Locally Lipschitz Functions and Bornological Derivatives

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Abstract

We study the relationships between Gateaux, Weak Hadamard and Fréchet differentiability and their bornologies for Lipschitz and for convex functions.

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1 Introduction

We begin by introducing the definitions and notation that will be used. Unless otherwise specified, $X$ is an infinite dimensional real Banach space with norm $\| \cdot \|$ and dual space $X^*$. A bornology $B$ on $X$ is a family of bounded subsets of $X$ such that $\bigcup \{B : B \in B\} = X$. We will focus on the following bornologies: $G = \{\text{singletons}\}$, $H = \{\text{compact sets}\}$, $WH = \{\text{weakly compact sets}\}$ and $F = \{\text{bounded sets}\}$. Observe that $G \subset H \subset WH \subset F$.

A function $f : X \to \mathbb{R}$ is called $\mathcal{B}$-differentiable at $x \in X$ if there is $\Lambda \in X^*$ such that for each $B \in \mathcal{B}$,

$$
\frac{1}{t} \left[ f(x + th) - f(x) - \langle \Lambda, th \rangle \right] \to 0 \quad \text{as} \quad t \downarrow 0
$$

uniformly for $h \in B$. Let $\mathcal{F}$ denote a family of real-valued locally Lipschitz functions on $X$; we will usually consider locally Lipschitz (loclip), Lipschitz (lip), distance (dist), continuous convex (conv) and norms (norm); it is, of course, easy to check that continuous convex functions are locally Lipschitz ([10], Proposition 1.6). For two bornologies on a fixed Banach space $X$, say $F$ and $G$ and a family of functions $\mathcal{F}$, we will write $F_{\mathcal{F}} = G_{\mathcal{F}}$, if for every $f \in \mathcal{F}$ and every $x \in X$, $f$ is $F$-differentiable at $x$ if and only if $f$ is $G$-differentiable at $x$. Since $G_{\text{locclip}} = H_{\text{locclip}}$, we will write $G$ and $H$ interchangeably.

In the paper [1], it was shown that $H_{\text{conv}} = F_{\text{conv}}$ if and only if $X$ is infinite dimensional. From this, one might be tempted to believe that various notions of differentiability for convex functions coincide precisely when the bornologies on the space coincide. However, this is far from being the case: for example, according to ([1], Theorem 2), $WH_{\text{conv}} = F_{\text{conv}}$ if and only if $X \not\cong l_1$. In contrast to this, we will show in Section 2 that differentiability notions coincide for Lipschitz functions precisely when the bornologies are the same (for the $H$, $WH$ and $F$ bornologies). In the third section we will study the relationship between $WH$-differentiability and $H$-differentiability for continuous convex functions. In particular, we show that if $B_{X^*}$ is $w^*$-sequentially compact, then $H_{\text{conv}} = WH_{\text{conv}}$ precisely when $H = WH$. However, there are spaces for which $H_{\text{conv}} = WH_{\text{conv}}$ and yet $H \not= WH$. This leads to examples showing that one cannot always extend a convex function from a space to a superspace while preserving $G$-differentiability at a prescribed point. Some characterizations of the Schur and Dunford-Pettis properties are also obtained in terms of differentiability of continuous $w^*$-lower semicontinuous convex functions on the dual space.

2 Lipschitz functions and bornologies

As mentioned in the introduction, there are spaces for which $WH \not= F$ but $WH_{\text{conv}} = F_{\text{conv}}$. However, this is not the case for Lipschitz functions.

**Theorem 2.1** For a Banach space $X$, the following are equivalent.

(a) $X$ is reflexive.

(b) $WH_{\text{lip}} = F_{\text{lip}}$.

(c) $WH_{\text{dist}} = F_{\text{dist}}$. 


In order to prove this theorem, we will need a special type of sequence in nonreflexive Banach spaces. Namely, we will say \( \{x_k\}_{k=1}^\infty \subset X \) is a special sequence if there is an \( \epsilon > 0 \) such that \( \{z_k\} \subset X \) has no weakly convergent subsequence whenever \( \|x_k - z_k\| < \epsilon \).

**Remark** (a) There are examples of sequences \( \{x_k\} \) such that \( \{x_k\} \) has no weakly convergent subsequence but \( \{x_k\} \) is not special.

Indeed, let \( X = \ell_1 \) and consider \( y_{n,m} = c_n + \frac{1}{n}c_m \) for \( m, n \in \mathbb{N}, m > n \). Let \( \{x_k\} \) be any sequential arrangement of \( \{y_{n,m}\} \). It is not hard to verify \( \{x_k\} \) has the desired properties. Another example is \( X = c_0 \) and \( y_{n,m} = \sum_{k=1}^n e_k + \sum_{k=n+1}^{n+m} \frac{1}{n}e_k \).

(b) If \( f \) is Lipschitz and \( WH \)-differentiable at 0 (with \( f'(0) = 0 \)) but \( f \) is not Fréchet differentiable at 0, it is not hard to construct a special sequence \( \{x_k\} \).

Indeed, because \( f \) is not Fréchet differentiable, we can choose \( \{x_k\} \) in the unit sphere \( S_X \) of \( X \) and \( t_k \downarrow 0 \) which satisfy

\[
\frac{|f(t_kx_k) - f(0)|}{t_k} \geq \epsilon \quad \text{for some } \epsilon > 0.
\]

Using the fact that \( f \) is Lipschitz and \( WH \)-differentiable at 0, one can easily show that \( \{x_k\} \) is special.

