Random Walk Integrals

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Abstract

We study the expected distance of a two-dimensional walk in the plane with unit steps in random directions. A series evaluation and recursions are obtained making it possible to explicitly formulate this distance for small number of steps. Closed form expressions for all the moments of a 2-step and a 3-step walk are given, and a formula is conjectured for the 4-step walk. Heavy use is made of the analytic continuation of the underlying integral.

1 Introduction, History and Preliminaries

We consider, for various values of $s$, the $n$-dimensional integral

$$W_n(s) := \int_{[0,1]^n} \left| \sum_{k=1}^n e^{2\pi i x_k} \right|^s \, dx$$

which occurs in the theory of uniform random walk integrals in the plane, where at each step a unit-step is taken in a random direction, see Figure 1. As such, the integral (1) expresses the $s$-th moment of the distance to the origin after $n$ steps. Particularly interesting is the special case of the expected distance after $n$ steps for which we just write $W_n := W_n(1)$.

A lot is known about the one-dimensional random walk. For instance, its expected distance after $n$ unit-steps is $(n-1)!!/(n-2)!!$ when $n$ is even, and $n!!/(n-1)!!$ when $n$ is odd. Here $n!!$ is the double factorial. Asymptotically this distance behaves like $\sqrt{2n}/\pi$.

For the two-dimensional walk no such explicit expressions were known, though the expected value for the root-mean-square distance is well known to be just $\sqrt{n}$; in this case the implicit square root in (1) disappears which greatly simplifies the problem.
The term “random walk” first appears in a question by Karl Pearson in *Nature* in 1905 [Pea1905]. He asked for the probability distribution of a two-dimensional random walk couched in the language of how far a “rambler” (hill walker) might walk. This triggered a response by Lord Rayleigh [Ray1905] just one week later. Raleigh replied that he had considered the problem earlier in the context of composition of vibrations of random phases, and gave the probability distribution $\frac{2x}{n}e^{-x^2/n}$ for large $n$. This quickly leads to a good approximation for $W_n(s)$ for large $n$ and fixed $s = 1, 2, 3, \ldots$.

Another week later, Pearson again wrote in *Nature*, see [Pea1905b], to note that G. J. Bennett had given a solution for the probability distribution for $n = 3$ which can be written in terms of the complete elliptic integral of the first kind $K$. This density function can be written as

$$p_3(x) = \text{Re} \left( \frac{\sqrt{x}}{\pi^2} K \left( \sqrt{\frac{(x + 1)^3(3 - x)}{16x}} \right) \right),$$  \hspace{1cm} (2)

see, e.g., [Hug95] and [Pea1906]. Unfortunately this solution is quite clumsy to work with. Pearson concluded that there was still great interest in the case of small $n$ which as he had noted is dramatically different from that of large $n$.

In further response to Pearson, Kluyver [Klu1906] made a lovely analysis of the cumulative distribution function of the distance traveled by a rambler in the plane for various choices of step length.

![Random walks in the plane](image)

(a) Several 4-step walks  \hspace{1cm} (b) A 500-step walk

Figure 1: Random walks in the plane.

Although much had been done in generalizing the problem, for instance allowing walks in three dimensions—where the analysis is somewhat simpler, varying the step lengths, confining the walks to different kinds of lattices, and calculating whether and when the walker would return to the origin (an excellent source of these results can be found in
\[\begin{array}{|c|c|c|c|c|c|}
\hline
n & s = 2 & s = 4 & s = 6 & s = 8 & s = 10 \\
\hline
2 & 2 & 6 & 20 & 70 & 252 \quad \text{A000984} \\
3 & 3 & 15 & 93 & 639 & 4653 \quad \text{A002893} \\
4 & 4 & 28 & 256 & 2716 & 31504 \quad \text{A002895} \\
5 & 5 & 45 & 545 & 7885 & 127905 \\
6 & 6 & 66 & 996 & 18306 & 384156 \\
\hline
\end{array}\]

Table 1: \(W_n(s)\) at even integers.

[Hug95]), apparently virtually no new light was shed on the original problem after the very productive period aforementioned. The moments, in particular for \(n = 3\) and \(n = 4\) have not been studied, and closed forms had not been obtained before. We wish to stress that progress was made because the techniques we used, for instance analysis of Meijer G-functions and their relationship with generalized hypergeometric series, were fully developed only much later in the 20th century. In fact, our results provide some striking closed forms for a large class of non-trivial Meijer G-functions as is further studied in [BSW10].

Applications of two-dimensional random walks are numerous and well known; for instance, [Hug95] mentions that they may be used to model the random migration of an organism possessing flagella; analysing the superposition of waves (e.g., from a laser beam bouncing off an irregular surface); and vibrations of arbitrary frequencies. The subject also finds use in Brownian motion and quantum chemistry.

We learned of the special case for \(s = 1\) of (1) from the whiteboard in the common room at the University of New South Wales. It had been written down by Peter Donovan as a generalization of a discrete cryptographic problem, as discussed in [Don09]. Some numerical values of \(W_n\) evaluated at integers are shown in Tables 1 and 2. One immediately notices the apparent integrality of the sequences for the even moments—which are the moments of the squared expected distance, and that the square root for \(s = 2\) gives the root-mean-square distance \(\sqrt{n}\). Several of these sequences were found in the Online Encyclopedia of Integer Sequences [Slo09]. By experimentation and some sketchy arguments we quickly conjectured and strongly believed that, for \(k\) a nonnegative integer

\[W_3(k) = \Re\, _3F_2\left(\frac{1}{2}, -\frac{k}{2}, -\frac{k}{2}; \frac{1}{4}\right).\]  \hspace{1cm} (3)

Appropriately defined, (3) also holds for negative odd integers. The reason for (3) was long a mystery, but it will be explained at the end of the paper.

In Section 2 we present an infinite series expression for \(W_n(s)\). From this it then follows that the even moments \(W_n(2k)\) are given by integer sequences. The combinatorial features of \(f_n(k) := W_n(2k)\), \(k\) a nonnegative integer, are then studied in Section 3. We show that there is a recurrence relation for the numbers \(f_n(k)\) and confirm the observation from
Table 2: \( W_n(s) \) at odd integers.

