Incomplete Rational Approximation in the Complex Plane$^*$

by

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ABSTRACT

We consider rational approximations of the form

\[
\left\{ (1 + z)^{\alpha n + 1} \frac{p_{cn}(z)}{q_n(z)} \right\}
\]

in certain natural regions in the complex plane where \( p_{cn} \) and \( q_n \) are polynomials of degree \( cn \) and \( n \) respectively. In particular we construct natural maximal regions (as a function of \( \alpha \) and \( c \)) where the collection of such rational functions is dense in the analytic functions. So from this point of view we have rather complete analogue theorems to the results concerning incomplete polynomials on an interval.

The analysis depends on an examination of the zeros and poles of the Padé approximants to \((1 + z)^{\alpha n + 1}\). This is effected by an asymptotic analysis of certain integrals. In this sense it mirrors the well known results of Saff and Varga on the zeros and poles of the Padé approximant to exp. Results that, in large measure, we recover as a limiting case.

In order to make the asymptotic analysis as painless as possible we prove a fairly general result on the behavior, in \( n \), of integrals of the form

\[
\int_0^1 [t(1 - t)f_z(t)]^n dt
\]

where \( f_z(t) \) is analytic in \( z \) and a polynomial in \( t \). From this we can and do analyze automatically (by computer) the limit curves and regions that we need.

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Padé approximation, incomplete rationals, incomplete polynomials, "steepest descent, zeros, poles.
§1. Introduction.

In his remarkable paper of 1924, Szegő [12] considered the zeros of the partial sums $s_\alpha(z) := \sum_{k=0}^\alpha z^k/k!$ of the MacLaurin expansion for $e^z$. Szegő [12] established that $\hat{z}$ is a limit point of zeros of the sequence of normalized partial sums, $\{s_\alpha(nz)\}_{n=0}^\infty$, if and only if

$$\hat{z} \in \{ z : |ze^{1-z}| = 1, |z| \leq 1 \}.$$  \hspace{1cm} (1.1)

Moreover, Szegő [12] showed that $\hat{z}$ is a nontrivial limit point of zeros of the normalized remainder $\{e^{nz} - s_\alpha(nz)\}_{n=1}^\infty$ if and only if

$$\hat{z} \in \{ z : |ze^{1-z}| = 1, |z| \geq 1 \}.$$  \hspace{1cm} (1.2)

Saff and Varga [11] established sharp generalizations of Szegő’s results to the asymptotic distribution of zeros and poles of more general sequences of the Padé approximants to $e^z$.

In this paper, we consider the Padé approximants to $(1 + z)^{\alpha + 1}$, and locate the limit points of the zeros and poles of the Padé approximants. The Padé approximation to $e^z$ is a limiting case of the Padé approximations to $(1 + z)^{\alpha + 1}$ (see §6). The approach is to obtain some general theorems (Theorem 2.4 and 2.5) concerning the zeros of the limit function of the integrals

$$\int_0^1 [t(1 - t)f_z(t)]^n dt$$  \hspace{1cm} (1.3)

where $f_z(t)$ is a polynomial in $t$ and analytic in $z$. These theorems can be applied to many other cases. As a consequence of these theorems, we not only determine the limit points of the zeros and poles of the Padé approximants to $(1 + z)^{\alpha + 1}$, but also obtain, for example,

**Theorem.** The set of functions $\{(1 + z)^{\alpha}r_n(z)/s_n(z) : r_n(z), s_n(z) \in \pi_n, \}_{n=1}^\infty$ is dense in $A(K)$, the analytic functions on $K$, where $K$ is an arbitrary compact subset of $R_3$ and not in any region strictly containing $R_3$ ($R_3$ is the region in Fig.1).

This can be thought of a generalization of the now numerous results on denseness of incomplete polynomials on an interval to the complex plane.

In section 2, we give the explicit formulas for the Padé approximants to $(1 + z)^{\alpha + 1}$ in the form of (1.3), and then prove our main theorems. The distribution of the limit points of the zeros and poles of the Padé approximants to $(1 + z)^{\alpha + 1}$ is established in section 3.

One advantage of our method is that one can carry out this procedure automatically on a computer. So after some theoretical results are obtained the messy algebra of determining the limit curves and regions is entirely automatic (using Maple or some other symbolic algebra package, see section 3).