Part (b) of the above remark shows that in order to prove Theorem 2.1, it is necessary to show each nonreflexive Banach space has a special sequence. On the other hand, part (a) shows that such sequences must be chosen carefully.

**Lemma 2.2** Suppose \( \{x_n\} \subset X \) has no weakly convergent subsequence. Then some subsequence of \( \{x_n\} \) is a special sequence.

**Proof.** If some subsequence of \( \{x_n\} \) is special, then there is nothing more to do. So we will suppose this is not so and arrive at a contradiction by producing a weakly convergent subsequence of \( \{x_n\} \).

Given \( \epsilon = 1 \), by our supposition, we choose \( N_1 \subset X \) and \( \{z_{1,i}\}_{i \in N_1} \) such that

\[
\|x_i - z_{1,i}\| < 1 \quad \text{for } i \in N_1 \quad \text{and} \quad w-\lim_{i \in N_1} z_{1,i} = z_1.
\]

Supposing \( N_{k-1} \) has been chosen, we choose \( N_k \subset N_{k-1} \) and \( \{z_{k,i}\}_{i \in N_k} \subset X \) satisfying

\[
\|x_i - z_{k,i}\| < \frac{1}{k} \quad \text{for } i \in N_k \quad \text{and} \quad w-\lim_{i \in N_k} z_{k,i} = z_k.
\]  \( \quad \) (2.1)

In this manner we construct \( \{z_{k,i}\}_{i \in N_k} \) and \( N_k \) for all \( k \in \mathbb{N} \).

Notice that \( z_n - z_m = w-\lim_{i \in N_n} (z_{n,i} - z_{m,i}) \) for \( n > m \). Thus by the \( w \)-lower semicontinuity of \( \| \cdot \| \) and (2.1) we obtain

\[
\|z_n - z_m\| \leq \liminf_{i \in N_n} \|z_{n,i} - z_{m,i}\| \leq \liminf_{i \in N_n} (\|z_{n,i} - x_i\| + \|x_i - z_{m,i}\|) \leq \frac{1}{n} + \frac{1}{m} \leq \frac{2}{m}.
\]

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Thus $z_n$ converges in norm to some $z_\infty \in X$.

Now for each $n \in \mathbb{N}$ choose integers $i_n \in N_\Lambda$ with $i_n > n$. We will show $x_{i_n} \overset{w}{\to} z_\infty$. So let $\Lambda \in B_X^*$ and $\epsilon > 0$ be given. We select an $n_0 \in \mathbb{N}$ which satisfies

$$\frac{1}{n_0} < \frac{\epsilon}{3} \quad \text{and} \quad \|z_m - z_\infty\| < \frac{\epsilon}{3} \quad \text{for} \quad m \geq n_0. \quad (2.2)$$

Because $z_{n_0,i} \overset{w}{\to} z_{n_0}$, we can select $m_0$ so that

$$|\langle \Lambda, z_{n_0,i} - z_{n_0} \rangle| < \frac{\epsilon}{3} \quad \text{for all} \quad i \geq m_0. \quad (2.3)$$

For $m \geq \max\{n_0,m_0\}$, we have

$$|\langle \Lambda, x_{i_m} - z_\infty \rangle| \leq |\langle \Lambda, x_{i_m} - z_{n_0,i_m} \rangle| + |\langle \Lambda, z_{n_0,i_m} - z_{n_0} \rangle| + |\langle \Lambda, z_{n_0} - z_\infty \rangle|$$

$$< \|x_{i_m} - z_{n_0,i_m}\| + \frac{\epsilon}{3} + \|z_{n_0} - z_\infty\| \quad \text{(by (2.3) since $i_m > m \geq m_0$)}$$

$$< \frac{1}{n_0} + \frac{\epsilon}{3} \quad \text{(by (2.2) and (2.1))}$$

Therefore $x_{i_m} \overset{w}{\to} z_\infty$.

**Proof of Theorem 2.1.** Notice that (a) $\implies$ (b) $\implies$ (c) is trivial. It remains to prove (c) $\implies$ (a). Suppose $X$ is not reflexive, hence $B_X$ is not weakly compact and so there exists \{ $x_n$ \} $\subset S_X$ with no weakly convergent subsequence. By Lemma 2.2 there is a subsequence, again denoted by \{ $x_n$ \}, and a $\Delta \in (0,1)$ such that \{ $z_n$ \} $\subset X$ has no weakly convergent subsequence whenever \| $z_n - x_n$ \| $< \Delta$.

By passing to another subsequence if necessary we may assume \| $x_n - x_m$ \| $> \delta$ for all $n \neq m$, with some $0 < \delta < 1$.

For $n = 1, 2, \ldots$, let $B_n = \{ x \in X : \| x - 4^{-n}x_n \| \leq \delta \Delta 4^{-n-1} \}$ and put $C = X \setminus \cup_{n=1}^\infty B_n$. Because $4^{-m} + \delta \Delta 4^{-m-1} < 4^{-n} - \delta \Delta 4^{-n-1}$ for $m > n$, we have that $B_n \cap B_m = \emptyset$ whenever $n \neq m$.