<table>
<thead>
<tr>
<th>( n )</th>
<th>( s = 1 )</th>
<th>( s = 3 )</th>
<th>( s = 5 )</th>
<th>( s = 7 )</th>
<th>( s = 9 )</th>
</tr>
</thead>
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<tr>
<td>2</td>
<td>1.27324</td>
<td>3.39531</td>
<td>10.8650</td>
<td>37.2514</td>
<td>132.449</td>
</tr>
<tr>
<td>3</td>
<td>1.57460</td>
<td>6.45168</td>
<td>36.7052</td>
<td>241.544</td>
<td>1714.62</td>
</tr>
<tr>
<td>4</td>
<td>1.79909</td>
<td>10.1207</td>
<td>82.6515</td>
<td>822.273</td>
<td>9169.62</td>
</tr>
<tr>
<td>5</td>
<td>2.00816</td>
<td>14.2896</td>
<td>152.316</td>
<td>2037.14</td>
<td>31393.1</td>
</tr>
<tr>
<td>6</td>
<td>2.19386</td>
<td>18.9133</td>
<td>248.759</td>
<td>4186.19</td>
<td>82718.9</td>
</tr>
</tbody>
</table>

Table 1 that the last digit in the column for \( s = 10 \) is always \( n \mod 10 \). The discovery of (3) was precipitated by the form of \( f_3 \) given in (9).

In Section 4 some analytic results are given, and the recursions for \( f_n(k) \) are lifted to \( W_n(s) \) by the use of Carlson’s theorem. The recursions for \( n = 2, 3, 4, 5 \) are given explicitly as an example. These recursions then give further information regarding the pole structure of \( W_n(s) \). Plots of the analytic continuation of \( W_n(s) \) on the negative real axis are given in Figure 2. From here we conjecture the recursion

\[
W_{2n}(s) = \sum_{j \geq 0} \left( \frac{s}{2} \right)^j W_{2n-1}(s-2j). 
\]

In Section 6 we explore the underlying probability model more closely starting with work of Kluyver [Klu1906]. This leads to an alternative tractable form for \( W_3(s) \) that eventually allows us to prove (3).

The ability to compute the moments to reasonably high accuracy was central to our work. This is not easy, and so some comments on obtaining high precision numerical evaluations of \( W_n(s) \) are given in Appendix A.2.

2  A Series Evaluation of \( W_n(s) \)

In this section a series evaluation of \( W_n(s) \) is presented. For our purposes, its main interest lies in the fact that it specializes to an expression for \( W_n(2k) \), \( k \geq 0 \) an integer, as a finite sum. This sum and its combinatorics will be studied in Section 3. We therefore restrict ourselves here to give a short formal proof of the series evaluation which ignores any convergence issues; note that these don’t arise when \( s \) is an even integer. A more complete and elementary proof is given in Appendix A.1. Chronologically our first one, as a side-product it yields other interesting integral evaluations.

**Theorem 1.** For \( n = 1, 2, \ldots \) and \( Re \ s \geq 0 \) one has

\[
W_n(s) = n^s \sum_{m \geq 0} (-1)^m \binom{s/2}{m} \sum_{k=0}^{m} \frac{(-1)^k}{n^{2k}} \binom{m}{k} \sum_{a_1 + \cdots + a_n = k} \left( \frac{k}{a_1, \ldots, a_n} \right)^2.
\]
Proof. In the spirit of the residue theorem of complex analysis, if \( f(x_1, \ldots, x_n) \) has a
Laurent expansion around the origin then

\[
\text{ct} f(x_1, \ldots, x_n) = \int_{[0,1]^n} f(e^{2\pi i x_1}, \ldots, e^{2\pi i x_n}) \, dx
\]  

where ‘ct’ denotes the operator which extracts from an expression the constant term of its
Laurent expansion. In light of (5), the integral definition (1) of \( W_n(s) \) may be restated as

\[
W_n(s) = \text{ct} \left( (x_1 + \cdots + x_n)(1/x_1 + \cdots + 1/x_n) \right)^{s/2}.
\]  

Formally expanding the right-hand side, we obtain

\[
W_n(s) = \text{ct} \sum_{m \geq 0} \frac{(-1)^m}{n^{2m}} \binom{s/2}{m} \left( n^2 - (x_1 + \cdots + x_n)(1/x_1 + \cdots + 1/x_n) \right)^m,
\]

and the claim follows from the next proposition. \( \square \)

**Proposition 1.** For \( n, m \) positive integers the constant term of

\[
(n^2 - (x_1 + \cdots + x_n)(1/x_1 + \cdots + 1/x_n))^m
\]
Proof. After expanding \((n^2 - (x_1 + \cdots + x_n)(1/x_1 + \cdots + 1/x_n))^m\) by the binomial theorem, it remains to show that the constant term of \(((x_1 + \cdots + x_n)(1/x_1 + \cdots + 1/x_n))^k\) is

\[
\sum_{a_1 + \cdots + a_n = k} \binom{k}{a_1, \ldots, a_n}^2.
\]

On using the multinomial theorem,

\[
(x_1 + \cdots + x_n)^k (1/x_1 + \cdots + 1/x_n)^k
= \sum_{a_1 + \cdots + a_n = k} \binom{k}{a_1, \ldots, a_n} x_1^{a_1} \cdots x_n^{a_n} \sum_{b_1 + \cdots + b_n = k} \binom{k}{b_1, \ldots, b_n} x_1^{-b_1} \cdots x_n^{-b_n},
\]

and the constant term is now obtained by matching \(a_1 = b_1, \ldots, a_n = b_n\).

From Theorem 1 we learn additionally that the even moments are integer sequences—actually sums of squares of multinomials—as detailed by the following corollary.

**Corollary 1.** For nonnegative integers \(k\) and \(n\),

\[
W_n(2k) = \sum_{a_1 + \cdots + a_n = k} \binom{k}{a_1, \ldots, a_n}^2.
\]

**Proof.** This follows easily from (6) since in that case the right-hand side can be finitely expanded.

Alternatively, the result is a consequence of Theorem 1 and the fact that the binomial transform, \(\{b_m\}_{m \geq 0}\), of a sequence \(\{a_k\}_{k \geq 0}\), where

\[
b_m := \sum_{k=0}^{m} (-1)^k \binom{m}{k} a_k,
\]

is an involution [CG96].