We consider incomplete rationals and incomplete polynomials in sections 4 and 5, respectively. In our last section, section 6, we discuss the Padé approximation to $e^z$ as a limiting case of the Padé approximation to $(1 + z)^{\alpha + 1}$.
§2. The main theorems.

In this section, we discuss the Padé approximation to \((1+z)^{\alpha n+1}\) at 0, and obtain the corresponding \(p, q\) and error term explicitly in the following integral form

\[
(2.1) \quad \int_0^1 [t(1-t)f_z(t)]^n dt
\]

where \(f_z(t)\) is a polynomial in \(t\) and analytic in \(z\) on some compact set \(K\). In the second half of this section, we establish a general theorem concerning the limit function of the above integral form (2.1) as \(n \to \infty\). From that limit function, we determine the limit points of the zeros of the integral forms (2.1) for \(n = 1, 2, \ldots\).

**Theorem 2.1.** For the \((m, n)\) Padé approximation to \((1+z)^{\alpha n+1}\) at 0, we have

(a) \[ (1+z)^{\alpha n+1} - \frac{p_m(z)}{q_m(z)} = \frac{z^{m+n+1} \int_0^1 (1-t)^m t^n (1+tz)^{\alpha n-m} dt}{q_n(z)}, \]

(b) \[ p_m(z) = \int_0^1 (t-1)^n t^{\alpha n-m} (1+z-t)^m dt, \]

and

(c) \[ q_n(z) = \int_0^1 (1-t)^m t^{\alpha n-m} (t(z+1)-1)^n dt. \]

**Proof.** We write

\[
(1+z)^{\alpha n+1} - \frac{p_m(z)}{q_m(z)} = z^{m+n+1} \epsilon_{\alpha n-m}(z)/q_n(z),
\]

Then

\[
[(1+z)^{\alpha n+1} q_n(z) - p_m(z)]^{(m+1)} = z^n T(z),
\]

where \(T(z)\) is a polynomial in \(z\) and \((1+z)^{\alpha n}\). Also

\[
[(1+z)^{\alpha n+1} q_n(z) - p_m(z)]^{(m+1)} = (1+z)^{\alpha n-m} S(z)
\]

where \(S(z)\) is polynomial of degree \(n\) in \(z\).

So we deduce that

\[
(2.2) \quad [(1+z)^{\alpha n+1} q_n(z) - p_m(z)]^{(m+1)} = C z^n (1+z)^{\alpha n-m},
\]

which implies

\[
\epsilon_{\alpha n-m}(z) = \frac{(1+z)^{\alpha n+1} q_n(z) - p_m(z)}{z^{m+n+1}}
\]

\[
= C^n \int_0^1 (1-t)^m t^n (1+tz)^{\alpha n-m} dt.
\]
In the last equality we used the following fact: if
\[ G(z) = \frac{z^{m+1}}{m!} \int_0^1 (1 - t)^m f(tz) dt, \]
then given suitable smoothness of \( f \),
\[ G^{(m+1)}(z) = f(z). \]

On the other hand, from (2.2) we have
\[ [(1 + z)^{\alpha n + 1} q_n(z)]^{(m+1)} = C z^n (1 + z)^{\alpha n-m}. \]
So integrating from \(-1\) gives back the correct initial terms. Let \( y = 1 + z \), from (2.3) we obtain
\[ [y^{\alpha n + 1} q_n(y - 1)]^{(m+1)} = C (y - 1)^n y^{\alpha n-m}. \]
Therefore,
\[ y^{\alpha n + 1} q_n(y - 1) = C^* y^{m+1} \int_0^1 (1 - t)^m (yt)^{\alpha n-m} (yt - 1)^n dt \]
or
\[ q_n(y - 1) = C^* \int_0^1 (1 - t)^m (yt - 1)^n dt, \]
which implies (c). (There is one free normalization constant.)

