof.** Let \(V_i\) be the space spanned by the \(i^{th}\) rows of the elements of \(V\), and let \(V^R_i\) be the space spanned by the \(i^{th}\) columns of the elements of \(V^R\). Then \(V^R_i \subseteq \text{Nullspace of }A_i\) where \(A_i\) is a matrix whose rows form a basis of \(V_i\). This implies that

\[
\dim V_i + \dim V^R_i \leq n.
\]

But now

\[
\dim V + \dim V^R \leq \sum_{i=1}^m \dim V_i + \sum_{i=1}^m \dim V^R_i
\]

\[
= \sum_{i=1}^m (\dim V_i + \dim V^R_i)
\]

\[
\leq mn
\]

We note that Lemmas 1 and 2 hold for any scalar field \(k\).

The following known result is needed in a later section. It was first proved by Flanders [3]. The first step of our proof is similar to that of Flanders but then our argument is considerably shorter because of Theorem 1 and Lemmas 1 and 2. We also note that the restriction on the size of the scalar field \(k\) has been removed by Meshulam [5].

**Theorem 2.** Let \(V\) be a space of \(n \times n\) matrices over a field \(k\) of \(\text{card }k \geq n\). If \(\text{rank }A \leq r\) for all \(A \in V\) then \(\dim V \leq nr\).
Proof. As in the proof of Theorem 1 we can assume that each $B \in V$ is of the form
\[
\begin{pmatrix}
B_1 & B_2 \\
B_3 & B_4
\end{pmatrix}
\]  
(1)
where $B_4$ is $r \times r$, and we showed there that $B_1 = 0$. There we considered the determinant of any $(r+1) \times (r+1)$ submatrix of $P^{-1}(B + xA)Q$ containing $B_4 + xI_r$. The fact that the coefficient of $x^{n-1}$ in this polynomial must be 0 implies that each row of $B_2$ is orthogonal (in the usual sense) to each column of $B_3$. Hence $B_2B_3 = 0$.

Let $W = \{(B_4)\}$ and $U = \left\{ \begin{pmatrix} 0 & B_2 \\ B_3 & 0 \end{pmatrix} \right\} \cap V$.

In $U$, let $W_1 = \{(B_3)\}$ and $U_1 = \left\{ \begin{pmatrix} 0 & B_2 \\ 0 & 0 \end{pmatrix} \right\} \cap U$.

If $B = \begin{pmatrix} 0 & B_2 \\ 0 & 0 \end{pmatrix} \in U_1$ and $C = \begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix} \in U$, then $B + C \in U$, which implies that $B_2Y = 0$, and so $W_1 \subseteq \{(B_2)\}^R$. Now by Lemma 2
\[
dim W_1 + \dim \{(B_2)\} = \dim W_1 + \dim U_1 \leq (n-r)r.
\]

Then by Lemma 1
\[
dim U = \dim W_1 + \dim U_1 \leq (n-r)r.
\]
and
\[
dim W + \dim U = \dim V \leq r^2 + (n-r)r = nr.
\]

Using the techniques of the previous lemmas and theorems, we show how to construct (up to equivalence) all spaces of $n \times n$ matrices of bounded rank $r$ and dimension $nr$. The derivation is somewhat tedious, but we include it here because we will use the same method in Section 4 to characterize the spaces of matrices of nilindex 2 having maximum dimension.

Let $V$ be such a space. Then if $A \in V$ we may assume that
\[
A = \begin{pmatrix} 0 & A_2 \\ A_3 & A_4 \end{pmatrix}
\]
where $A_2A_3 = 0$, $\dim W = \dim \{(A_4)\} = r^2$ and
\[
\dim U = \dim \left\{ \begin{pmatrix} 0 & A_2 \\ A_3 & 0 \end{pmatrix} \right\} \cap V = r(n-r).
\]

Suppose there exists a $B \in U$ of the form
\[
B = \begin{pmatrix} 0 & B_2 \\ B_3 & 0 \end{pmatrix}
\]
such that there is a non-zero entry \( b = b_{i,j} \) in the \( B_2 \) corner of \( B \). Let \( c = b_{k,l} \) be any entry in the \( B_3 \) corner of \( B \).

