Consecutive Integers with Equally Many Principal Divisors

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1 Introduction

Classifying the positive integers as primes, composites, and the unit, is so familiar that it seems inevitable. However, other classifications can bring interesting relationships to our attention. In that spirit, let us classify positive integers by the number of principal divisors they possess, where we define a principal divisor of a positive integer \( n \) to be any prime-power divisor \( p^a \mid n \) which is maximal (so \( p \) is prime, \( a \) is a positive integer, and \( p^{a+1} \) is not a divisor of \( n \)). The standard notation \( p^a \| n \) can be read as “\( p^a \) is a principal divisor of \( n \).”

For each integer \( n \geq 0 \), let \( P_n \) be the set of all positive integers with exactly \( n \) principal divisors, so \( P_0 = \{1\} \), and

\[
\begin{align*}
P_1 & = \{2, 3, 4, 5, 7, 8, 9, 11, 13, 16, 17, 19, 23, 25, 27, 29, 31, 32, 37, \cdots \}, \\
P_2 & = \{6, 10, 12, 14, 15, 18, 20, 21, 22, 24, 26, 28, 33, 34, 35, 36, 38, \cdots \}, \\
P_3 & = \{30, 42, 60, 66, 70, 78, 90, 102, 105, 110, \cdots \}, \\
P_4 & = \{210, 330, 390, 420, 462, 510, \cdots \}, \text{etc.}
\end{align*}
\]

In particular, \( P_1 \) comprises the prime-powers, or principal integers; \( P_2 \) comprises the products of two coprime principal integers, or rank 2 integers; and so on. Collectively, we call \( \{P_n : n \geq 0\} \) the rank sets of positive integers.
Clearly the rank sets are a partition of the positive integers, by the Fundamental Theorem of Arithmetic. Thus it is interesting to look at the occurrence of runs of consecutive integers within each rank set: this is one of the relationships immediately brought into focus by the classification.

For compactness, let us write \([a, r]\) to denote the run of \(r\) consecutive integers beginning with \(a\), where \(a\) and \(r\) are positive integers, so 

\[ [a, r] = \{a + i : 0 \leq i < r\} \]

We call \(r\) the size of the run. In particular, the run \([a, r]\) is nontrivial if \(r \geq 2\), and \([a, r]\) is a maximal run in \(P_n\) if it is nontrivial and \(P_n\) contains neither \(a - 1\) nor \(a + r\). Thus, the first few maximal runs in \(P_1\) are \([2, 4]\), \([7, 3]\), \([16, 2]\), \([31, 2]\), \([127, 2]\) and \([256, 2]\); the first few maximal runs in \(P_2\) are \([14, 2]\), \([20, 3]\), \([33, 4]\), \([38, 3]\), \([44, 3]\), \([50, 3]\) and \([54, 5]\).

It is easy to see that the size of runs in \(P_n\) is bounded. For if \(M\) is the product of the first \(n + 1\) primes, then \(M \in P_{n+1}\) and any run of \(M\) consecutive integers contains a multiple of \(M\), so any run in \(P_n\) has size less than \(M\). Thus, for each integer \(n \geq 0\) there is a positive integer \(r(n)\) which is the maximum size attained by runs in \(P_n\). Trivially, \(r(0) = 1\). We have already seen that \(r(1) \geq 4\) and \(r(2) \geq 5\), and we shall soon see that in fact \(r(1) = 4\). Our main objective is to study \(r(2)\), which we shall determine “within 1”. Subsequently, we shall also discuss \(r(n)\) for \(n \geq 3\).

2 Maximal runs in \(P_1\) and \(P_2\)

Returning to the principal integers, it is clear that any nontrivial run in \(P_1\) contains an even integer so, being principal, any such integer must be a power of 2. Since 2 and 4 are the only powers of 2 that differ by 2, any maximal run of principal integers greater than 5 must contain exactly one even integer, so \([2, 4]\) is the unique longest run in \(P_1\), and \(r(1) = 4\).

The long-standing conjecture credited to Catalan, that 8 and 9 are the only two consecutive integers which are nontrivial powers, was recently proved by Mihailescu [10]. From this it follows that any maximal run of principal integers greater than 9 must contain a power of 2 and any adjacent number in the run must be a prime. It is well known that \(2^n - 1\) can only be prime when \(n\) itself is prime, and \(2^n + 1\) can only be prime when \(n\) is a power of 2: when \(n \geq 3\) these two conditions are mutually exclusive, so any maximal run of principal integers greater than 9 has just two members.

Primes of the form \(2^n + 1\) are Fermat primes. The only known Fermat primes are 3, 5, 17, 257 and 65537, but no proof is known that there are
no others. Primes of the form $2^n - 1$ are Mersenne primes. Currently 43 Mersenne primes are known [6]. A distributed computing project known as GIMPS (the Great International Mersenne Prime Search) has made numerous additions to this list in recent years, but no proof is known that infinitely many such primes exist. Consequently, although we know [2, 4] and [7, 3] are the only maximal runs of more than two principal integers, it is not known whether $P_1$ contains infinitely many runs of size 2.

What happens with maximal runs of rank 2 numbers? This is less familiar territory, so one does not know quite what to expect. We shall prove:

**Theorem** There is no run of 10 consecutive integers in $P_2$.

Hence $r(2) \leq 9$. Is this a “sharp” result? It turns out to be “within 1” of the exact value of $r(2)$. Our methods appear unable to decide on the existence of runs of size 9 in $P_2$, but our results strongly suggest the following:

**Conjecture 1** In $P_2$ there is no run of size 9 and the only maximal runs of size greater than 6 are $[141, 8]$, $[212, 8]$, $[323, 7]$ and $[2302, 7]$.

It may also be true that $[91, 6]$ is the only maximal run of size 6 in $P_2$, but we have less information about runs of size 6 than about longer runs.

### 3 Størmer’s Theorem

One of the main tools we shall use to prove our theorem is a result of Carl Størmer [12], which follows from a remarkable earlier theorem of his on fundamental solutions to Pell equations [11].

**Theorem 1 (Størmer)** For given positive integers $A, B, m, n, a_1, \ldots, a_m, b_1, \ldots, b_n$, there are at most finitely many sequences of positive integers $x_1, \ldots, x_m, y_1, \ldots, y_n$ such that

$$|Aa_1^{x_1} \cdots a_m^{x_m} - Bb_1^{y_1} \cdots b_n^{y_n}| \leq 2.$$ 

All solutions follow from the fundamental solutions to a finite set of Pell equations determined by $A, B, m, n, a_1, \ldots, a_m, b_1, \ldots, b_n$.