Now, consider \( p_m(z) \), from (a) and (c) we can write
\[ p_m(z) = (1 + z)^{\alpha n + 1} \int_0^1 (1 - t)^m t^{\alpha n-m} (t(z + 1) - 1)^n dt \]
\[ - z^{m+n+1} \int_0^1 (1 - t)^m t^n (1 + tz)^{\alpha n-m} dt. \]
Let \( t = (s(1 + z) - 1)/z \), we can rewrite
\[ z^{m+n+1} \int_0^1 (1 - t)^m t^n (1 + tz)^{\alpha n-m} dt \]
\[ = z^{m+n+1} \int_{1+z}^1 \left( \frac{1 + z}{z} \right)^m (1 - s)^m \frac{1}{z^n} [s(1 + z) - 1]^n s^{\alpha n-m} (1 + z)^{\alpha n-m} \frac{1 + z}{z} ds \]
\[ = (1 + z)^{\alpha n + 1} \int_{1+z}^1 (1 - s)^m [s(1 + z) - 1]^n s^{\alpha n-m} ds. \]
Therefore,
\[ p_m(z) = (1 + z)^{\alpha n + 1} \int_0^1 (1 - t)^m t^{\alpha n-m} (t(1 + z) - 1)^n dt \]
\[ = (1 + z)^{\alpha n + 1} \int_0^1 \left( 1 - \frac{s}{1 + z} \right)^m \left( \frac{s}{1 + z} \right)^{\alpha n-m} (s - 1)^n \frac{ds}{1 + z} \]
\[ = \int_0^1 (1 + z - s)^m s^{\alpha n-m} (s - 1)^n dt, \]
which is (b). \( \square \)

If we let \( m = cn \) and suppose that \( cn \) is an integer, we have the following corollary.
Corollary 2.2. For the \((cn, n)\) Padé approximation to \((1 + z)^{\alpha + 1}\) at 0, we have
\[(a) \quad (1 + z)^{\alpha + 1} \frac{p_{cn}(z)}{q_{n}(z)} = \frac{z^{\alpha + n + 1}}{q_{n}(z)} \int_{0}^{1} [(1 - t)^{\alpha} t(1 + t z)^{\alpha - c}]^{n} dt,\]
\[(b) \quad p_{cn}(z) = \int_{0}^{1} [(t - 1) t^{\alpha - c}(1 + z - t)]^{n} dt,\]
and
\[(c) \quad q_{n}(z) = \int_{0}^{1} [(1 - t)^{\alpha} t^{\alpha - c}(1 + z - 1)]^{n} dt.\]

The relations among \(p_{m}(z), q_{n}(z)\) and \(e_{\alpha n - m}\) are considered in the next corollary for some specific \(\alpha\) and \(c\).

Corollary 2.3. When \(c = 1\), we have
\[(2.4) \quad (1 + z)^{n} p_{n} \left( \frac{-z}{1 + z} \right) = q_{n}(z).\]

And if \(\alpha - c = 1\), then
\[(2.5) \quad (-1)^{n} e_{(\alpha - c) n}(-1 + z) = q_{n}(z).\]

Proof. From Corollary 2.2, we have
\[p_{n} \left( \frac{-z}{1 + z} \right) = \int_{0}^{1} [(t - 1) t^{\alpha - 1} \left( 1 - \frac{z}{1 + z} - t \right)]^{n} dt\]
\[= \left( \frac{1}{1 + z} \right)^{n} \int_{0}^{1} [(1 - t) t^{\alpha - 1} t(1 + z - 1)]^{n} dt,\]
which implies (2.4).

When \(\alpha - c = 1\), from (a) of Corollary 2.2,
\[e_{(\alpha - c) n}(-1 + z) = \int_{0}^{1} [(1 - t)^{\alpha} t(1 - t(1 + z))]^{n} dt\]
\[= (-1)^{n} \int_{0}^{1} [(1 - t)^{\alpha} t(t(1 + z) - 1)]^{n} dt,\]
which completes the proof of the corollary. \(\square\)

Since \(p_{cn}(z), q_{n}(z)\) and \(e_{(\alpha - c) n}(z)\) can all be written in the integral form
\[(2.6) \quad \int_{0}^{1} [(t - 1) f_{z}(t)]^{n} dt\]
where \(f_{z}(t)\) is a polynomial in both \(z\) and \(t\), it is natural to investigate some properties of this integral form.
Theorem 2.4. Let

\[ I_n = \int_0^1 [t(1 - t)f(t)]^n dt = \int_0^1 [Q(t)]^n dt \]

where \( Q(t) = t(1 - t)f(t) \) is a polynomial of degree \( N \) in \( t \). Let \( t_1, t_2, \ldots, t_{N-1} \) be the \( N - 1 \) zeros of \( Q'(t) \). Suppose that

\[ |Q(t_i)| \neq |Q(t_j)|, \quad i \neq j. \]