Since \( \text{dim } W = r^2 \) there must exist a matrix \( D \) in \( V \) such that
\[
D = \begin{pmatrix}
0 & D_2 \\
D_3 & D_4
\end{pmatrix}
\]
in which the \( j^{th} \) column of \( D_4 \) and the \( k^{th} \) row of \( D_4 \) are filled with 0s and the remaining submatrix of \( D_4 \) consists of the identity \( I_{r-1} \). Now let \( S \) be the \((r + 1) \times (r + 1)\) submatrix of \( B + xD \) containing \( b, c \) and \( D_4 \). Then \( \det S = 0 \) and the constant term of the polynomial is \( \pm bc \) and so \( c = 0 \). Hence
\[
B = \begin{pmatrix}
0 & B_2 \\
0 & 0
\end{pmatrix}.
\]
But if \( X \in U \) then \( X + xB \in U \) and it follows that in fact
\[
U = \left\{ \begin{pmatrix}
0 & A_2 \\
0 & 0
\end{pmatrix} \right\}
\]

Finally let \( E \) be any matrix in \( V \) where
\[
E = \begin{pmatrix}
0 & E_2 \\
E_3 & E_4
\end{pmatrix}
\]
and let \( X \) be any matrix in \( U \). Then \( E + xX \in V \) and hence \( E_3 \subseteq \{(A_2)\}^R \).

But
\[
\text{dim } \{(A_2)\} = \text{dim } U = r(n - r)
\]
and so Lemma 2 implies that \( E_3 = 0 \); therefore
\[
V = \left\{ \begin{pmatrix}
0 & A_2 \\
0 & A_4
\end{pmatrix} \right\}
\]
where \( A_2 \) and \( A_3 \) are arbitrary. A similar argument shows that if no \( B \in U \) has a suitable \( b_{i,j} \neq 0 \) in the \( B_2 \) corner, then
\[
V = \left\{ \begin{pmatrix}
0 & 0 \\
A_3 & A_4
\end{pmatrix} \right\}
\]

We summarize this discussion in the following

**Theorem 3.** Let \( V \) be a space of \( n \times n \) matrices of rank at most \( r \) over a field \( k \) with \( \text{card } k \geq n \). If \( \text{dim } V = nr \) then up to equivalence \( V \) is of the form
\[
\left\{ \begin{pmatrix}
0 & A_2 \\
0 & A_4
\end{pmatrix} \right\} \text{ or } \left\{ \begin{pmatrix}
0 & 0 \\
A_3 & A_4
\end{pmatrix} \right\}
\]
where \( A_4 \) is \( r \times r \).
The Main Result

**Theorem 4.** Let $V$ be a space of $n \times n$ nilpotent matrices of index 2 over a field $k$ where $\text{card } k > n/2$. Suppose $\text{rank } X \leq r$ for all $X \in V$. Then $\dim V \leq r(n - r)$.

**Proof.** Let $Q \in V$ such that $\text{rank } Q = r$. By rearranging a Jordan basis it is easy to show that $Q$ is similar to \[
\begin{pmatrix}
0 & I_r \\
0 & 0
\end{pmatrix}
\]
where $I_r$ is an $r \times r$ identity matrix. Without loss of generality we replace $Q$ by the above matrix. Let $A \in V$. In the proof of Theorem 1 we actually require that $\text{card } k > r$ but we know that $r \leq n/2$ so we can apply Theorem 1 to assume that $A = \begin{pmatrix}
A_1 & A_2 & A_3 \\
0 & 0 & A_4 \\
0 & 0 & A_5
\end{pmatrix}$
where $A_1, A_3$, and $A_5$ are $r \times r$ matrices. But $(Q + A)^2 = QA + AQ = 0$ and this shows that $A_5 = -A_1$. We proceed in 3 cases depending on whether any of $A_1$ or $A_2$ and $A_4$ are 0.

**Case 1.** Suppose that $A_1 = 0$ for every $A \in V$ so that each $A \in V$ is of the form \[
\begin{pmatrix}
0 & A_2 & A_3 \\
0 & 0 & A_4 \\
0 & 0 & 0
\end{pmatrix}.
\]
Now let $W = \{(A_3)\}$ and $U = \left\{ \begin{pmatrix} 0 & A_2 & 0 \\
0 & 0 & A_4 \\
0 & 0 & 0 \end{pmatrix} \right\} \cap V$. Clearly $\dim W \leq r^2$.

In $U$, let $W_1 = \{(A_2)\}$ and $U_1 = \left\{ \begin{pmatrix} 0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & A_4 \end{pmatrix} \right\} \cap U$.

Suppose $B = \begin{pmatrix} 0 & B_2 & 0 \\
0 & 0 & B_4 \\
0 & 0 & 0 \end{pmatrix} \in U$ and $C = \begin{pmatrix} 0 & 0 & 0 \\
0 & 0 & C_4 \\
0 & 0 & 0 \end{pmatrix} \in U_1$.