3 Combinatorial Features

In light of Corollary 1, we consider the combinatorial sums

\[
f_n(k) := \sum_{a_1 + \cdots + a_n = k} \binom{k}{a_1, \ldots, a_n}^2
\]

(7)
of multinomial coefficients squared. These numbers also appear in [RS09] in the following way: \( f_n(k) \) counts the number of abelian squares of length \( 2k \) over an alphabet with \( n \) letters (that is strings \( xx' \) of length \( 2k \) from an alphabet with \( n \) letters such that \( x' \) is a permutation of \( x \)). It is not hard to see that

\[
fn_1+n_2(k) = \sum_{j=0}^{k} \binom{k}{j}^2 fn_1(j) fn_2(k-j),
\]

for two non-overlapping alphabets with \( n_1 \) and \( n_2 \) letters. In particular, we may use (8) to obtain \( f_1(k) = 1 \), \( f_2(k) = \binom{2k}{k} \), as well as

\[
f_3(k) = \sum_{j=0}^{k} \binom{k}{j}^2 \binom{2j}{j} = 3F_2 \left( \frac{1}{2}, -k, -k \mid 1 \right) = \binom{2k}{k} 3F_2 \left( -k, -k, -k \mid 1, -k + \frac{1}{2} \right),
\]

\[
f_4(k) = \sum_{j=0}^{k} \binom{k}{j}^2 \binom{2j}{j} \binom{2(k-j)}{k-j} = \binom{2k}{k} 4F_3 \left( \frac{1}{2}, -k, -k, -k \mid 1 \right).
\]

Here and below \( _pF_q \) notates the generalised hypergeometric function. In general, (8) can be used to write \( fn \) as a sum with at most \( \lceil n/2 \rceil - 1 \) summation indices.

We recall a generating function for \( (fn(k))_{k=0}^\infty \) used in [BBBG08]. Let \( I_n(z) \) denote the modified Bessel function of the first kind. Then

\[
\sum_{k \geq 0} fn(k) \frac{z^k}{(k!)^2} = \left( \sum_{k \geq 0} \frac{z^k}{(k!)^2} \right)^n = _0F_1(1; z)^n = I_0(2\sqrt{z})^n.
\]

The following theorem—which we conjectured and then found in the literature [Bar64]—will prove very useful.

**Theorem 2.** For fixed \( n \geq 2 \), the sequence \( fn(k) \) satisfies a recurrence of order \( \lambda = \lceil n/2 \rceil \) with polynomial coefficients of degree \( n - 1 \):

\[
c_{n,0}(k) fn(k) + \cdots + c_{n,\lambda}(k) fn(k + \lambda) = 0
\]

where

\[
c_{n,0}(k) = (-1)^\lambda (n!!)^2 \left( k + \frac{n}{4} \right)^{n+1-2\lambda} \prod_{j=1}^{\lambda-1} (k + j)^2,
\]

and \( c_{n,\lambda}(k) = (k + \lambda)^{n-1} \).

**Proof.** The result is established in [Bar64]. In that paper the recursions for \( 3 \leq n \leq 6 \) are given explicitly. \( \square \)
The recursions for \( n = 2, 3, 4, 5 \) are listed in Example 1 in Section 4.3.

Remark 1. For fixed \( k \), the map \( n \mapsto f_n(k) \) is a polynomial of degree \( k \). This follows from

\[
f_n(k) = \sum_{j=0}^{k} \binom{n}{j} \sum_{a_1 + \cdots + a_j = k, a_i \geq 0} \left( \prod_{a_j \geq 1} \binom{k}{a_1, \ldots, a_j} \right)^2,
\]

because the right-hand side is a linear combination (with positive coefficients only depending on \( k \)) of the polynomials \( \binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{k} \) of respective degrees 0, 1, \ldots, \( k \). From (13) the coefficient of \( \binom{n}{k} \) is seen to be \( k!^2 \). We therefore obtain the first-order approximation

\[
W_n(s) \approx n^{s/2} \Gamma(s/2 + 1)
\]

for \( n \) going to infinity, see also [Klu1906].

In particular, \( W_n(1) \approx \sqrt{n\pi}/2 \). Similarly, the coefficient of \( \binom{n}{k-1} \) is \( \frac{k-1}{4}(k!)^2 \) which gives rise to the second-order approximation

\[
(k!)^2 \binom{n}{k} + \frac{k - 1}{4} (k!)^2 \binom{n}{k-1} = k!n^k - \frac{k(k-1)}{4} k!n^{k-1} + O(n^{k-2})
\]

of \( f_n(k) \). We therefore obtain

\[
W_n(s) \approx n^{s/2-1} \left\{ \left( n - \frac{1}{2} \right) \Gamma \left( \frac{s}{2} + 1 \right) + \Gamma \left( \frac{s}{2} + 2 \right) - \frac{1}{4} \Gamma \left( \frac{s}{2} + 3 \right) \right\},
\]

which is exact for \( s = 0, 2, 4 \). In particular, \( W_n(1) \approx \sqrt{n\pi}/2 + \sqrt{\pi/n}/32 \). More general approximations are given in [Cra09].

Remark 2. It follows straight from (7) that, for primes \( p \), \( f_n(p) \equiv n \mod p \). Further, for \( k \geq 1 \), \( f_n(k) \equiv n \mod 2 \). This may be derived inductively from the recurrence (8) since, assuming that \( f_n(k) \equiv n \mod 2 \) for some \( n \) and all \( k \geq 1 \),

\[
f_{n+1}(k) = \sum_{j=0}^{k} \binom{k}{j} f_n(j) \equiv k + \sum_{j=1}^{k} \binom{k}{j} j \equiv k + \sum_{j=1, j \mod odd}^{k} \binom{k}{j} = k + 2^{k-1} \equiv k \pmod{2}.
\]

Hence for odd primes \( p \),

\[
f_n(p) \equiv n \pmod{2p}.
\]

The congruence (14) also holds for \( p = 2 \) since \( f_n(2) = (2n-1)n \), compare (13). In particular, (14) confirms that indeed the last digit in the column for \( s = 10 \) is always \( n \mod 10 \)—an observation from Table 1.
Remark 3. The integers $f_3(k)$ (respectively $f_4(k)$) also arise in physics, see for instance [BBBG08], and are referred to as hexagonal (respectively diamond) lattice integers. The corresponding entries in Sloane’s online encyclopedia [Slo09] are A002893 and A002895. We recall the following formulae [BBBG08, (186)–(188)], relating these sequences in non-obvious ways:

$$\left( \sum_{k \geq 0} f_3(k)(-x)^k \right)^2 = \sum_{k \geq 0} f_2(k)^3 \frac{x^{3k}}{((1 + x)^3(1 + 9x))^{k+\frac{1}{2}}}$$

$$= \sum_{k \geq 0} f_2(k)f_3(k) \frac{(-x(1 + x)(1 + 9x))^{k}}{((1 - 3x)(1 + 3x))^{2k+1}}$$

$$= \sum_{k \geq 0} f_4(k) \frac{x^k}{((1 + x)(1 + 9x))^{k+1}}.$$  

It would be instructive to similarly engage $f_5(k).$

\[\diamondsuit\]