In particular, for any set of primes $P$, let $S(P)$ be the set of all positive integers with all their prime factors in $P$, that is, $S(P)$ is the multiplicative semigroup of positive integers generated by $P$. If $P$ is finite, then by
Størmer’s theorem there are only finitely many pairs of consecutive integers in \( S(P) \). D. H. Lehmer [8] gave a new proof of Størmer’s theorem for the case of consecutive integers, and explicitly computed the last pairs of consecutive integers in \( S(P) \) when \( P \) is any initial subset of the primes with largest member 41 or less [9]. Some related tabulations are given in [3], and some elementary arguments establishing special instances of Størmer’s result are given in [4] and [7]. The latter paper, by Halsey and Hewitt, discusses the fascinating connection between fundamental frequency ratios in Western music and consecutive pairs of integers in \( S(2,3,5) \).

<table>
<thead>
<tr>
<th>( S(P) )</th>
<th>Last maximal runs</th>
</tr>
</thead>
<tbody>
<tr>
<td>( S(2,3) )</td>
<td>[8, 2], [2, 3]</td>
</tr>
<tr>
<td>( S(2,3,5) )</td>
<td>[80, 2], [8, 3], [3, 4]</td>
</tr>
<tr>
<td>( S(2,3,5,7) )</td>
<td>[4374, 2], [48, 3], [7, 4]</td>
</tr>
</tbody>
</table>

Table 1: Last maximal runs in \( S(P) \)

For our present purposes, we note in Table 1 the last maximal runs of various sizes in \( S(P) \) when \( P \) is an initial subset of the primes not greater than 7. We shall also need the following consequence of Størmer’s theorem:

**Corollary** The only pairs of integers satisfying \( |3^a - 5^b| = 2 \) are \((3,5)\) and \((25,27)\), and the only pair satisfying \( |3^a - 7^b| = 2 \) is \((7,9)\).

**Proof.** Størmer’s method applied to pairs of integers \( \{3^a, 5^b\} \) satisfying \( |3^a - 5^b| = 2 \) puts them equal to \( x - 1 \) and \( x + 1 \), so their product is of the form \( x^2 - 1 = Dy^2 \), where \( y \in S(3,5) \) and \( D \in \{3^\alpha 5^\beta \mid 1 \leq \alpha, \beta \leq 2\} = \{15, 45, 75, 225\} \). By Størmer’s theorem [11], for each \( D \) there is at most one such positive integer \( y \); if it exists, the corresponding pair \((x, y)\) is the fundamental solution of \( x^2 - 1 = Dy^2 \). For the perfect square \( D = 225 \) there is no solution \((x, y)\) in positive integers. For \( D = 15, 45 \) and 75, the fundamental solutions are \((x, y) = (55, 12), (8, 1)\) and \((26, 3)\) respectively; since \( y \in S(3,5) \) when \( y = 1 \) or 3, the corresponding pairs \((x - 1, x + 1)\) are \((3, 5)\) and \((25, 27)\).

Similarly, pairs of integers \( \{3^a, 7^b\} \) satisfying \( |3^a - 7^b| = 2 \) must correspond to fundamental solutions of \( x^2 - 1 = Dy^2 \), where \( y \in S(3,7) \) and \( D \in \{21, 63, 147, 441\} \). The first three cases have fundamental solutions \((x, y) = (55, 12), (8, 1)\) and \((97, 8)\); since \( y \in S(3,7) \) only when \( y = 1 \), the corresponding pair \((x - 1, x + 1)\) is \((7,9)\).
4 Constraints on runs in $P_2$

We shall establish several properties of runs in $P_2$, and then deduce our theorem. The first property concerns the intersection of $S(2,3)$ with $P_2$.

**Property 1** Any run of consecutive integers in $P_2$ contains at most one multiple of 6.

**Proof.** Since 89 and 97 are the first two consecutive primes with difference greater than 6, it follows that between any two consecutive multiples of 6 less than 96 there is at least one prime. Hence no run in $P_2$ can contain two consecutive multiples of 6 less than 96. On the other hand, 8 and 9 are the last two consecutive integers in $S(2,3)$, so 48 and 54 are the last two consecutive multiples of 6 in $P_2$. The property follows. ■

The next three properties concern multiples of 5 in $P_2$ that occur within runs which contain a multiple of 6.

**Property 2** Only one maximal run of consecutive integers in $P_2$ contains a multiple of 6 and a multiple of 5 which differ by 2 or 3; that run is $[158, 5]$.

**Proof.** If $6a$ and $5b$ are members of $P_2$ that differ by 2, then $6a \in S(2,3)$ and $5b = 10c \in S(2,5)$ for some integer $c$. Then $3a$ and $5c$ are consecutive integers in $S(2,3,5)$, and $[80,2]$ is the last such pair. The corresponding pairs in $P_2$ are $(10,12), (18,20), (48,50)$ and $(160,162)$. For the first three pairs, the intervening number is in $P_1$; hence the only maximal run in $P_2$ that contain integers $6a$ and $5b$ with $|6a - 5b| = 2$ is $[158,5]$.

Similarly, if $6a$ and $5b$ are members of $P_2$ that differ by 3, then $6a \in S(2,3)$ and $5b = 15c \in S(3,5)$ for some integer $c$, so $2a$ and $5c$ are consecutive integers in $S(2,3,5)$. The corresponding pairs in $P_2$ are $(12,15), (15,18), (45,48)$ and $(72,75)$. In each case there is an intervening prime, so no maximal run in $P_2$ contains integers $6a$ and $5b$ with $|6a - 5b| = 3$. ■

**Property 3** No run of consecutive integers in $P_2$ contains a multiple of 6 and a multiple of 5 which differ by 4.

**Proof.** First suppose $6a$ and $20b$ are members of $P_2$ that differ by 4, so $6a = 12c \in S(2,3)$ for some integer $c$, and $20b \in S(2,5)$. Then $5b$ and $3c$ are consecutive integers in $S(2,3,5)$. Since $[80,2]$ is the last such pair, the corresponding pairs in $P_2$ are $(20,24), (36,40), (96,100)$ and $(320,324)$. The first three pairs have an intervening prime, while the last pair is not in
a run in $P_2$ because $322 \in P_3$. Thus, no run of consecutive integers in $P_2$
contains a multiple of 6 and a multiple of 20 differing by 4.