Then $(B + C)^2 = 0$ implies that $B_2C_4 = 0$. So in $U_1$, if $T = \{(A_4)\}$ then $T \subseteq W_1^r$ and by Lemma 2, $\dim W_1 + \dim T = \dim W_1 + \dim U_1 \leq (n - 2r)r$.

So by Lemma 1

\[\dim U = \dim W_1 + \dim U_1 \leq (n - 2r)r\]  \hspace{1cm} (3)

and

\[\dim V = \dim W + \dim U \leq r^2 + (n - 2r)r = nr - r^2\]
Case 2. Suppose $A_2$ and $A_4$ do not exist. In this case $r = n/2$ and each $A \in V$ is of the form $A = \begin{pmatrix} A_1 & A_2 \\ 0 & -A_1 \end{pmatrix}$.

If each $A_1 = 0$ then $\dim V \leq r^2 = nr - r^2$ and so we assume there exists an $A_1 \neq 0$. Let $W = \{ (A_1) \}$. Let $r_1$ be the largest rank of any matrix in $W$. Then $W$ is a space of nilpotent matrices of index 2 and bounded rank $r_1$ so by induction we may assume $\dim W \leq r_1(r - r_1)$. As above, there exists a matrix in $V$ which is similar to

$$Q_1 = \begin{pmatrix} 0 & I_{r_1} & A''_2 & A''_4 \\ 0 & 0 & A''_2 & A''_4 \\ 0 & 0 & 0 & -I_{r_1} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

where $A''_2$ is an $r_1 \times (r - r_1)$ matrix.

Also $Q^2_1 = 0$ implies that $A''_2 = 0$. Now let

$$U = \left\{ \begin{pmatrix} 0 & 0 & B_1 & B_2 \\ 0 & 0 & B_3 & B_4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\} \cap V$$

If $B \in U$ then $(Q_1 + B)^2 = 0$ implies that $B_3 = 0$ and so $\dim U \leq r^2 - r_1(r - r_1)$. Hence by Lemma 1

$$\dim V = \dim W + \dim U \leq rr_1 - r_1^2 + r^2 - r_1(r - r_1) = r^2 = nr - r^2$$

Case 3. Suppose there exists an $A_1 \neq 0$ and $A_2$ and $A_4$ do exist. Note that $r < n/2$. Each $A \in V$ is of the form

$$A = \begin{pmatrix} A_1 & A_2 & A_3 \\ 0 & 0 & A_4 \\ 0 & 0 & -A_1 \end{pmatrix}$$

Let $W = \{ (A_1) \}$. Then as above, by induction we may assume $\dim W \leq rr_1 - r_1^2$ where $r_1$ is the largest rank of any matrix in $W$. Also we may assume there is a matrix in $V$ which is similar to

$$Q_1 = \begin{pmatrix} 0 & I_{r_1} & A'_2 & A'_3 & A''_2 & A''_3 & A''_4 \\ 0 & 0 & A''_2 & A''_3 & A''_4 \\ 0 & 0 & 0 & A''_2 & A''_4 \\ 0 & 0 & 0 & 0 & -I_{r_1} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

where $A'_2$ is $(r - r_1) \times (n - 2r)$, $A''_2$ is $r_1 \times (n - 2r)$, $A'_4$ is $(n - 2r) \times r_1$ and $A''_4$ is $(n - 2r) \times (r - r_1)$. Then $Q^2_1 = 0$ implies that $A''_2 = A''_4 = 0$. Let
\( U = \left\{ \begin{pmatrix} 0 & 0 & A_2' & A_3' & A_3'' \\ 0 & 0 & A_2'' & A_3'' & A_3''' \\ 0 & 0 & 0 & A_4'' & A_4''' \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \right\} \cap V \)

As above, if \( B \in U \) then \((Q_1 + B)^2 = 0\) implies that
\[
B = \begin{pmatrix} 0 & 0 & B_2' & B_3' & B_3'' \\ 0 & 0 & 0 & B_3'' & B_3''' \\ 0 & 0 & 0 & 0 & B_4' \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}
\]
where \( B_3'' \) is \((r - r_1) \times (r - r_1)\). Also if \( x \in k \) then \( \text{rank}(Q_1 + xB) \leq r \) and this implies that the \( \text{rank} B_3'' \leq (r - 2r_1) \). Hence if in \( U, S = \{ (B_3'') \} \) then \( \text{dim} \ S \leq (r - r_1)(r - 2r_1) \) by Theorem 2. Now in \( U \) let
\[
W_1 = \{ \begin{pmatrix} B_3' & B_3'' \\ B_3'' & B_3''' \end{pmatrix} \} \quad \text{and} \quad T = \{ \begin{pmatrix} B_3' & B_3'' \\ 0 & 0 \end{pmatrix} \}
\]