4 Analytic Results

4.1 Analyticity

We start with a preliminary investigation of the analyticity of $W_n(s)$ for a given $n.$

Proposition 2. $W_n(s)$ is analytic at least for $\text{Re } s \geq 0.$

Proof. Let $s_0$ be a real number such that the integral in (1) converges for $s = s_0.$ Then we claim that $W_n(s)$ is analytic in $s$ for $\text{Re } s > s_0.$ To this end, let $s$ be such that $s_0 < \text{Re } s \leq s_0 + \lambda$ for some real $\lambda > 0.$ For any real $0 \leq a \leq n,$

$$|a^s| = a^{\text{Re } s} \leq n^\lambda a^{s_0},$$

and therefore

$$\sup_{s_0 < \text{Re } s \leq s_0 + \lambda} \int_{[0,1]^n} \left| \sum_{k=1}^{n} e^{2\pi i k \cdot x} \right|^s d\mathbf{x} \leq n^\lambda W_n(s_0) < \infty.$$  

This local boundedness implies, see for instance [Mat01], that $W_n(s)$ as defined by the integral in (1) is analytic in $s$ for $\text{Re } s > s_0.$ \[\square\]

This result will be extended in Theorem 5 and Corollary 2.
4.2 $n = 1$ and $n = 2$

The case $n = 1$ is trivial: it follows straight from the integral definition (1) that $W_1(s) = 1$.

In the case $n = 2$, direct integration of (30) with $n = 2$ yields

$$W_2(s) = 2^{s+1} \int_0^{1/2} \cos(\pi t)^s \, dt = \left( \frac{s}{s/2} \right),$$

(15)

which may also be obtained using (4). In particular, $W_2(1) = 4/\pi$. It may be worth noting that neither Maple 13 nor Mathematica 7 can evaluate $W_2(1)$ if it is entered naively, each returning the symbolic answer 0.

4.3 Functional Equations

We may lift the recursive structure of $f_n$, defined in Section 3, to $W_n$ to a fair degree on appealing to Carlson’s theorem [Tit39, 5.81]. We recall that a function $f$ is of exponential type in a region if $|f(z)| \leq Me^{d|z|}$ for some constants $M$ and $c$.

**Theorem 3** (Carlson). Let $f$ be analytic in the right half-plane $\Re z \geq 0$ and of exponential type with the additional requirement that $|f(z)| \leq Me^{d|z|}$ for some $d < \pi$ on the imaginary axis $\Re z = 0$. If $f(k) = 0$ for $k = 0, 1, 2, \ldots$ then $f(z) = 0$ identically.

**Theorem 4.** Given that $f_n(k)$ satisfies a recurrence

$$c_{n,0}(k)f_n(k) + \cdots + c_{n,\lambda}(k)f_n(k + \lambda) = 0$$

with polynomial coefficients $c_{n,j}(k)$ (see Theorem 2) then $W_n(s)$ satisfies the corresponding functional equation

$$c_{n,0}(s/2)W_n(s) + \cdots + c_{n,\lambda}(s/2)W_n(s + 2\lambda) = 0,$$

for $\Re s \geq 0$.

**Proof.** Let

$$U_n(s) := c_{n,0}(s)W_n(2s) + \cdots + c_{n,\lambda}(s)W_n(2s + 2\lambda).$$

Since $f_n(k) = W_n(2k)$ by Corollary 1, $U_n(s)$ vanishes at the nonnegative integers by assumption. Consequently, $U_n(s)$ is zero throughout the right half-plane and we are done—once we confirm that Theorem 3 applies. By Proposition 2, $W_n(s)$ is analytic for $\Re s \geq 0$. Clearly, $|W_n(s)| \leq n^{\Re s}$. Thus

$$|U_n(s)| \leq \left( |c_{n,0}(s)| + |c_{n,1}(s)|n^2 + \cdots + |c_{n,\lambda}(s)|n^{2\lambda} \right) n^{2\Re s}.$$

In particular, $U_n(s)$ is of exponential type. Further, $U_n(s)$ is polynomially bounded on the imaginary axis $\Re s = 0$. Thus $U_n$ satisfies the growth conditions of Theorem 3.
Example 1. For \( n = 2, 3, 4, 5 \) we find

\[
\begin{align*}
(s + 2)W_2(s + 2) - 4(s + 1)W_2(s) &= 0, \\
(s + 4)^2W_3(s + 4) - 2(5s^2 + 30s + 46)W_3(s + 2) + 9(s + 2)^2W_3(s) &= 0, \\
(s + 4)^3W_4(s + 4) - 4(s + 3)(5s^2 + 30s + 48)W_4(s + 2) + 64(s + 2)^3W_4(s) &= 0, \\
(s + 6)^4W_5(s + 6) - (35(s + 5)^4 + 2(5s^2 + 30s + 48)W_4(s + 2) + 64(s + 2)^3W_5(s) &= 0.
\end{align*}
\]

We note that in each case the recursion lets us determine significant information about the nature and position of any poles of \( W_n(s) \). We exploit this in the next theorem for \( n > 3 \). The case \( n = 2 \) is transparent since as determined above \( W_2(s) = (s^2)/2 \) which has simple poles at the negative odd integers.

Theorem 5. Let an integer \( n \geq 3 \) be given. The recursion guaranteed by Theorem 4 provides an analytic continuation of \( W_n(s) \) to all of the complex plane with poles of at most order two at certain negative integers.

Proof. Proposition 2 proves analyticity in the right halfplane. It is clear that the recursion given by Theorem 4 ensures an analytic continuation with poles only possible at negative integer values compatible with the recursion. Indeed, with \( \lambda = \lfloor n/2 \rfloor \) we have

\[
W_n(s) = -\frac{c_{n,1}(s/2)W_n(s + 2) + \cdots + c_{n,\lambda}(s/2)W_n(s + 2\lambda)}{c_{n,0}(s/2)}
\]

with the \( c_{n,j} \) as in (11). We observe that the right side of (16) only involves \( W_n(s + 2k) \) for \( k \geq 1 \). Therefore the least negative pole can only occur at a zero of \( c_{n,0}(s/2) \) which is explicitly given in (12). We then note that the recursion forces poles to be simple or of order two, and to be replicated as claimed.

Corollary 2. If \( n \geq 3 \) then \( W_n(s) \), as given by (1), is analytic for \( \text{Re } s > -2 \).