Now suppose $6a$ and $10d$ are members of $P_2$ that differ by 4, and $d$ is odd.
Then $|3a - 5d| = 2$, so $a$ is also odd. But $6a \in S(2, 3)$ and $10d \in S(2, 5)$,
so $a$ is a power of 3 and $d$ is a power of 5. By our corollary to Størmer’s
Theorem, $(3, 5)$ and $(25, 27)$ are the only pairs of proper powers of 3 and 5
that differ by 2. The corresponding pairs in $P_2$ are $(6, 10)$ and $(50, 54)$, but
each has an intervening prime, so neither pair is contained in a run in $P_2$.
The property follows.

It is noteworthy that the pair $(320, 324)$, appearing in the proof of Prop-
erty 3, actually corresponds to a near miss: $P_2$ contains the two maximal
runs $[319, 3]$ and $[323, 7]$, and the only intervening integer is $322 \in P_3$. The
two bordering integers are $318 \in P_3$ and $330 \in P_4$ (both multiples of 6), and
their neighbours 317 and 331 are consecutive primes.

**Property 4** Any run of consecutive integers in $P_2$ contains at most one
multiple of 5.

**Proof.** On the contrary, suppose there is a run of consecutive integers
in $P_2$ that contains two multiples of 5. Let $R$ be that portion of the run
which begins and ends with two consecutive multiples of 5. Since $R$ has size
6, we have $6a \in R$ for some integer $a$. But $6a \in P_2$ so it is distinct from the
multiples of 5. If $6a$ differs by 2 from the nearer multiple of 5, these two
members of $R$ belong to the maximal run $[158, 5]$, by Property 2. But this
does not contain two multiples of 5, so is disjoint from $R$. Hence $6a$ must be
adjacent to the nearer multiple of 5. But then it must differ by 4 from the
other multiple of 5 in $R$, and Property 3 shows that no run in $P_2$ contains
two such numbers. Hence, by contradiction, $R$ does not exist.

Since any run of 10 consecutive integers contains two multiples of 5,
Property 4 immediately implies our target result:

**Theorem 2** There is no run of 10 consecutive integers in $P_2$.

**Corollary** Runs in $P_2$ have maximum size $8 \leq r(2) \leq 9$.

Although our methods do not seem to be strong enough to decide whether
runs of size 9 exist in $P_2$, consideration of multiples of 7 yields further prop-
erties, revealing more of the structure of $P_2$. In particular, we are led to
discover the examples of runs of size 8 in $P_2$ which are incorporated in
Conjecture 1. We shall pursue this in the next section.
5 Further constraints on runs in $P_2$

Property 5 Only two maximal runs of consecutive integers in $P_2$ contain a multiple of 6 and a multiple of 7 which differ by 2 or 3: those runs are $[54, 5]$ and $[141, 8]$.

Proof. If $6a$ and $7b$ are members of $P_2$ that differ by 2, then $6a \in S(2, 3)$ and $7b = 14c \in S(2, 7)$ for some integer $c$. Then $3a$ and $7c$ are consecutive integers in $S(2, 3, 7)$. From the fact that $[4374, 2]$ is the last pair in $S(2, 3, 5, 7)$, a straightforward calculation verifies that $[63, 2]$ is the last pair in $S(2, 3, 7)$. The corresponding pairs in $P_2$ are $(12, 14)$, $(54, 56)$ and $(96, 98)$. For the first and third pair, the intervening number is prime; hence the only maximal run in $P_2$ that contains integers $6a$ and $7b$ with $|6a - 7b| = 2$ is $[54, 5]$.

Similarly, if $6a$ and $7b$ are members of $P_2$ that differ by 3, then $6a \in S(2, 3)$ and $7b = 21c \in S(3, 5)$ for some integer $c$, so $2a$ and $7c$ are consecutive integers in $S(2, 3, 7)$. The corresponding pairs in $P_2$ are $(18, 21)$, $(21, 24)$, $(81, 84)$, $(144, 147)$ and $(189, 192)$. For all but one of these pairs, there is an intervening prime; hence the only maximal run in $P_2$ that contains integers $6a$ and $7b$ with $|6a - 7b| = 3$ is $[141, 8]$. $lacksquare$

Property 6 Any run of consecutive integers in $P_2$ contains at most one multiple of 7.

Proof. On the contrary, suppose there is a run $R$ of consecutive integers in $P_2$ that contains two consecutive multiples of 7. Since they are in $P_2$, neither is a multiple of 6, so there is a multiple of 6 between them but not adjacent to either of them. Thus the multiple of 6 and the nearer multiple of 7 differ by 2 or 3, so $R$ must be contained in $[54, 5]$ or $[141, 8]$, by Property 5. But each of these runs contains only one multiple of 7, so by contradiction it follows that $R$ does not exist. $lacksquare$

Property 7 There are no runs of consecutive integers in $P_2$ that contain a multiple of 6 and a multiple of 7 which differ by 4.

Proof. First suppose $6a$ and $28b$ differ by 4 and belong to some run of consecutive integers in $P_2$. Then $6a = 12c \in S(2, 3)$ for some integer $c$, and $28b \in S(2, 7)$, so $7b$ and $3c$ are consecutive integers in $S(2, 3, 7)$. As noted in the proof of Property 5, $[63, 2]$ is the last pair in $S(2, 3, 7)$. The
corresponding pairs in $P_2$ are $(24, 28), (108, 112)$ and $(192, 196)$, but none of them is contained in a run in $P_2$.

Now suppose, for some odd integer $d$, that $6a \in S(2, 3)$ and $14d \in S(2, 7)$ differ by 4 and belong to some run of consecutive integers in $P_2$. Then $|3a - 7d| = 2$, so $a$ is odd; hence $3a$ is a power of 3 and $7d$ is a power of 7. By our corollary to Størmer’s Theorem, $(7, 9)$ is the only pair of proper powers of 3 and 7 that differ by 2; the corresponding pair $(14, 18)$ in $P_2$ is not contained in a run in $P_2$. The property follows.

**Property 8** Exactly two maximal runs of consecutive integers in $P_2$ contain a multiple of 6 and a multiple of 35: these are $[33, 4]$ and $[4374, 2]$.