For $x \in X$, let $f(x)$ be the distance of $x$ from $C$. Thus $f$ is a Lipschitz function on $X$ with $f(0) = 0$.

We will check that $f$ is $WH$-differentiable at $0$ but not $F$-differentiable at $0$.

Let us first observe that $f$ is $G$-differentiable at $0$. So fix any $h \in X$ with \| $h$ \| $= 1$. Then $[0,\infty)h$ meets at most one ball $B_n$. In fact assume $t_m, t_n > 0$ are such that \| $t_m - 4^{-i}x_i$ \| $\leq \delta \Delta 4^{-i-1}$ for $i = n, m$. Then $|4^it_i - 1| < \frac{\delta \Delta}{4}$ for $i = n, m$ and

$$\| x_n - x_m \| \leq \| x_n - 4^n t_n h \| + \| 4^n t_n h - 4^m t_m h \| + \| 4^m t_m h - x_m \|$$

$$< \frac{\delta \Delta}{4} + \frac{\delta \Delta}{4} + \frac{\delta \Delta}{4} = \delta \Delta < \delta.$$

This means that $n = m$. It thus follows that for $t > 0$ small enough, we have $f(th) = 0$. Therefore $f$ is $G$-differentiable at $0$, with $f'(0) = 0$. Let us further check that $f$ is not $F$-differentiable at $0$.

Indeed,

$$\frac{f(4^{-n}x_n)}{\| 4^{-n}x_n \|} = \frac{\delta \Delta}{4} \quad \text{for all} \quad n,$$

while $\| 4^{-n}x_n \| \to 0$.

Finally assume that $f$ is not $WH$-differentiable at $0$. Then there are a weakly compact set $K \subset B_X$, $\epsilon > 0$, and sequences \{ $k_m$ \} $\subset K, t_m \downarrow 0$ such that

$$\frac{f(t_m k_m)}{t_m} > \epsilon \quad \text{for all} \quad m \in \mathbb{N}.$$
Hence, as \( f \) is 1-Lipschitz, we have \( \inf \| k_n \| \geq \varepsilon > 0 \). Further, because \( f(t_m k_m) > 0 \), there are \( n_m \in \mathbb{N} \) such that 
\[
\| t_m k_m - 4^{-n_m} x_{n_m} \| < \Delta \delta 4^{-m_n - 1}, \quad m = 1, 2, \ldots .
\]
Consequently,
\[
\| 4^{n_m} t_m k_m - x_{n_m} \| < \frac{\Delta \delta}{4} < \Delta \quad \text{and} \quad |4^{n_m} t_m||k_m| - 1| < \frac{\Delta \delta}{4}.
\] (2.4)
Because \( \{ x_n \} \) is a special sequence with \( \Delta \), the first inequality in (2.4) says that \( \{ 4^{n_m} t_m k_m \} \) does not have a weakly convergent subsequence. However the second inequality in (2.4) together with \( \inf \| k_n \| > 0 \) ensures that \( 4^{n_m} t_m \) is bounded and so, since \( \{ k_m \} \) is weakly compact, \( 4^{n_m} t_m k_m \) has a weakly convergent subsequence, a contradiction. This proves \( f \) is \( WH \)-differentiable at 0.

Recall that a Banach space has the Schur property if \( H = WH \), that is, weakly convergent sequences are norm convergent.

**Theorem 2.3** For a Banach space \( X \), the following are equivalent.

(a) \( X \) has the Schur property.

(b) \( H_{\text{lip}} = WH_{\text{lip}} \).

(c) \( H_{\text{dist}} = WH_{\text{dist}} \).

**Proof.** It is clear that (a) \( \iff \) (b) \( \iff \) (c), thus we prove (c) \( \iff \) (a). Suppose \( X \) is not Schur and choose \( \{ x_n \} \subset S_X \) such that \( x_n \rightrightarrows 0 \) but \( \| x_n \| \not\rightrightarrows 0 \). Since \( \{ x_n \} \) is not relatively norm compact, we may assume by passing to a subsequence if necessary that \( \| x_i - x_j \| > \delta \) for some \( \delta \in (0,1) \) whenever \( i \neq j \).

As in the proof of Theorem 2.1, let \( B_n = \{ x \in X : \| x - 4^{-n} x_n \| \leq \delta 4^{-n-1} \} \), \( C = X \setminus \bigcup_{n=1}^{\infty} B_n \) and let \( f(x) = d(x,C) \). Now \( f(0) = 0 \) and the argument of Theorem 2.1, shows that \( f \) is \( G \)-differentiable at 0 with \( f'(0) = 0 \). However,
\[
f\left(\frac{4^{-n} x_n}{4^{-n}}\right) = \frac{\delta}{4} \quad \text{for all} \quad n \in \mathbb{N}.
\]
Since \( \{ x_n \} \cup \{ 0 \} \) is weakly compact, it follows that \( f \) is not \( WH \)-differentiable at 0.

**Remark.** Using the technique from the proof of Theorem 2.1, one can also prove the following statement. If a nonreflexive Banach space \( X \) admits a Lipschitzian \( C^k \)-smooth bump function, then it admits a Lipschitz function which is \( C^k \)-smooth on \( X \setminus \{ 0 \} \), \( WH \)-differentiable at 0, but not \( F \)-differentiable at 0. A corresponding remark holds for non-Schur spaces.