Proof. This follows directly from Theorem 5, the fact that \( c_{n,0}(s/2) \) given in (12) has no zero for \( s = -1 \), and the proof of Proposition 2.

In Figure 2 the analytic continuations for each of \( W_3, W_4, W_5, \) and \( W_6 \) are plotted on the real line.

Example 2. Using the recurrence given in Example 1 we find that \( W_3(s) \) has simple poles at \( s = -2, -4, -6, \ldots \), compare Figure 2(a). Moreover, the residue at \( s = -2 \) is given by \( \text{Res}_{-2}(W_3) = \frac{2}{\sqrt{3}\pi} \), and all other residues of \( W_3 \) are rational multiples thereof. This may be obtained from the integral representation given in (19) observing that, at \( s \) a negative even integer, the residue contributions are entirely from the first term.

\[
\square
\]
Example 3. Similarly, we find that $W_4$ has double poles at $-2, -4, -6, \ldots$, compare Figure 2(b). With more work, it is also possible to show that

$$\lim_{s \to -2} (s + 2)^2 W_4(s) = \frac{3}{2\pi^2}.$$  

♢

Remark 4. More generally, it would appear that Theorem 5 can be extended to show that

- for $n$ odd $W_n$ has simple poles at $-2p$ for $p = 1, 2, 3 \ldots$, while
- for $n$ even $W_n$ has simple poles at $-2p$ and $2(1 - p) - n/2$ for $p = 1, 2, 3 \ldots$ which will overlap when $4|n$.  

♢

4.4 Convolution Series

Next, we wish to lift the convolution sum (8) to $W_n(s)$. Our conjecture is:

Conjecture 1. For positive integers $n$ and complex $s$,

$$W_{2n}(s) = \sum_{j \geq 0} \left(\frac{s}{2j}\right)^2 W_{2n-1}(s - 2j).$$  \hfill (17)

It is understood that the right-hand side of (17) refers to the analytic continuation of $W_n$ as guaranteed by Theorem 5. Conjecture 1, which is consistent with the pole structure described in Remark 4, has been confirmed by David Broadhurst [Bro09] using a Bessel integral representation for $W_n$, given in (18), for $n = 2, 3, 4, 5$ and odd integers $s < 50$ to a precision of 50 digits. By (8) the conjecture clearly holds for $s$ an even positive integer. For $n = 1$ it is confirmed next.

Example 4. For $n = 1$ we obtain from (17) using $W_1(s) = 1$,

$$W_2(s) = \sum_{j \geq 0} \left(\frac{s}{2j}\right)^2 = \begin{pmatrix} s/2 \end{pmatrix} = \begin{pmatrix} s \end{pmatrix}$$

which agrees with (15).  

♢
5 Bessel integral representations

As noted such problems have a long lineage. In response to the questions posed by Pearson in *Nature*, Kluyver [Klu1906] made a lovely analysis of the cumulative distribution function of the distance traveled by a “rambler” in the plane for various fixed step lengths. In particular, for our uniform walk Kluyver provides the Bessel function representation

\[ P_n(t) = t \int_0^\infty J_1(xt) J_n'(x) \, dx. \]

Thus, \( W_n(s) = \int_0^s t^n p_n(t) \, dt \), where \( p_n = P_n' \). From here, David Broadhurst [Bro09] obtains

\[ W_n(s) = 2^{s+1-k} \frac{\Gamma(1 + \frac{s}{2})}{\Gamma(k - \frac{s}{2})} \int_0^\infty x^{2k-s-1} \left( -\frac{1}{x} \frac{d}{dx} \right)^k J_n^0(x) \, dx \]

for real \( s \); valid as long as \( 2k > s > \max(-2, -\frac{n}{2}) \). Equation (18) enabled Broadhurst [Bro09] to verify Conjecture 1 for \( n = 2, 3, 4, 5 \) and odd positive \( s < 50 \) to a precision of 50 digits.

**Remark 5.** For \( n = 3, 4 \), symbolic integration in *Mathematica* of (18) leads to interesting analytic continuations [Cra09] such as

\[ W_3(s) = \frac{1}{2^{2s+1}} \tan \left( \frac{\pi s}{2} \right) \left( \frac{s}{s-1} \right)^2 _3F_2 \left( \frac{1}{2}, \frac{1}{2} + \frac{s}{2}, \frac{1}{2}; \frac{1}{4} \right) + \left( \frac{s}{s-1} \right) \_3F_2 \left( -\frac{s}{2}, -\frac{s}{2}, -\frac{s}{2}; 1, -\frac{s-1}{2}; \frac{1}{4} \right) \],

(19)

and

\[ W_4(s) = \frac{1}{2^{2s}} \tan \left( \frac{\pi s}{2} \right) \left( \frac{s}{s-1} \right)^3 _4F_3 \left( \frac{1}{2}, \frac{1}{2} + 1, \frac{s}{2} + 1; \frac{s+3}{2}, \frac{s+3}{2}, \frac{s+3}{2}; 1 \right) + \left( \frac{s}{s-1} \right) _4F_3 \left( -\frac{s}{2}, -\frac{s}{2}, -\frac{s}{2}, -\frac{s}{2}; 1, 1, -\frac{s-1}{2}; 1 \right) \],

(20)

We note that for \( s = 2k = 0, 2, 4, \ldots \) the first term in (19) (resp. (20)) is zero and the second is a formula given in (9) (resp. (10)). Thence, one can in principle prove (19) and (20) by applying Carlson’s theorem—after showing the singularities at \( 1, 3, 5, \ldots \) are removable. A rigorous proof and more details appear in [BSW10].