**Proof.** If $6a$ and $35b$ are members of some run of consecutive integers in $P_2$, then $6a \in S(2, 3)$ and $35b \in S(5, 7)$, so $|6a - 35b|$ cannot be a multiple of 2, 3, 5 or 7. But every run in $P_2$ has size less than 10, by our Theorem, so $|6a - 35b| = 1$, and $6a$ and $35b$ are consecutive integers in $S(2, 3, 5, 7)$. Since $[4374, 2]$ is the last nontrivial run in $S(2, 3, 5, 7)$, the corresponding pairs in $P_2$ are $(35, 36)$ and $(4374, 4375)$. The corresponding maximal runs in $P_2$ are $[33, 4]$ and $[4374, 2]$.

**Property 9** Any run of size at least 7 in $P_2$ contains exactly one multiple of 6, exactly one multiple of 5, and exactly one multiple of 7, and these are three distinct members of the run. The multiple of 5 is always adjacent to the multiple of 6. If the run size is 8 or more, the multiple of 7 is also adjacent to the multiple of 6, except in the case of the maximal run $[141, 8]$.

**Proof.** Uniqueness of the multiples of 6, 5 and 7 follows from Properties 1, 4 and 6 respectively. By Property 8, the only two runs in $P_2$ containing a multiple of 6, and a multiple of 5 which is also a multiple of 7, have size less than 7. Hence, in any run of size 7 or more, all three must be distinct integers. By Properties 2 and 3, in any run of size at least 6 in $P_2$ the multiple of 5 must be adjacent to the multiple of 6. In a run of size 8 or more, the multiple of 7 must occur in the intersection of the first 7 integers and the last 7, by Property 6; similarly the multiple of 6 must occur in the intersection of the first 6 integers and the last 6, by Property 1. Hence the multiple of 6 and multiple of 7 differ by at most 4. By Property 7, there is no run in which the difference is 4. By Property 5, $[141, 8]$ is the only run of size at least 8 in which the difference is 2 or 3. Hence, in every other run of size 8 or more, the difference must be 1. ■
Property 10  Except for the maximal run $[141,8]$, in any run of size 8 or more in $P_2$ the multiple of 6 is of the form $6^a2^b$ or $6^a3^b$, where $a \geq 1$ and $b \geq 0$ are integers of opposite parity.

Proof. Let $R \neq [141,8]$ be a run of size 8 in $P_2$, and let $n$ be its multiple of 6. In some order, its multiples of 5 and 7 are $n-1$ and $n+1$, by Property 9. Put $n = 2^c3^d$, where $c$ and $d$ are positive integers.

Suppose $5|n-1$ and $7|n+1$. Then $2^c3^d \equiv 3^{2c+d} \equiv 1 \pmod{5}$ and $2^c3^d \equiv 3^{2c+d} \equiv -1 \pmod{7}$, so $3c+d \equiv 0 \pmod{4}$ and $2c+d \equiv 3 \pmod{6}$. Hence $c \equiv d \pmod{12}$ and $c \equiv d \equiv 1 \pmod{2}$. Put $a = \min\{c,d\}$, where $a \geq 1$ is odd. Also let $|c-d| = 12s$, and put $\max\{c,d\} = a+6b$, where $b = 2s \geq 0$ is even. Then $n = 6^a2^b$ or $6^a3^b$.

Now consider $5|n+1$ and $7|n-1$. In this case $3c+d \equiv 2 \pmod{4}$ and $2c+d \equiv 0 \pmod{6}$, so $c \equiv d+6 \pmod{12}$ and $c \equiv d \equiv 0 \pmod{2}$. Put $a = \min\{c,d\}$, where $a \geq 2$ is even. Also let $|c-d| = 12s+6$, and put $\max\{c,d\} = a+6b$, where $b = 2s+1 \geq 1$ is odd. Again we have $n = 6^a2^b$ or $6^a3^b$. The property follows. ■

Property 10 places a strong restriction on the possible multiples of 6 in any run of size 8 or 9 in $P_2$. Below $10^{25}$ there are just 90 of these special multiples of 6 with $a$ odd, and 84 multiples with $a$ even. It is a straightforward computation to check beside these 174 numbers. We find one gem, the maximal run $[212,8]$. No later run of size 8 or 9 occurs up to $10^{25}$, providing strong evidence for Conjecture 1. Indeed, among runs in $P_2$ that contain three consecutive integers which are multiples of 5, 6 and 7, the only other instances of size greater than 3 below $10^{25}$ are $[2302,7]$ and $[24575,5]$.

6  Computing runs in $P_n$ for $n \geq 3$

Computation sheds some interesting light on runs in $P_n$ with $n \geq 3$, and turns up some delightful gems.

If $\{a,a+1\} \subset P_n$, then $a$ and $a+1$ are coprime, and each has $n$ principal divisors, so $a(a+1) \geq p_1p_2\cdots p_{2n}$, the product of the first $2n$ primes. But $(a+1)^2 > a(a+1)$, so $a+1 > (p_1p_2\cdots p_{2n})^{1/2}$, and the ceiling of this square root is a lower bound for $a+1$. The product of primes is never a square, so the floor of the square root gives the lower bound

$$a \geq \lfloor (p_1p_2\cdots p_{2n})^{1/2} \rfloor.$$

Table 2 lists the first maximal run in $P_n$ for $3 \leq n \leq 7$, together with the corresponding factorisations. It is interesting to notice how the small primes
crowd in and form the majority of principal divisors in the first maximal run of each rank set. It is possible that the first maximal run in \( P_n \) always occurs in the interval \([A, 2^n A]\), where \( A = \lfloor (p_1 p_2 \cdots p_{2^n})^{1/2} \rfloor \). Indeed, each instance in Table 2 occurs in the interval \([A, 2^n A/2]\).