### 3 Differentiability properties of convex functions

We begin by summarizing some known results. First recall that a Banach space \( X \) has the Dunford-Pettis property if \( \langle x_n, x_n \rangle \to 0 \) whenever \( x_n \rightrightarrows 0 \) and \( x_n^* \rightrightarrows 0 \). For notational purposes we will say
X has the $DP^*$ if $\langle x_n^*, x_n \rangle \to 0$ whenever $x_n^* \stackrel{w^*}{\to} 0$ and $x_n \stackrel{w}{\to} 0$; see ([4], p. 177) for more on the Dunford-Pettis property. Note that a completely continuous operator takes weakly convergent sequences to norm convergent sequences. The proof of the next result is essentially in [1].

**Theorem 3.1 ([1])**

(a) $X$ does not contain a copy of $\ell_1$ if and only if $WH_{\text{conv}} = F_{\text{conv}}$ if and only if $WH_{\text{norm}} = F_{\text{norm}}$ if and only if each completely continuous linear $T : X \to c_0$ is compact.

(b) $X$ has the $DP^*$ if and only if $H_{\text{conv}} = WH_{\text{conv}}$ if and only if $H_{\text{norm}} = WH_{\text{norm}}$ if and only if each continuous linear $T : X \to c_0$ is completely continuous.

(c) $X$ is finite dimensional if and only if $G_{\text{conv}} = F_{\text{conv}}$ if and only if $G_{\text{norm}} = F_{\text{norm}}$ if and only if each continuous linear $T : X \to c_0$ is compact.

**Proof.** Let us mention that (a) is contained in ([1], Theorem 2) and (c) is from ([1], Theorem 1). Whereas (b) can be obtained by following the proofs of ([1], Proposition 1 and Theorem 1).

If $WH_{\text{conv}} \neq G_{\text{conv}}$, for example, we can be somewhat more precise.

**Proposition 3.2** Suppose $WH_{\text{conv}} \neq G_{\text{conv}}$ on $X$. Then there is a norm $\| \cdot \|$ on $X$ such that $\| \cdot \|$ is not $WH$-differentiable at $x_0 \neq 0$ but $\| \cdot \| ^*$ is strictly convex at $\Lambda_0 \in X^* \setminus \{0\}$ satisfying $\langle \Lambda_0, x_0 \rangle = \| x_0 \| \| \Lambda_0 \| ^* .$

**Proof.** Following the techniques of [1], one obtains a norm $\| \cdot \|$ on $X$ such that $\| \cdot \|$ is $G$-differentiable at $x_0 \neq 0$ but $\| \cdot \|$ is not $WH$-differentiable at $x_0$. Now define $\| \cdot \|$ on $X$ by

$$\| x \| = (\| x \|^2 + d^2(x, x_0))^\frac{1}{2}$$

Clearly $d^2(\cdot, x_0)$ is $F$-differentiable at $x_0$ and so it follows that $\| \cdot \|$ is $G$-differentiable at $x_0$ but $\| \cdot \|$ is not $WH$-differentiable at $x_0$ because $\| \cdot \|$ is not. Suppose now that $\{ x_n \}$ satisfies

$$2\| x_n \|^2 + 2\| x_0 \|^2 - \| x_n + x_0 \|^2 \to 0. \quad (3.1)$$

Then by convexity one obtains

$$\| x_n \| \to \| x_0 \|, \quad d^2(x_n, x_0) \to d^2(x_0, x_0) = 0.$$ 

From this one easily sees that $\| x_n - x_0 \| \to 0$.

Now take $\Lambda_0 \in X^*$ such that $\| \Lambda_0 \| = 1$ and $\langle \Lambda_0, x_0 \rangle = \| x_0 \|$. We show that $\| \cdot \|$ is strictly convex at $\Lambda_0$. Suppose that $\| x^* \| = 1$ and $\| x^* + \Lambda_0 \| = 2$, then choose $\{ x_n \}$ with $\| x_n \| = \| x_0 \|$ so that

$$\langle x^* + \Lambda_0, x_n \rangle \to 2\| x_0 \|. \quad (3.2)$$

Consequently $\langle \Lambda_0, x_n \rangle \to \| x_0 \|$ and thus $\langle \Lambda_0, x_n + x_0 \rangle \to 2\| x_0 \|$. But then $\{ x_n \}$ satisfies (3.1) and so $\| x_n - x_0 \| \to 0$. This with (3.2) shows that $\langle x^*, x_0 \rangle = \| x_0 \|$. Because
\( \| \cdot \| \) is \( G \)-differentiable at \( x_0 \), we conclude that \( x^* = \Lambda_0 \). This proves the strict convexity of \( \| \cdot \| \) at \( \Lambda_0 \).

One can also formulate similar statements (and proofs) for the cases \( G_{\text{conv}} \neq F_{\text{conv}} \) and \( WH_{\text{conv}} \neq F_{\text{conv}} \).

We now turn our attention to spaces for which \( WH_{\text{conv}} = H_{\text{conv}} \). Let us recall that a Banach space \( X \) has the Grothendieck property if \( w^* \)-convergent sequences in \( X^* \) are weakly convergent; see ([4], p. 179). The following corollary is an immediate consequence of Theorem 3.1 (b).