6 A Probabilistic Approach

In this section, we will take a probabilistic approach so as to be able to express our quantities of interest in terms of special functions which eventually allows us to explicitly evaluate \( W_3(s) \) at odd integers.
It is elementary to express the distance $y$ of an $(n+1)$-step walk conditioned on a given distance $x$ of an $n$-step walk. By a simple application of the cosine rule we find

$$y^2 = x^2 + 1 + 2x \cos(\theta),$$

where $\theta$ is the outside angle of the triangle with sides of lengths $x$, 1, and $y$:

$$\begin{array}{c}
\triangle \\
\hline
x & \theta & 1 \\
y & \end{array}$$

It follows that the $s$-th moment of an $(n+1)$-step walk conditioned on a given distance $x$ of an $n$-step walk is

$$g_s(x) := \frac{1}{\pi} \int_0^\pi y^s \, d\theta = |x - 1|^s \, _2F_1 \left( \frac{1}{2}, -\frac{s}{2}; 1; -\frac{4x}{(x-1)^2} \right). \quad (21)$$

Here we appealed to symmetry to restrict the angle to $\theta \in [0, \pi)$. We then evaluated the integral in hypergeometric form which, for instance, can be done with the help of Mathematica. Observe that $g_s(x)$ does not depend on $n$. Since $W_{n+1}(s)$ is the $s$-th moment of the distance of an $(n + 1)$-step walk, we obtain

$$W_{n+1}(s) = \int_0^n g_s(x) \, p_n(x) \, dx, \quad (22)$$

where $p_n(x)$ is the density of the distance $x$ for an $n$-step walk. Clearly, for the 1-step walk we have $p_1(x) = \delta_1(x)$, a Dirac delta function at $x = 1$. It is also easily shown that the probability density for a 2-step walk is given by $p_2(x) = 2(\pi \sqrt{4 - x^2})^{-1}$ for $0 \leq x \leq 2$ and 0 otherwise. The density $p_3(x)$ is given in (2).

For $n = 3$, based on (9) we define

$$V_3(s) := _3F_2 \left( \frac{1}{2}, -\frac{s}{2}, -\frac{s}{2}; 1, 1; \frac{4}{4} \right), \quad (23)$$

so that by Corollary 1 and (9), $W_3(2k) = V_3(2k)$ for nonnegative integers $k$. This led us to explore $V_3(s)$ more generally numerically and so to conjecture and eventually prove the following:

**Theorem 6.** For nonnegative even integers and all odd integers $k$:

$$W_3(k) = \Re V_3(k).$$
Remark 6. Note that, for all complex $s$, the function $V_3(s)$ also satisfies the recursion given in Example 1 for $W_3(s)$—as is routine to prove symbolically. However, $V_3$ does not satisfy the growth conditions of Carlson’s Theorem 3. Thus, it yields a rather nice illustration that the hypotheses can fail.

Proof of Theorem 6. It remains to prove the result for odd integers. Since, as noted in Remark 6, for all complex $s$, the function $V_3(s)$ defined in (23) also satisfies the recursion given in Example 1, it suffices to show that the values given for $s = 1$ and $s = -1$ are correct. From (22), we have the following expression for $W_3$:

$$W_3(s) = \frac{2}{\pi} \int_{0}^{\frac{\pi}{2}} \frac{g_s(x)}{\sqrt{4 - x^2}} \, dx = \frac{2}{\pi} \int_{0}^{\frac{\pi}{2}} g_s(2\sin(t)) \, dt.$$  \hspace{1cm} (24)

For $s = 1$: equation (21), [BB87, Exercise 1c), p. 16], and Jacobi’s imaginary transformations [BB87, Exercises 7a) & 8b), p. 73] allow us to write

$$\frac{\pi}{2} g_1(x) = (x+1) E\left(\frac{2\sqrt{x}}{x+1}\right) = \text{Re} \left(2E(x) - (1-x^2)K(x)\right)$$  \hspace{1cm} (25)

where $K$ and $E$ denote the complete elliptic integrals of the first and second kind. Thus, from (24) and (25),

$$W_3(1) = \frac{4}{\pi^2} \text{Re} \int_{0}^{\pi/2} \left(2 E(2\sin(t)) - (1 - 4\sin^2(t))K(2\sin(t))\right) \, dt$$

$$= \frac{4}{\pi^2} \text{Re} \int_{0}^{\pi/2} \int_{0}^{\pi/2} \frac{2\sqrt{1 - 4\sin^2(t) \sin^2(r)}}{\sqrt{1 - 4\sin^2(t) \sin^2(r)}} \, dt \, dr$$

$$- \frac{4}{\pi^2} \text{Re} \int_{0}^{\pi/2} \int_{0}^{\pi/2} \frac{1 - 4\sin^2(t)}{\sqrt{1 - 4\sin^2(t) \sin^2(r)}} \, dt \, dr.$$

Amalgamating the two last integrals and parameterizing, we consider

$$Q(a) := \frac{4}{\pi^2} \int_{0}^{\pi/2} \int_{0}^{\pi/2} \frac{1 + a^2 \sin^2(t) - 2a^2 \sin^2(t) \sin^2(r)}{\sqrt{1 - a^2 \sin^2(t) \sin^2(r)}} \, dt \, dr.$$  \hspace{1cm} (26)

We now use the binomial theorem to integrate (26) term-by-term for $|a| < 1$ and substitute $\frac{2}{\pi} \int_{0}^{\pi/2} \sin^2m(t) \, dt = (-1)^m (-1/2)^m$ throughout. Moreover, $(-1)^m \left(\frac{a}{m}\right) = (a)_m/m!$ where the later denoted the Pochhammer symbol. Evaluation of the consequent infinite
sum produces:

\[
Q(a) = \sum_{k \geq 0} (-1)^k \left( \frac{-1/2}{k} \right)^2 \left( a^2 \left( \frac{-1/2}{k} \right) \right)^2 - a^{2k+2} \left( \frac{-1/2}{k+1} \right) \left( \frac{-1/2}{k+1} \right) - 2a^{2k+2} \left( \frac{-1/2}{k+1} \right)^2 \\
= \sum_{k \geq 0} (-1)^k a^{2k} \left( \frac{-1/2}{k} \right)^3 \frac{1}{(1-2k)^2} \\
= {_3F_2} \left( \frac{-1/2, -1/2, 1}{1, 1} \right) a^2 .
\]

Analytic continuation to \( a = 2 \) yields the claimed result as per for \( s = 1 \).

**For** \( s = -1 \): we similarly and more easily use (21) and (24) to derive

\[
W_3(-1) = \text{Re} \frac{4}{\pi^2} \int_0^{\pi/2} K(2 \sin(t)) \, dt \\
= \text{Re} \frac{4}{\pi^2} \int_0^{\pi/2} \int_0^{\pi/2} \frac{1}{\sqrt{1 - 4 \sin^2(t) \sin^2(r)}} \, dt \, dr = V_3(-1).
\]

\[\square\]

**Example 5.** Theorem 6 allows us to establish the following equivalent expressions for \( W_3(1) \):

\[
W_3(1) = \frac{4\sqrt{3}}{3} \left( _3F_2 \left( \frac{-1/2, -1/2, -1/2}{1, 1} \frac{1}{1} \right) - \frac{1}{\pi} \right) + \frac{\sqrt{3}}{24} \cdot \frac{1}{3} F_2 \left( \frac{1/2, 1/2, 1/2}{1/2, 1/2} \frac{1}{1} \right) \\
= 2\sqrt{3} \frac{K^2(k_3)}{\pi^2} + \sqrt{3} \frac{1}{K^2(k_3)} \\
= 3 \frac{2^{1/3}}{16} \pi^4 \Gamma^6 \left( \frac{1}{3} \right) + 27 \frac{2^{2/3}}{4} \pi^4 \Gamma^6 \left( \frac{2}{3} \right) .
\]

These rely on using Legendre’s identity and several Clausen-like product formulae, plus Legendre’s evaluation of \( K(k_3) \) where \( k_3 := \frac{\sqrt{3}-1}{2\sqrt{2}} \) is the third singular value as in [BB87].