<table>
<thead>
<tr>
<th>( P_n )</th>
<th>First maximal run</th>
<th>( a )</th>
<th>( a + 1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P_3 )</td>
<td>[230, 2]</td>
<td>2.5.23</td>
<td>3.7.11</td>
</tr>
<tr>
<td>( P_4 )</td>
<td>[7314, 2]</td>
<td>2.3.23.53</td>
<td>5.7.11.19</td>
</tr>
<tr>
<td>( P_5 )</td>
<td>[254540, 2]</td>
<td>( 2^2 \cdot 5 \cdot 11 \cdot 13.89 )</td>
<td>3.7.17.23.31</td>
</tr>
<tr>
<td>( P_6 )</td>
<td>[11243154, 2]</td>
<td>2.3.13.17.61.139</td>
<td>5.7.11.19.29.53</td>
</tr>
<tr>
<td>( P_7 )</td>
<td>[965009045, 2]</td>
<td>5.7.11.13.23.83.101</td>
<td>2.3.17.29.41.73.109</td>
</tr>
</tbody>
</table>

Table 2: First maximal runs

Table 3 lists the starters of successive maximal runs in \( P_n \) for \( 3 \leq n \leq 7 \). In each case the interval \([A, 2^n A/2]\) contains at least 5 maximal runs. Moreover, for \( P_8 \), the corresponding interval contains the maximal run \([a, 2]\) with \( a = 68971338435 \), since \( a = 3.5.17.23.29.31.103.127 \) and \( a + 1 = 2^2 \cdot 7 \cdot 11 \cdot 13.19.37.107.229 \), but we do not know whether this is the first maximal run in \( P_8 \), nor how many maximal runs occur in this interval.

<table>
<thead>
<tr>
<th>( P_n )</th>
<th>Starters of successive maximal runs of size 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P_3 )</td>
<td>230, 285, 429, 434, 455, 494, 560, 594, 609, 615, 645, \cdots</td>
</tr>
<tr>
<td>( P_4 )</td>
<td>7314, 8294, 8645, 9009, 10659, 11570, 11780, 11934, 13299, \cdots</td>
</tr>
<tr>
<td>( P_5 )</td>
<td>254540, 310155, 378014, 421134, 432795, 483405, 486590, \cdots</td>
</tr>
<tr>
<td>( P_6 )</td>
<td>11243154, 13516580, 16473170, 16701684, 17348330, 19286805, \cdots</td>
</tr>
<tr>
<td>( P_7 )</td>
<td>965009045, 1068044054, 1168027146, 1177173074, 1209907985, \cdots</td>
</tr>
</tbody>
</table>

Table 3: Successive maximal runs of size 2

We have also computed the first maximal runs of various sizes \( r \geq 3 \) in \( P_n \) for \( 3 \leq n \leq 6 \). Table 4 summarises this data.

From Tables 2 and 4 we have \( r(3) \geq 16, r(4) \geq 12, r(5) \geq 5, r(6) \geq 3 \) and \( r(7) \geq 2 \). We have also seen that \( r(8) \geq 2 \). It is noteworthy that the data in Table 4 is not monotonic: the first maximal run of size 14 in \( P_3 \) precedes the first maximal runs of sizes 12 and 13, and the first maximal run of size 7 in \( P_4 \) precedes the first maximal run of size 6. We have already seen this phenomenon in \( P_2 \), where the first maximal run of size 8 precedes the first maximal run of size 7.
Table 4: First maximal runs of increasing size

<table>
<thead>
<tr>
<th>$r$</th>
<th>$P_3$</th>
<th>$P_4$</th>
<th>$P_5$</th>
<th>$P_6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>230</td>
<td>7314</td>
<td>254540</td>
<td>11243154</td>
</tr>
<tr>
<td>3</td>
<td>644</td>
<td>37960</td>
<td>1042404</td>
<td>481098980</td>
</tr>
<tr>
<td>4</td>
<td>1308</td>
<td>134043</td>
<td>21871365</td>
<td>...</td>
</tr>
<tr>
<td>5</td>
<td>2664</td>
<td>357642</td>
<td>129963314</td>
<td>...</td>
</tr>
<tr>
<td>6</td>
<td>6850</td>
<td>2720780</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>10280</td>
<td>1217250</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>39693</td>
<td>14273478</td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>44360</td>
<td>44939642</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>48919</td>
<td>76067298</td>
<td></td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>218972</td>
<td>163459742</td>
<td></td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>534078</td>
<td>547163235</td>
<td></td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>2699915</td>
<td>...</td>
<td></td>
<td></td>
</tr>
<tr>
<td>14</td>
<td>526095</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>17233173</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>16</td>
<td>127890362</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Let us briefly consider lower bounds for the starters of runs of size $r \geq 3$. Of course, the square root lower bound $A$ for runs of size 2 is also a lower bound for longer runs, but we want something stronger. Suppose $\{a, a+1, a+2\} \subset P_n$. The only divisor that can be common to two of these integers is 2. If $2|a+1$ there is no common divisor, so $a(a+1)(a+2) \geq p_1p_2\cdots p_{3n}$, the product of the first 3n primes. If $2|a$ then 2 is a common divisor and $8|a(a+2)$, so $a(a+1)(a+2) \geq 4p_1p_2\cdots p_{3n-1}$. Since $p_{3n} > 4$, and $(a+1)^2 > a(a+2)$ we deduce the general lower bound

$$a \geq [2(p_2p_3\cdots p_{3n-1})^{1/3}]$$

Let us denote this bound by $B$. Similar reasoning yields lower bounds for the starters of longer runs, but we shall confine our discussion to $B$.

If $n = 2$ then $A = 14$ and $B = 20$, and these are precisely the starters of the first runs of sizes 2 and 3 in $P_2$. If $n = 3$ then $[230, 2] \subset [A, 2A]$ and $[644, 3] \subset [B, 2B]$ where $A = 173$ and $B = 338$. And so on. Perhaps the first maximal run of size 3 in $P_n$ always occurs in the interval $[B, 2^nB]$. 

11
7 Nontrivial runs in $P_n$ for $n \geq 3$

Our computational results certainly confirm that $r(n) \geq 2$ for $n \leq 8$ But it is not obvious that there are nontrivial runs in $P_n$ for every $n$. In this section we shall show how to make new runs from old, in particular, how to use suitable sets of 4 neighbouring integers in $P_n$ to produce pairs of consecutive integers in $P_{2n-1}$.

For positive integers $a, s, t$ with $s < t$, the integers $a(a + s + t)$ and $(a + s)(a + t)$ differ by $st$. By imposing appropriate conditions on the divisors of $a, a + s, a + t$ and $a + s + t$, we can ensure that

$$b = \frac{1}{st}a(a + s + t), \quad b + 1 = \frac{1}{st}(a + s)(a + t)$$

are consecutive integers with equally many principal divisors. The simplest case is when $s = 1$ and $t = 2$:

**Theorem 3** If $\{a, a+1, a+2, a+3\} \subset P_n$ and $12|a(a+3)$ and $b + 1 = \frac{1}{2}(a+1)(a+2)$ are consecutive integers in $P_{2n-1}$.