**Corollary 3.3** If \( X \) has the Dunford-Pettis property and the Grothendieck property, then \( WH_{\text{conv}} = H_{\text{conv}} \).

In particular, note that \( \ell_\infty \) has the Grothendieck property (cf [3], p.103) and the Dunford-Pettis property (cf [4], p. 177). Thus, unlike the case for Lipschitz functions, one can have \( WH_{\text{conv}} = H_{\text{conv}} \) for non-Schur spaces. It will follow from the next result that these non-Schur spaces must be quite large, though.

**Theorem 3.4** For a Banach space \( X \), the following are equivalent.

(i) \( X \) has the \( DP^* \)

(ii) \( H_{\text{conv}} = WH_{\text{conv}} \)

(iii) If \( B_{Y^*} \) is \( w^* \)-sequentially compact, then any continuous linear \( T : X \rightarrow Y \) is completely continuous.

**Proof.** By Theorem 3.1(b) we know that (i) and (ii) are equivalent and that (iii) implies (i). We will show (i) implies (iii) by contraposition. Suppose (iii) fails, that is, there is an operator \( T : X \rightarrow Y \) which is not completely continuous for some \( Y \) with \( B_{Y^*} \) \( w^* \)-sequentially compact. Hence we choose \( \{x_n\} \subset X \) such that \( x_n \overset{w^*}{\rightarrow} 0 \) but \( \|Tx_n\| \neq 0 \). Because \( Tx_n \overset{w^*}{\rightarrow} 0 \), we know that \( \{Tx_n\} \) is not relatively norm compact. Hence letting \( E_n = \text{span} \{y_k : k \leq n\} \) with \( y_k = Tx_k \) we know there is an \( \epsilon > 0 \) such that \( \sup d(y_k, E_n) > \epsilon \) for each \( n \).

By passing to a subsequence, if necessary, we assume \( d(y_n, E_{n-1}) > \epsilon \) for each \( n \). Now choose \( \Lambda_n \in B_{Y^*} \) such that \( \langle \Lambda_n, x \rangle = 0 \) for all \( x \in E_{n-1} \) and \( \langle \Lambda_n, x_n \rangle \geq \epsilon \). Because \( B_{Y^*} \) is \( w^* \)-sequentially compact, there is a subsequence \( \Lambda_{n_k} \) such that \( \Lambda_{n_k} \overset{w^*}{\rightarrow} \Lambda \in B_{Y^*} \). Observe that \( \langle \Lambda_n, y_k \rangle = 0 \) for \( n > k \) and consequently \( \langle \Lambda, y_k \rangle = 0 \) for all \( k \). Now let \( z_n^* = T^*(\Lambda_{n_k} - \Lambda) \) and \( z_k = x_{n_k} \). Certainly \( z_k^* \overset{w^*}{\rightarrow} 0 \) and \( z_k \overset{w^*}{\rightarrow} 0 \) while \( \langle z_k^*, z_k \rangle = \langle \Lambda_{n_k} - \Lambda, Tx_{n_k} \rangle = \langle \Lambda_{n_k} - \Lambda, y_{n_k} \rangle \geq \epsilon \) for all \( k \). This shows that \( X \) fails the \( DP^* \).

**Corollary 3.5** If \( X \) has a \( w^* \)-sequentially compact dual ball or, more generally, if every separable subspace of \( X \) is a subspace of a complemented subspace with \( w^* \)-sequentially compact dual ball, then the following are equivalent.
(a) \( WH_{\text{conv}} = H_{\text{conv}} \).

(b) \( X \) has the Schur property.

**Proof.** Note that (b) \( \implies \) (a) is always true, so we show (a) \( \implies \) (b). If \( B_X \) is \( w^* \)-sequentially compact and \( X \) is not Schur then \( I : X \to X \) is not completely continuous and Theorem 3.4 applies. More generally, suppose \( x_n \xrightarrow{w} 0 \) but \( \| x_n \| \not\to 0 \) and \( \text{span} \{ x_n \} \subset Y \) with \( B_Y \) \( w^* \)-sequentially compact. If there is a projection \( P : X \to Y \), then \( P \) is not completely continuous since \( P|_Y \) is the identity on \( Y \).

We can say more in the case that \( X \) is weakly countably determined (WCD); see [9] and Chapter VI of [2] for the definition and further properties of WCD spaces.

**Corollary 3.6** For a Banach space \( X \), the following are equivalent.

(a) \( X \) is WCD and \( WH_{\text{conv}} = H_{\text{conv}} \)

(b) \( X \) is separable and has the Schur property.

**Proof.** It is obvious that (b) \( \implies \) (a), so we show (a) \( \implies \) (b). First, since \( B_X \) is \( w^* \)-sequentially compact (see e.g. [9], Corollary 4.9 and [7], Theorem 11), it follows from Corollary 3.5 that \( X \) has the Schur property. But WCD Schur spaces are separable (see e.g. [9], Theorem 4.3).

**Remark**

(a) Corollary 3.5 is satisfied, for instance, by GDS spaces (see [7], Theorem 11) and spaces with countably norming \( M \)-basis (see [11], Lemma 1). Notice that \( \ell_1(\Gamma) \) has a countably norming \( M \)-basis for any \( \Gamma \), thus spaces with countably norming \( M \)-bases and the Schur property need not be separable.