More simply but similarly, we have

\[
W_3(-1) = 2\sqrt{3} \frac{K^2(k_3)}{\pi^2} = \frac{3}{16} \frac{2^{1/3}}{\pi^4} \Gamma^6 \left( \frac{1}{3} \right) .
\]

Using the recurrence presented in Example 1 it follows that similar expressions can be given for \( W_3 \) evaluated at odd integers.

In [BSW10] corresponding hypergeometric closed forms for \( W_4 \) are presented. \[\diamond\]
7 Conclusion

The behaviour of these two-dimensional walks provides a fascinating blend of probabilistic, analytic, algebraic and combinatorial challenges.

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A Appendix

A.1 A rigorous proof of Theorem 1

We begin with:

Proposition 3. For complex $s$ with $\text{Re } s \geq 0$,

$$W_n(s) = n^s \sum_{m \geq 0} (-1)^m \left( \frac{s/2}{m} \right) \left( \frac{2}{n} \right)^{2m} \int_{[0,1]^n} \left( \sum_{1 \leq i < j \leq n} \sin^2(\pi(x_j - x_i)) \right)^m \, dx. \quad (27)$$

Proof. Start with

$$\left| \sum_{k=1}^{n} e^{2\pi x_k} \right|^2 = \left( \sum_{k=1}^{n} \cos(2\pi x_k) \right)^2 + \left( \sum_{k=1}^{n} \sin(2\pi x_k) \right)^2$$

$$= \left( \sum_{i<j} \left( \cos(2\pi x_i) + \cos(2\pi x_j) \right)^2 + \left( \sin(2\pi x_i) + \sin(2\pi x_j) \right)^2 \right) - n(n-2)$$

$$= 4 \left( \sum_{i<j} \cos^2(\pi(x_j - x_i)) \right) - n(n-2)$$

$$= n^2 - 4 \left( \sum_{i<j} \sin^2(\pi(x_j - x_i)) \right).$$

Therefore, noting that binomial expansion may be applied to the integrand outside a set
of $n$-dimensional measure zero,
\[
W_n(s) = \int_{[0,1]^n} \left( n^2 - 4 \left( \sum_{i<j} \sin^2(\pi(x_j - x_i)) \right) \right)^{s/2} \, dx
\]
\[
= n^s \int_{[0,1]^n} \sum_{m \geq 0} (-1)^m \left( \frac{s/2}{m} \right) \left( \frac{2}{n} \right)^{2m} \left( \sum_{i<j} \sin^2(\pi(x_j - x_i)) \right)^m \, dx.
\]
Thus the result follows once changing the order of integration and summation is justified. Observe that if $s$ is real then $(-1)^m \left( \frac{s/2}{m} \right)$ has a fixed sign for $m > s/2$ and we can apply monotone convergence. On the other hand, if $s$ is complex then we may use
\[
\lim_{m \to \infty} \left| \frac{(s/2)_m}{(\Re s/2)_m} \right| = \left| \frac{\Gamma(-\Re s/2)}{\Gamma(-s/2)} \right|,
\]
which follows from Stirling’s approximation, and apply dominated convergence using the real case for comparison. \hfill \Box

We next evaluate the integrals in (27):

**Theorem 7.** For nonnegative integers $m$,
\[
\int_{[0,1]^n} \left( \sum_{i<j} \sin^2(\pi(x_j - x_i)) \right)^m \, dx = \left( \frac{n}{2} \right)^{2m} \sum_{k=0}^{m} \frac{(-1)^k}{n^{2k}} \binom{m}{k} \sum_{a_1 + \cdots + a_n = k} \binom{k}{a_1, \ldots, a_n}^2.
\]

**Proof.** Denote the left-hand by $I_{n,m}$. Using Proposition 1 we note that the claim is equivalent to asserting that $2^{2m}I_{n,m}$ is the constant term of
\[
(n^2 - (x_1 + \cdots + x_n)(1/x_1 + \cdots + 1/x_n))^m.
\]
Observe that
\[
(n^2 - (x_1 + \cdots + x_n)(1/x_1 + \cdots + 1/x_n))^m = \left( \sum_{1 \leq i < j \leq n} \frac{2 - x_j - x_i}{x_j - x_i} \right)^m
\]
\[
= (-1)^m \left( \sum_{1 \leq i < j \leq n} \frac{(x_j - x_i)^2}{x_i x_j} \right)^m.
\]
The result therefore follows from the next proposition. \hfill \Box

As before, we denote by ‘ct’ the operator which extracts from an expression the constant term of its Laurent expansion.
Proposition 4. For any integers $1 \leq i_1 \neq j_1, \ldots, i_m \neq j_m \leq n,$
\[
\int_{[0,1]^n} \prod_{k=1}^{m} 4 \sin^2(\pi(x_{j_k} - x_{i_k})) \, dx = (-1)^m \text{ct} \prod_{k=1}^{m} \frac{(x_{j_k} - x_{i_k})^2}{x_{i_k} x_{j_k}}.
\]

Proof. We prove this by evaluating both sides independently. First, we have
\[
\text{LHS} := \int_{[0,1]^n} \prod_{k=1}^{m} 4 \sin^2(\pi(x_{j_k} - x_{i_k})) \, dx
\]
\[
= (-1)^m \int_{[0,1]^n} \prod_{k=1}^{m} \left( e^{\pi i(x_{j_k} - x_{i_k})} - e^{-\pi i(x_{j_k} - x_{i_k})} \right)^2 \, dx
\]
\[
= (-1)^m \sum_{a,b} (-1)^{\sum_k (a_k + b_k - 2)/2} \int_{[0,1]^n} e^{\pi \sum_k (a_k + b_k)(x_{j_k} - x_{i_k})} \, dx
\]
\[
= \sum_{a,b} (-1)^{\sum_k (a_k + b_k)/2} \int_{[0,1]^n} \cos \left( \pi \sum_k (a_k + b_k)(x_{j_k} - x_{i_k}) \right) \, dx
\]
where the last two sums are over all sequences $a, b \in \{\pm 1\}^m$. In the last step the summands corresponding to $(a, b)$ and $(-a, -b)$ have been combined.