Then using Lemma 1 again:
\[
\text{dim} \ W_1 = \text{dim} \ S + \text{dim} \ T \\
\leq (r - r_1)(r - 2r_1) + 2rr_1 - r_1^2 \\
= r^2 - rr_1 + r_1^2.
\]

Finally let
\[
U_1 = \left\{ \begin{pmatrix} 0 & 0 & B_2' & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \right\} \cap U.
\]

Using Lemma 2 and the argument as found in Case 1 it can be shown that \( \text{dim} \ U_1 \leq (n - 2r)(r - r_1) \). Now using Lemma 1
\[
\text{dim} \ V = \text{dim} \ W + \text{dim} \ U \\
= \text{dim} \ W + \text{dim} \ W_1 + \text{dim} \ U_1 \\
\leq rr_1 - r_1^2 + r^2 - rr_1 + r_1^2 + (n - 2r)(r - r_1)
\]

Simplifying:
\[
\text{dim} \ V \leq nr - r^2 + 2rr_1 - nr_1.
\]

But \( r < n/2 \) so \( 2rr_1 < nr_1 \) and hence
\[
\text{dim} \ V < r(n - r).
\]

\[\square\]
There are some important consequences that follow immediately from this theorem. Indeed, it was questions like these that originally interested us in spaces of nilpotent matrices.

**Corollary 1.** Let $V$ be a space of anticommuting $n \times n$ matrices over a field $k$ where $\text{card } k > n/2$ and $\text{char } k \neq 2$. If $\text{rank } A \leq r$ for all $A \in k$ then $\dim V \leq r(n - r)$.

*Proof.* Since $\text{char } k \neq 2$ the matrices in $V$ must be nilpotent of index 2.

Let $A$ be the algebra of all $n \times n$ matrices over a field $k$ where $\text{char } k \neq 2$. Define a new multiplication $\circ$ as

$$X \circ Y = \frac{1}{2}(XY + YX)$$

Then $A$ with its new operation $\circ$ is a Jordan Algebra. It is often called a special Jordan Algebra.

**Corollary 2.** Let $A$ be a special Jordan Algebra constructed from $n \times n$ matrices over a field $k$ where $\text{card } k > n/2$ and $\text{char } k \neq 2$. If $A_1$ is a zero subalgebra of $A$ then $\dim A_1 \leq r(n - r)$ where $r$ is the maximum rank of any matrix in $A_1$.

*Proof.* In a zero subalgebra $X \circ Y = 0$ and the result follows directly from Corollary 1.

### 4 The Spaces of Maximum Dimension

We now show that the inequality in our main result is sharp by constructing spaces which have the maximum dimension $r(n - r)$. In addition, the spaces constructed below are, up to similarity, the only ones reaching the maximum dimension. Again we consider the three cases.

**Case 1**

Let $V_1 = \left\{ \begin{pmatrix} 0 & A_2 & A_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}$ where $A_2$ is any $r \times (n - 2r)$ matrix and $A_3$ is any $r \times r$ matrix. Clearly $V_1$ is a space of nilpotent matrices of index 2 and bounded rank $r$ and $\dim V_1 = nr - r^2$. Similarly $V_2 = \left\{ \begin{pmatrix} 0 & 0 & A_3 \\ 0 & 0 & A_4 \\ 0 & 0 & 0 \end{pmatrix} \right\}$ where $A_4$ is any $(n - 2r) \times r$ matrix and $A_3$ is any $r \times r$ matrix is also a space of nilpotent matrices of index 2 and bounded rank $r$ and $\dim V_2 = nr - r^2$.

If a space is of this type has maximum dimension $r(n - r)$, we can show that these are the only such subspaces. The argument is very similar to that in the derivation of Theorem 3, so we omit the details.

**Case 2**

Let $V_3 = \left\{ \begin{pmatrix} 0 & A_2 \\ 0 & 0 \end{pmatrix} \right\}$ where $A_2$ is any $n/2 \times n/2$ matrix ($n$ is even).

Then $V_3$ is a space of nilpotent matrices of index 2 and bounded rank $r = n/2$ and $\dim V_3 = r(n - r) = n^2/4$. 

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Again we can show that these are the only subspaces of this type that achieve maximum dimension. The argument is similar to that of Theorem 3 and we omit it.

Case 3 Note that in this case we showed that $\dim V < nr - r^2$ and so no subspaces of maximum dimension of this type exist.

References