(b) If \( X^* \) satisfies \( WH_{\text{conv}} = F_{\text{conv}} \), then \( X \) also does (because \( L_1 \subset X^* \) if \( \ell_1 \subset X \) (see [5] Proposition 4.2)) but not conversely (\( c_0 \) and \( \ell_1 \); cf. Theorem 3.1(a)).

(c) Let \( X \) be a space such that \( X \) is Schur but \( X^* \) does not have the Dunford-Pettis property (cf. [4], p 178). Then \( X \) satisfies \( H_{\text{conv}} = WH_{\text{conv}} \) but \( X^* \) does not satisfy \( H_{\text{conv}} = WH_{\text{conv}} \).

(d) There are spaces with the \( DP^* \) that are neither Schur nor have the Grothendieck property; for example \( \ell_1 \times \ell_\infty \).

(e) It is well-known that \( \ell_\infty \) has \( \ell_2 \) as a quotient ([8], p. 111). Thus quotients of spaces with the \( DP^* \) need not have the \( DP^* \). It is clear that superspaces of spaces with the \( DP^* \) need not have the \( DP^* \); the example \( c_0 \subset \ell_\infty \) shows that subspaces need not inherit the \( DP^* \).

(f) Haydon ([6]) has constructed a nonreflexive Grothendieck \( C(K) \) space which does not contain \( \ell_\infty \). Using the continuum hypothesis, Talagrand ([12]) constructed a nonreflexive Grothendieck \( C(K) \) space \( X \) such that \( \ell_\infty \) is neither a subspace nor a quotient of \( X \). Since \( C(K) \) spaces have the Dunford-Pettis property (see [3], p. 113), both these spaces have the \( DP^* \).
As a byproduct of Corollaries 3.3 and 3.5 we obtain the following example which is related to results from ([13]).

**Example.** Let $X$ be a space with the Grothendieck and Dunford-Pettis properties such that $X$ is not Schur (e.g. $\ell_\infty$). Then there is a separable subspace $Y$ (e.g. $c_0$) of $X$ and a continuous convex function $f$ on $Y$ such that $f$ is $G$-differentiable at 0 (as a function on $Y$), but no continuous convex extension of $f$ to $X$ is $G$-differentiable at 0 (as a function on $X$); there also exist $y_0 \in Y \setminus \{0\}$ and an equivalent norm $\| \cdot \|$ on $Y$ whose dual norm is strictly convex but no extension of $\| \cdot \|$ to $X$ is $G$-differentiable at $y_0$.

**Proof.** Let $Y$ be a separable non-Schur subspace of $X$. By Corollary 3.5, there is a continuous convex function $f$ on $Y$ which is $G$-differentiable at 0, but is not $WH$-differentiable at 0. Since any extension $\tilde{f}$ of $f$ also fails to be $WH$-differentiable at 0, it follows that $\tilde{f}$ is not $G$-differentiable at 0 because $X$ has the $DP^*$. Because $Y$ fails the $DP^*$, there is a sequence $\{\Lambda_n\} \subset X^*$ such that $\Lambda_n$ converges $w^*$ but not Mackey to 0. By the proof of ([1], Theorem 3), there is a norm $\| \cdot \|$ on $Y$ whose dual is strictly convex that fails to be $WH$-differentiable at some $y_0 \in Y \setminus \{0\}$; as above, no extension of $\| \cdot \|$ to $X$ can be $G$-differentiable at $y_0$.

We close this note by relating the Schur and Dunford-Pettis properties to some notions of differentiability for dual functions.

**Theorem 3.7** For a Banach space $X$, the following are equivalent.

(a) $X$ has the Schur property.

(b) $G$-differentiability and $F$-differentiability coincide for $w^*$-lsc continuous convex functions on $X^*$.

(c) $G$-differentiability and $F$-differentiability coincide for dual norms on $X^*$.

(d) $H_{lip} = WH_{lip}$.

**Proof.** Of course (a) and (d) are equivalent according to Theorem 2.3.

(a) $\implies$ (b): Suppose (b) does not hold. Then for some continuous convex $w^*$-lsc $f$ on $X^*$, there exists $\Lambda_0 \in X^*$ such that $f$ is $G$-differentiable at $\Lambda_0$ but $f$ is not $F$-differentiable at $\Lambda_0$. Let $f'(\Lambda_0) = x^{**} \in X^{**}$. We also choose $\delta > 0$ and $K > 0$ such that for $x_1^*, x_2^* \in B(\Lambda_0, \delta)$ we have $|f(x_1^*) - f(x_2^*)| \leq K\|x_1^* - x_2^*\|$ (since $f$ is locally Lipschitz). Because $f$ is not $F$-differentiable at $\Lambda_0$, there exist $t_n \downarrow 0, t_n < \frac{\delta}{2}, \Lambda_n \in S_{X^*}$ and $\epsilon > 0$ such that

$$f(\Lambda_0 + t_n \Lambda_n) - f(\Lambda_0) - \langle x^{**}, t_n \Lambda_n \rangle \geq \epsilon t_n.$$  \hspace{1cm} (3.3)