Now note that, for $a$ an even integer,
\[
\int_0^1 \cos(\pi(ax + b)) \, dx = \begin{cases} 
\cos(\pi b) & \text{if } a = 0, \\
0 & \text{otherwise}.
\end{cases}
\tag{28}
\]
Since $a_k + b_k$ is even, we may apply (28) iteratively to obtain
\[
\int_{[0,1]^n} \cos \left( \pi \sum_k (a_k + b_k)(x_{j_k} - x_{i_k}) \right) \, dx = \begin{cases} 
1 & \text{if } a, b \in S, \\
0 & \text{otherwise},
\end{cases}
\]
where $S$ denotes the set of sequences $a, b \in \{\pm 1\}^m$ such that
\[
\sum_{k=1}^{m} (a_k + b_k)(x_{j_k} - x_{i_k}) = 0
\]
as a polynomial in $x$. It follows that
\[
\text{LHS} = \sum_{a,b \in S} (-1)^{\sum_k (a_k + b_k)/2}
\tag{29}
\]

On the other hand, consider
\[
\text{RHS} := (-1)^m \text{ct} \prod_{k=1}^{m} \frac{(x_{j_k} - x_{i_k})^2}{x_{i_k} x_{j_k}}.
\]
and observe that, by a similar argument as above,
\[
(-1)^m \prod_{k=1}^{m} \left( \frac{x_{j_k} - x_{i_k}}{x_{i_k} x_{j_k}} \right)^2 = \sum_{a, b} \prod_{k=1}^{m} (-1)^{(a_k + b_k)/2} \left( \frac{x_{j_k}}{x_{i_k}} \right)^{(a_k + b_k)/2}
\]
where the sum is again over all sequences \(a, b \in \{\pm 1\}^m\). From here, it is straight-forward to verify that RHS is equivalent to the expression given for LHS in (29).

The desired evaluation is now available. On combining Theorem 7 and Proposition 3 we obtain that for \(\text{Re } s > 0\),
\[
W_n(s) = n^s \sum_{m \geq 0} (-1)^m \binom{s/2}{m} \sum_{k=0}^{m} \frac{(-1)^k}{n^{2k}} \binom{m}{k} \sum_{a_1 + \cdots + a_n = k} \left( \frac{k}{a_1, \ldots, a_n} \right)^2.
\]
This is Theorem 1.

**Remark 7.** We briefly outline the experimental genesis of the evaluation given in Theorem 7. The sequence \(2^m I_{3,m}\) is Sloane’s, [Slo97], A093388 where a link to [Ver99] is given. This paper contains the sum
\[
2^m I_{3,m} = (-1)^m \sum_{k=0}^{m} \binom{m}{k} (-8)^k \sum_{j=0}^{m-k} \binom{m-k}{j}^3
\]
and further mentions that \(2^m I_{3,m}\) is therefore the coefficient of \((xyz)^m\) in
\[
(8xyz - (x + y)(y + z)(z + x))^m.
\]
Observe also that \(2^m I_{2,m}\) is the coefficient of \((xy)^m\) in
\[
(4xy - (x + y)(y + x))^m.
\]
It was then noted that
\[
8xyz - (x + y)(y + z)(z + x) = 3^2 xyz - (x + y + z)(xy + yz + zx)
\]
and this line of extrapolation led to the correct conjecture, so that the next case would involve
\[
4^2 wxyz - (w + x + y + z)(wxy + xyz + yzw + zwx),
\]
which was what we have now proven.

**Remark 8.** Dyson’s long-proven constant rank conjecture [BBC07, p. 296] implies that the multinomial coefficient
\[
\binom{a_1 + a_2 + \cdots + a_n}{a_1, a_2, \ldots, a_n} = \text{ct} \prod_{1 \leq i \neq j \leq n} \left( 1 - \frac{x_i}{x_j} \right)^{a_i},
\]
but we have been unable to connect this further with our work.
A.2 Numerical Evaluations

A one-dimensional reduction of the integral (1) may be achieved by taking periodicity into account:

\[ W_n(s) = \int_{[0,1]^{n-1}} \left| 1 + \sum_{k=1}^{n-1} e^{2\pi i x_k} \right|^s \, d(x_1, \ldots, x_{n-1}). \]  

(30)

From here, we note that quick and rough estimates are easily obtained using the Monte Carlo method. Moreover, since the integrand function is periodic this seems like an invitation to use lattice sequences—a quasi-Monte Carlo method. E.g., the lattice sequence from [CKN06] can be straightforwardly employed to calculate an entire table in one run by keeping a running sum over different values of \( n \) and \( s \). A standard stochastic error estimator can then be obtained by random shifting.

Generally, however, Broadhurst’s representation (18) seems to be the best available for high precision evaluations of \( W_n(s) \). We close by commenting on the special cases \( n = 3, 4 \).

**Example 6.** The first high precision evaluations of \( W_3 \) were performed by David Bailey who confirmed the initially only conjectured Theorem 6 for \( s = 2, \ldots, 7 \) to 175 digits. This was done on a 256-core LBNL system in roughly 15 minutes by applying tanh-sinh integration to

\[ W_3(s) = \int_0^1 \int_0^1 (9 - 4(\sin^2(\pi x) + \sin^2(\pi y) + \sin^2(\pi(x - y))))^{s/2} \, dy \, dx, \]

which is obtained from (30) as in Proposition 3. More practical is the one-dimensional form (24) which can deliver high precision results in minutes on a simple laptop. For integral \( s \), Theorem 6 allows extremely high precision evaluations. ◊

**Example 7.** Assuming that Conjecture 1 holds for \( n = 2 \), Theorem 6 implies that for nonnegative integers \( k \)

\[ W_4(k) = \Re \sum_{j \geq 0} \left( \frac{s/2}{j} \right)^2 \, _3F_2 \left( \frac{1}{2}, -\frac{k}{2} + j, -\frac{k}{2} + j \, \mid 1, 1 \right). \]

This representation is very suitable for high precision evaluations of \( W_4 \) since, roughly, one correct digit is added by each term of the sum. Formula (20) by Crandall also lends itself quite well for numerical work (by slightly perturbing \( s \) even for integral arguments). ◊

**References**