**Proof.** Since $a$ and $a + 3$ differ by 3, if $12|a(a+3)$ then exactly one of $a$ and $a + 3$ is divisible by 4, the other is odd and both are divisible by 3. Hence the principal divisors of $b = \frac{1}{2}a(a+3)$ include a power of 2, and the product of powers of 3 in $a$ and $a + 3$, so $b \in P_{2n-1}$. Also $a + 1$ and $a + 2$ are relatively prime and $b + 1$ is odd, so $b + 1 \in P_{2n-1}$. $\blacksquare$

For example, with $a = 33$ and $(s, t) = (1, 2)$, the run $[33, 4] \subset P_2$ yields $b = 594 = \frac{1}{2}33.36 = 2.3^2.11$ and $b + 1 = 595 = \frac{1}{2}34.35 = 5.7.17$, so $\{b, b + 1\} \subset P_3$.

By taking other choices for $a, s, t$ so that $\{a, a + s, a + t, a + s + t\} \subset P_n^*$, we can construct $\{b, b + 1\} = \{(a + s + t)/st, (a + s)(a + t)/st\} \subset P_m$, where $P_m = P_{2n-1}$ when $P_n^* = P_n$, and $P_m = P_{2n}$ when $P_n^*$ is a suitable union of two or more rank sets. Table 5 summarises some illustrative examples, based on the maximal runs $[33, 4], [141, 8]$ and $[2302, 7]$ in $P_2$, and $[1308, 4]$, the first run of size 4 in $P_3$. Note that extending into the neighbourhood of $[2302, 7]$ yields some pairs $\{b, b + 1\} \subset P_4$.

The first maximal run of size 4 in $P_3$ is $[a, 4]$ where $a = 21871365$. Since $a + 3$ is a multiple of 12, we have

$$b = \frac{1}{2}a(a + 3) = 2^2.3^3.5.29.31.41.137.239.367$$

$$b + 1 = \frac{1}{2}(a + 1)(a + 2) = 7.11.17.23.37.61.97.131.277$$

so $\{b, b + 1\} \subset P_3$ with $b = 239178336288660$. Hence $r(9) \geq 2$. 

12
In this section we shall show how the simultaneous occurrence of primes in two arithmetic sequences leads to pairs of consecutive integers in $P_n$. Our approach is based on the famous 1837 theorem of Dirichlet (see [2]) on primes in an arithmetic sequence.

**Theorem 4 (Dirichlet)** For any coprime positive integers $m$ and $r$, the arithmetic sequence $\{km + r : k \geq 0\}$ contains infinitely many primes.

A generalisation of Dirichlet’s theorem, which has not yet been proven but appears very likely to be true, is:

**Conjecture 2** If $m, m', r, r'$ are positive integers with $\gcd\{m, r\} = 1$ and $\gcd\{m', r'\} = 1$, there are infinitely many positive integers $k$ such that $km + r$ and $km' + r'$ are both prime.

For example, $2k + 1$ and $3k + 2$ are simultaneously prime when $k = 1, 3, 5, 9, 15, \ldots$. The corresponding pairs are $(2k+1, 3k+2) = (3, 5), (7, 11), (11, 17), (19, 29), (31, 47), \ldots$. This is of interest in our present context because $b = 3(2k + 1)$ and $b + 1 = 2(3k + 2)$ are consecutive integers, so $\{b, b + 1\} \subset P_2$ whenever $2k + 1$ and $3k + 2$ are primes greater than 3. We deduce that $(21, 22), (33, 34), (57, 58), (93, 94), \ldots$ are pairs of consecutive integers in $P_2$. Generalising this example, we have

<table>
<thead>
<tr>
<th>$P_n^*$</th>
<th>$s$</th>
<th>$t$</th>
<th>$a$</th>
<th>$b$</th>
<th>$P_m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_2$</td>
<td>1</td>
<td>2</td>
<td>33</td>
<td>594</td>
<td>$P_3$</td>
</tr>
<tr>
<td>$P_2$</td>
<td>1</td>
<td>2</td>
<td>141</td>
<td>10152</td>
<td>$P_3$</td>
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<tr>
<td>$P_2$</td>
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<td>2</td>
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<td>10584</td>
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</tr>
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<td>3</td>
<td>141</td>
<td>6815</td>
<td>$P_3$</td>
</tr>
<tr>
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<td>3</td>
<td>144</td>
<td>7104</td>
<td>$P_3$</td>
</tr>
<tr>
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<td>2</td>
<td>2304</td>
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</tr>
<tr>
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<td>3</td>
<td>2303</td>
<td>1771007</td>
<td>$P_3$</td>
</tr>
<tr>
<td>$P_2$</td>
<td>1</td>
<td>3</td>
<td>2304</td>
<td>1772554</td>
<td>$P_3$</td>
</tr>
<tr>
<td>$P_2$</td>
<td>1</td>
<td>4</td>
<td>2303</td>
<td>1328831</td>
<td>$P_3$</td>
</tr>
<tr>
<td>$P_2 \cup P_3$</td>
<td>1</td>
<td>3</td>
<td>2300</td>
<td>1766400</td>
<td>$P_4$</td>
</tr>
<tr>
<td>$P_2 \cup P_3$</td>
<td>2</td>
<td>3</td>
<td>2310</td>
<td>891275</td>
<td>$P_4$</td>
</tr>
<tr>
<td>$P_2 \cup P_3$</td>
<td>2</td>
<td>3</td>
<td>2313</td>
<td>893589</td>
<td>$P_4$</td>
</tr>
<tr>
<td>$P_3$</td>
<td>1</td>
<td>2</td>
<td>1308</td>
<td>857394</td>
<td>$P_5$</td>
</tr>
</tbody>
</table>

Table 5: New runs from old

8 Matched primes in arithmetic sequences
Theorem 5 If Conjecture 2 is true, there are infinitely many pairs of consecutive integers in $P_n$, for each $n \geq 2$.