Because $f$ is convex and $w^*$-lsc, using the separation theorem we can choose $x_n \in X$ satisfying

$$\langle x_n, x^* \rangle \leq f(\Lambda_0 + t_n \Lambda_n + x^*) - f(\Lambda_0 + t_n \Lambda_n) + \frac{\epsilon t_n}{2} \quad \text{for all } x^* \in X^*;$$  \hspace{1cm} (3.4)

$$\langle x_n, x^* \rangle \leq f(\Lambda_0 + t_n \Lambda_n + x^*) - f(\Lambda_0 + t_n \Lambda_n) + \frac{\epsilon t_n}{2} \quad \text{for all } x^* \in X^*;$$  \hspace{1cm} (3.4)
Putting $x^* = -t_n \Lambda_n$ in (3.4) and using (3.3) one obtains

$$\langle x_n, t_n \Lambda_n \rangle \geq f(\Lambda_0 + t_n \Lambda_n) - f(\Lambda_0) - \frac{\epsilon_f}{2}$$

$$\geq \langle x^{**}, t_n \Lambda_n \rangle + \frac{\epsilon_f}{2}.$$ 

And hence, $\|x_n - x^{**}\| \geq \frac{\eta}{2}$ for all $n$.

Let $\eta > 0$ and fix $x^* \in S_{X^*}$. Since $f$ is $G$-differentiable at $\Lambda_0$, there is a $0 < t_0 < \frac{\eta}{2}$ such that for $|t| \leq t_0$ we have

$$\langle x^{**}, tx^* \rangle - f(\Lambda_0 + tx^*) + f(\Lambda_0) \geq -\frac{\eta}{2}t_0.$$ 

(3.5)

Using (3.4) with the fact that $f$ has Lipschitz constant $K$ on $B(\Lambda_0, \delta)$, for $|t| \leq t_0$ we obtain

$$\langle x_n, tx^* \rangle \leq f(\Lambda_0 + t_n \Lambda_n + tx^*) - f(\Lambda_0 + t_n \Lambda_n) + \frac{\epsilon_f}{2}$$

$$\leq f(\Lambda_0 + tx^*) - f(\Lambda_0) + \frac{\epsilon_f}{2} + 2Kt_n.$$ 

Choosing $n_0$ so large that $\frac{\epsilon_f}{2} + 2Kt_n \leq \frac{\eta}{2}t_0$ for $n \geq n_0$, the above inequality yields

$$f(\Lambda_0 + tx^*) - f(\Lambda_0) - \langle x_n, tx^* \rangle \geq -\frac{\eta}{2}t_0 \quad \text{for} \quad n \geq n_0, \; |t| \leq t_0.$$ 

(3.6)

Adding (3.5) and (3.6) results in

$$\langle x^{**} - x_n, tx^* \rangle \geq -\eta t_0 \quad \text{for} \quad n \geq n_0, \; |t| \leq t_0.$$ 

Hence $|\langle x^{**} - x_n, x^* \rangle| \leq \eta$ for $n \geq n_0$. This shows that $x_n \rightharpoonup x^{**}$. Combining this with the fact that $\|x_n - x^{**}\| \to 0$ shown above, we conclude that for some $\delta > 0$ and some subsequence we have $\|x_{n_i} - x_{n_{i+1}}\| > \delta$ for all $i$. However $x_{n_i} - x_{n_{i+1}} \rightharpoonup 0$ (in $X$) because $x_{n_i} - x_{n_{i+1}} \rightharpoonup 0$ (in $X^{**}$). This shows that $X$ is not Schur.

Since (b) $\implies$ (c) is obvious, we show that (c) $\implies$ (a). Write $X = Y \times$ and suppose that $X$ is not Schur. Then we can choose $\{y_n\} \subset Y$ such that $y_n \rightharpoonup 0$ but $\|y_n\| = 1$ for all $n$. Let $\{\gamma_n\} \subset (\frac{1}{2}, 1)$ be such that $\gamma_n \uparrow 1$ and define $\|\cdot\|$ on $X^* = Y^* \times$ by

$$\|\langle \Lambda, t \rangle\| = \sup\{|\langle \Lambda, y_n \rangle + \gamma_n t|\} \geq \frac{1}{2}(\|\Lambda\| + |t|).$$

This norm is dual since it is a supremum of $w^*-\ell sc$ functions and the proof of ([1], Theorem 1) shows that $\|\cdot\|$ is Gateaux but not Fréchet differentiable at $(0, 1)$.

If $X$ is not Schur, then the previous theorem ensures the existence of a $w^*-\ell sc$ convex continuous function on $X^*$ which is $G$-differentiable but not $F$-differentiable at some point. The following remark shows that we can be more precise if $X \not\cong \ell_1$.

**Remark.** If $X \not\cong \ell_1$ and $X$ is not reflexive, then there is a $w^*-\ell sc$ convex $f$ on $X^*$ and $\Lambda \in X^*$ such that $f$ is $G$-differentiable at $\Lambda$ and $f'(\Lambda) \in X^{**}\setminus X$ (and, a fortiori, $f$ is not Fréchet differentiable at $\Lambda$).