Proof. For some $n \geq 2$ let $m = p_1 \cdots p_{n-1}$ and $m' = p_n \cdots p_{2n-2}$ be the product of the first $n - 1$ primes and the next $n - 1$ primes, respectively. Let $x_0$ be the smallest positive solution of the simultaneous congruences $x \equiv 1 \pmod{m}$ and $x \equiv 0 \pmod{m'}$. The Chinese Remainder Theorem ensures that $x_0$ exists, and the general solution is $x \equiv x_0 \pmod{mm'}$. Then there are positive integers $r$ and $r'$ such that $x_0 - 1 = mr'$ and $x_0 = m'r$, so the general solution satisfies

$$x = m'r + kmm' = m'(km + r)$$
$$x - 1 = mr' + kmm' = m(km' + r')$$

where $k$ runs through the integers. Now $x_0$ and $x_0 - 1$ are coprime; hence $\text{gcd}\{mr', m'r\} = 1$, so $\text{gcd}\{m, r\} = \text{gcd}\{m', r'\} = 1$. If Conjecture 2 holds, it follows that there are infinitely many positive integers $k$ such that $p = km + r$ and $q = km' + r'$ are simultaneously prime. If $k$ is large enough, then $p$ and $q$ are not among the first $2n - 2$ primes, so $b = mq$ and $b + 1 = m'p$ are consecutive integers with $n$ principal divisors (all of which happen to be prime). □

For example, when $n = 10$ we have

$$m = 2.3.5.7.11.13.17.19.23 = 223092870$$
$$m' = 29.31.37.41.43.47.53.59.61 = 525737919635921.$$

The smallest positive solution to $x \equiv 1 \pmod{m}$ and $x \equiv 0 \pmod{m'}$ is $x_0 = 6949903578918639188851$,

so $x_0 - 1 = mr'$ and $x_0 = m'r$ give $r = 13219331$ and $r' = 31152513206355$. The sequences $\{km + r : k \geq 0\}$ and $\{km' + r' : k \geq 0\}$ are simultaneously prime when $k = 26, 38, 74, \cdots$, so the smallest pair of matched primes is $(p, q) = (5813633951, 13700338423740301)$ yielding $\{b, b+1\} = \{mq, m'p\} \subset P_{10}$ with $b = 3056447818923499884753870$. Hence $r(10) \geq 2$.

9 Upper bounds on $r(n)$ for $n \geq 3$

In the Introduction we noted that no run of consecutive integers in $P_n$ can contain a multiple of $M = p_1p_2 \cdots p_{n+1}$, the product of the first $n + 1$. 


primes, so \( r(n) < M \). Thus \( r(2) < 30 \). Eventually we proved \( 8 \leq r(2) \leq 9 \). As a first step toward this result, we showed that no run in \( P_2 \) contains more than one multiple of 6, which immediately implies the improved upper bound \( r(2) < 12 \). We shall now show that the same ideas yield corresponding results for \( n \geq 3 \).

**Theorem 6** For any positive integer \( n \), let \( N = p_1 p_2 \cdots p_n \) be the product of the first \( n \) primes, and let \( b \) be the largest integer such that no prime factor of the product \( b(b+1) \) exceeds \( p_n \). Then no run of consecutive integers greater than \( bN \) in \( P_n \) contains more than one multiple of \( N \).

**Proof.** For any integer \( b \), if no prime factor of the product \( b(b+1) \) exceeds \( p_n \), then \( \{b, b+1\} \subset S(p_1, p_2, \cdots, p_n) \) and conversely. By Størmer’s Theorem, there exists a largest integer \( b \) with this property. Then \( bN \) and \( (b+1)N \) are the last two consecutive multiples of \( N \) in \( P_n \), and suppose \( R \) contains at least one multiple of \( N \). Let \( aN \) be the smallest multiple of \( N \) in \( R \). Then \( aN \in P_n \) and \( N \in P_n \cap S(p_1, p_2, \cdots, p_n) \), so \( a \in S(p_1, p_2, \cdots, p_n) \). But \( a > b \), so \( a+1 \notin S(p_1, p_2, \cdots, p_n) \). Thus \( a+1 \) contains a prime factor \( p > p_n \), and \( (a+1)N \) has at least \( n+1 \) principal divisors. Thus \( (a+1)N \notin P_n \), so \( (a+1)N \notin R \), and the theorem follows.

**Corollary 1** No run of consecutive integers in \( P_3 \) contains more than one multiple of 30, and no run of consecutive integers in \( P_4 \) contains more than one multiple of 210.

**Proof.** Since \( N = 30 \) is the product of the first 3 primes, and \( b = 80 \) is the largest integer such that \( \{b, b+1\} \subset S(2, 3, 5) \), the theorem ensures that no run of consecutive integers greater than \( bN = 2400 \) contains more than one multiple of 30. On the other hand, the gap between each pair of consecutive primes up to 2411, the first prime greater than \( bN \), is less than 30, with one exception. The exceptional pair is \( (1327, 1361) \), which differ by 34. Since 1350 is the only multiple of 30 between 1327 and 1361, it follows that among the nonnegative integers up to \( (b+1)N = 2430 \), there is at least one prime between every pair of consecutive multiples of 30. Hence no run of consecutive integers in \( P_3 \) contains two consecutive multiples of 30.

A similar argument applies for \( P_4 \), with \( N = 210 \) and \( b = 4374 \). The gaps between consecutive primes up to 918563, the first prime greater than \( bN \), are all less than 210, so the claimed result follows. Indeed, the largest gap between consecutive primes up to 918563 is 114, achieved by the pair \( (492113, 492227) \).
Corollary 2 Upper bounds on the size of maximal runs in $P_3$ and $P_4$ are $r(3) \leq 59$ and $r(4) \leq 419$.

Proof. This follows immediately from Corollary 1. ■

10 Closing remarks

An extension of Størmer’s Theorem shows that for any finite set of primes $P$ and any positive constant $c$ there are only finitely many pairs of integers in $S(P)$ which differ by $c$. As noted in [7], this follows from a theorem of Alan Baker on logarithms of algebraic numbers [1].

One of the most intriguing questions left open in our discussion is whether there are pairs of consecutive integers in $P_n$ for every $n \geq 1$. It seems likely to us that in fact there are infinitely many such pairs for every $n \geq 1$, but we realize that this is a bold conjecture. We showed in Theorem 5 that this conjecture holds for every $n \geq 2$ if Conjecture 2 holds. Its truth for $n = 1$ is equivalent to the conjecture that either there are infinitely many Fermat primes (which seems unlikely) or else there are infinitely many Mersenne primes (which seems more likely). Conjecture 2 does not seem to imply that conjecture, but it certainly implies that there are infinitely many twin primes, a notorious unproven conjecture.

Perhaps it is also true for every $n \geq 1$ that there are only finitely many runs of size greater than $N$ in $P_n$, where $N$ is the product of the first $n$ primes.