**Proof.** Let $Y$ be a separable nonreflexive subspace of $X$. Let $y^* \in Y^*$ be such that $y^*$ does not attain its norm on $B_Y$. Let $y'^* \in S_{Y^{**}}$ be such that $\langle y'^*, y^* \rangle = 1$. Note that $y'^* \in S_{Y^{**}} \setminus Y$
because \( y^* \) does not attain its norm on \( B_Y \). By the Odell-Rosenthal theorem (see [3], p.236), choose \( \{y_n\} \subset B_Y \) such that \( y_n \xrightarrow{w^*} y^* \). Now \( Y^{**} = Y^{\perp \perp} \subset X^{**} \) and some careful “identification checks” show that \( y_n \xrightarrow{w^*} y^* \) as elements of \( X^{**} \) and \( y^* \in X^{**} \setminus X \). Let \( \Lambda \) be a norm preserving extension of \( y^* \), then \( \langle y^*, \Lambda \rangle = 1 \) and we define \( f \) on \( X^* \) by

\[
f(x^*) = \sup \{ \langle x^*, y_n \rangle - 1 - a_n : n \in \} \text{ where } a_n \downarrow 0.
\]

We now show that \( y^{**} \in \partial f(\Lambda) \). Indeed,

\[
f(\Lambda + x^*) - f(\Lambda) = f(\Lambda + x^*) - \sup \{ \langle \Lambda + x^*, y_n \rangle - 1 - a_n \} \geq \lim_{n \to \infty} \{ \langle \Lambda, y_n \rangle - 1 - a_n + \langle x^*, y_n \rangle \} = \langle y^{**}, x^* \rangle.
\]

To see that \( f \) is \( G \)-differentiable, fix \( x^* \in X^* \) and let \( \varepsilon > 0 \). Choose \( n_0 \) so that \( |\langle y^{**} - y_n, x^* \rangle| \leq \varepsilon \| x^* \| \) for \( n \geq n_0 \). Now if \( 2\| t x^* \| < \min \{ a_1, \ldots, a_{n_0} \} \), we have

\[
0 \leq f(\Lambda + t x^*) - f(\Lambda) - \langle y^{**}, t x^* \rangle = \sup \{ \langle \Lambda + t x^*, y_n \rangle - 1 - a_n \} - \langle y^{**}, t x^* \rangle \\
= \sup \{ \langle \Lambda, y_n \rangle - 1 + \langle y_n - y^{**}, t x^* \rangle - a_n \} \\
\leq \max \{ 0, \sup \{ \langle y_n - y^{**}, t x^* \rangle - a_n \} \} \\
\leq \sup \{ \| (y^{**} - y_n, t x^*) \| \} \leq \varepsilon \| t x^* \|.
\]

Thus \( f \) is \( G \)-differentiable at \( \Lambda \) with \( G \)-derivative \( y^{**} \in Y^{**} \setminus Y \).

Using the results of [1] and Šmulian’s test type arguments in a fashion similar to Theorem 3.7, one can also obtain the following result. We will not provide the details.

**Theorem 3.8** For a Banach space \( X \), the following are equivalent.

(a) \( X \) has the Dunford-Pettis property.

(b) \( G \)-differentiability and \( W H \)-differentiability coincide for \( w^* \)-lsc, continuous convex functions on \( X^* \).

(c) \( G \)-differentiability and \( WH \)-differentiability coincide for dual norms on \( X^* \).

We next consider what happens for \( \mathcal{F} \) a family of norms alone.

**Remark.**

(a) \( G_{\text{norm}} = F_{\text{norm}} \) on \( X \) implies \( G_{\text{dualnorm}} = F_{\text{dualnorm}} \) on \( X^* \), but not conversely.

(b) \( G_{\text{norm}} = WH_{\text{norm}} \) on \( X \) implies \( G_{\text{dualnorm}} = WH_{\text{dualnorm}} \), but not conversely.

(c) \( WH_{\text{norm}} = F_{\text{norm}} \) on \( X \) does not imply \( WH_{\text{dualnorm}} = F_{\text{dualnorm}} \) on \( X^* \).
Proof. (a) This is immediate from Theorem 3.1(c) and Theorem 3.7 (since there are Schur spaces that are not finite dimensional).

(b) Since the $DP^*$ implies the Dunford-Pettis property the first part follows from Theorem 3.1(b) and Theorem 3.8. However, if $X$ is separable, then by Corollary 3.6, $H_{conv} = WH_{conv}$ if and only if $X$ is Schur. Thus the separable space $C[0,1]$ does not satisfy $G_{norm} = WH_{norm}$ yet it has the Dunford-Pettis property, and thus by Theorem 3.8 satisfies $G_{dualnorm} = WH_{dualnorm}$.

(c) On $c_0$ one has $WH_{norm} = F_{norm}$ (see Theorem 3.1). But $\ell_1 = c_0^*$ is a separable dual space and so it admits a dual $G$-norm (see [2], Theorem II.6.7(ii) and Corollary II.6.9(ii)). This norm cannot be everywhere $F$-differentiable since $\ell_1$ is not reflexive (see [2], Proposition II.3.4). However, this dual norm is everywhere $WH$-differentiable since $\ell_1$ is Schur. Thus we do not have $WH_{dualnorm} = F_{dualnorm}$ on $\ell_1$.

In fact we can be more precise than we were in (c). Using Theorem 3.7, Theorem 3.8 and Theorem 3.1 along with results from [1] one can obtain the following chain of implications.

\[
X \text{ fails the Schur property but has the Dunford-Pettis property} \implies
WH_{dualnorm} \neq F_{dualnorm} \text{ on } X^* \implies
X \text{ fails the Schur property and } X^* \supset \ell_1.
\]

References