The reader interested in principal divisors of a positive integer will find another discussion of them in [5]. In that paper the focus is on the sum of principal divisors.

Acknowledgement The first author is grateful for hospitality, during much of the work on this paper, from the School of Mathematical and Physical Sciences of The University of Newcastle, where he holds a conjoint professorship.

References

[1] A. Baker, Linear forms in the logarithms of algebraic numbers (IV), Mathematika 15 (1968), 204-216.


11 Appendix (Not for publication)

To assist the refereeing of this paper, we append some further data and notes. Inclusion of these would unduly increase the length of the published paper, and probably would overdo the amount of detail the general reader might tolerate. However, making this information available for the refereeing process should help in the checking of details.

11.1 Lower bounds in Section 6

Corresponding to \( A \) and \( B \), the lower bound for \( \{a, a+1, a+2, a+3\} \subset P_n \) is \( a + 1 \geq \lceil (72p_3p_4 \cdots p_{4n-2})^{1/4} \rceil \). This follows by noting that two of the factors have common divisor 2, and two may have common divisor 3, so

\[
(a + 2)^4 > a(a + 1)(a + 2)(a + 3) \geq 4.3p_1p_2 \cdots p_{4n-2}.
\]

The corresponding lower bound for \( \{a, a+1, a+2, a+3, a+4\} \subset P_n \) is

\[
a + 1 \geq \lceil (144p_3p_4 \cdots p_{5n-3})^{1/5} \rceil
\]

because three of the factors can have common divisor 2, in which case \( 16|a(a + 2)(a + 4) \), and two of the factors can have common divisor 3, so

\[
(a + 2)^5 > a(a + 1)(a + 2)(a + 3)(a + 4) \geq 8.3p_1p_2 \cdots p_{5n-3}
\]

11.2 Computational details for Section 7

The run \([141, 8] \subset P_2\) yields two examples with \((s, t) = (1, 2)\) : we have \(\{10152, 10153\} \subset P_3\) since \(10152 = 1^3.141.144 = 2^4.3^3.47\) and \(10153 = 1^3.142.143 = 11.13.73\); also \(\{10584, 10585\} \subset P_3\) since \(10584 = 1^3.144.145 = 2^3.3^3.72\) and \(10585 = 1^3.145.146 = 5.29.73\). The same run yields two examples with \((s, t) = (1, 3)\) : we have \(\{6815, 6816\} \subset P_3\) since \(6815 = 1^3.141.145 = 5.29.47\) and \(6816 = 1^3.142.143 = 2^5.3.71\); also \(\{7104, 7105\} \subset P_3\) since \(7104 = 1^3.144.148 = 2^6.3.37\) and \(7105 = 1^3.145.147 = 5.7^2.29\).

The run \([2302, 7] \subset P_2\) yields several examples: with \((s, t) = (1, 2)\) and \(a = 2304\), with \((s, t) = (1, 3)\) and \(a = 2303\) or 2304, and with \((s, t) = (1, 4)\) and \(a = 2303\), we have \(\{b, b+1\} \subset P_4\) for \(b = 2657664, 1771007, 1772544\) and \(1328831\) respectively. By extending into the neighbourhood of \([2302, 7]\) we obtain three very interesting examples with \(\{b, b+1\} \subset P_4\) and using \(2300 \in P_3\), \(2301 \in P_3\), \(2310 \in P_2\), \(2312 \in P_2\), \(2313 \in P_2\), \(2315 \in P_2\), \(2316 \in P_3\), \(2318 \in P_3\).
With \((s, t) = (1, 3)\) and \(a = 2300\), we have

\[
\begin{align*}
b &= \frac{1}{3} \cdot 2300 \cdot 2304 = 2^{10} \cdot 3 \cdot 5^2 \cdot 23 = 1766400 \\
b + 1 &= \frac{1}{3} \cdot 2301 \cdot 2303 = 7^2 \cdot 13 \cdot 47.59.
\end{align*}
\]

With \((s, t) = (2, 3)\) and \(a = 2310\), we have

\[
\begin{align*}
b &= \frac{1}{6} \cdot 2310 \cdot 2315 = 5^2 \cdot 7 \cdot 11 \cdot 463 = 891275 \\
b + 1 &= \frac{1}{6} \cdot 2312 \cdot 2313 = 2^2 \cdot 3 \cdot 17 \cdot 257.
\end{align*}
\]

With \((s, t) = (2, 3)\) and \(a = 2313\), we have

\[
\begin{align*}
b &= \frac{1}{6} \cdot 2313 \cdot 2318 = 3 \cdot 19 \cdot 61 \cdot 257 = 893589 \\
b + 1 &= \frac{1}{6} \cdot 2315 \cdot 2316 = 2^5 \cdot 193 \cdot 463.
\end{align*}
\]

The first maximal run of size 4 in \(P_3\) is \([a, 4]\) with \(a = 1308\). Since \(12|a\), we have

\[
\begin{align*}
b &= \frac{1}{2} a(a + 3) = 2 \cdot 3^2 \cdot 19 \cdot 23 = 108348 \\
b + 1 &= \frac{1}{2} (a + 1)(a + 2) = 2 \cdot 5 \cdot 7 \cdot 11 \cdot 17 = 122755.
\end{align*}
\]

so \(\{b, b + 1\} \subset P_5\) with \(b = 857394\).

### 11.3 Proof of corollaries to Theorem 6

**Proof of Corollary 1, second part**

Similarly, \(N = 210\) is the product of the first 4 primes and \(b = 4374\) is the largest integer such that \(\{b, b + 1\} \subset S(2, 3, 5, 7)\). Hence the theorem implies that no run of consecutive integers greater than \(bN = 918540\) in \(P_4\) contains more than one multiple of 210.

On the other hand, the gap between each pair of successive primes up to 918563, the first prime greater than \(bN\), is less than 210. Indeed, the largest gap is 114, attained by the pair (492113, 492227). Thus, among the nonnegative integers less than \((b+1)N = 918750\), there is at least one prime between consecutive multiples of
210. Hence no run of consecutive integers in $P_4$ contains two consecutive multiples
of 210.

**Proof of Corollary 2**

Any 60 consecutive integers contain two multiples of 30, so Corollary 1 implies
that every run of consecutive integers in $P_3$ has size less than 60. Similarly, every
run of consecutive integers in $P_4$ has size less than $2 \times 210 = 420$. 