Then

\[ \lim_{n \to \infty} I_n^{1/n} = \arg(Q(t_i))|Q(t_i)| = Q(t_i) \quad \text{for some} \quad i. \]

Proof. In the proof we use the method of steepest descent and a saddle point argument. First, let’s recall the real case, see [7, p. 96, #198]. Suppose two functions \( \varphi(x) \) and \( g(x) \) are continuous and positive on the interval \([a, b]\). Then

\[ \lim_{n \to +\infty} \left\{ \int_a^b \varphi(x) [g(x)]^n dx \right\}^{1/n} \]

exists and is equal to the maximum of \( g(x) \) on \([a, b]\). (This is in fact a fairly easy exercise.)

Let’s return to the proof of the theorem. Observe that from any point there is a downhill contour (in decreasing magnitude) that terminates at one of the zeros of \( Q(t) \). Otherwise the path would end at a point of minimum modulus of \( Q(t) \) other than a zero which is impossible. So descent is always possible (and not to \( t = \infty \) because \( Q(t) \) is a polynomial in this case).

Now, suppose that \( A_i \) is a piece of arc of constant argument for \( Q(t) \) through \( t_i \) from \( \gamma_i \) to \( \delta_i \). Then by the above observation we can connect \( \gamma_i \) to one zero and \( \delta_i \) to another to form a contour \( B_i \) from one zero of \( Q(t) \) to another in a descending fashion. Do this procedure for all the \( t_i \). This forms no closed contours since if it did integrating around one of these is non-zero by a steepest descent argument but is zero by Cauchy’s Theorem.

Thus the contour \( B_1, B_2, \ldots, B_{N-1} \) in some order must connect all \( N \) zeros of \( Q(t) \) with exactly one link.

In particular there is a path from 0 to 1 via some of the saddle points of \( Q(t) \) (not necessarily all), say \( t_{i_k}, \quad k = 1, 2, \ldots, r \), where \( r \leq N - 1 \). So we may suppose that

\[ |Q(t_{i_1})| > |Q(t_{i_k})|, \quad k = 2, \ldots, r. \]

Now, we can apply the modified steepest descent argument in the real case (see [7, p. 287, #198]) since along \( A_{i_1} \) \( Q(t) \) has constant argument while for the remainder of the \( B_{i_1} \), the modulus of \( Q(t) \) is less than \( |Q(t_{i_1})| \) by the construction of \( B_{i_1} \). And along other \( B_{i_k}, \quad k = 2, \ldots, r \), we know that the modulus of \( Q(t) \) is less than \( |Q(t_{i_1})| \) too by (2.8) and the constructions of \( B_{i_k}, \quad k = 2, \ldots, r \). This gives us the desired result. \( \square \)

Theorem 2.4 is a point-wise version for \( z \) if \( f_z(t) \) is a polynomial in \( t \) and analytic in \( z \) on some compact set \( K \). From Theorem 2.4 we can prove the following uniform version of Theorem 2.4, which is the result we really need.
Theorem 2.5. Let

\[ I_n(z) = \int_0^1 [t(1-t)f_z(t)]^n dt = \int_0^1 [Q_z(t)]^n dt \]

where \( Q_z(t) = t(1-t)f_z(t) \) is a polynomial in \( t \) and analytic in \( z \) on an open connected set \( U \). Suppose

\[ |Q_z(t_i(z))| \neq |Q_z(t_j(z))| \]

for any \( i \neq j \), and any \( z \in U \), where \( t_i := t_i(z) \) are the zeros of the polynomial \( \frac{d}{dt} Q_z(t) \) (which by the above assumption can be given so that each \( t_i \) is analytic on \( U \)). Then

(a) \( I_n(z)^{1/n} \) converges to a non-zero limit pointwise on \( U \).

(b) \( |I_n(z)|^{1/n} \) is uniformly bounded on compact subsets of \( U \).

(c) \( I_n(z)^{1/n} \) converges uniformly to \( Q_z(t_i(z)) \) on compact subsets of \( U \), and \( Q_z(t_i(z)) \) is analytic on \( U \). Moreover, \( Q_z(t_i(z)) \neq 0 \) for all \( z \in U \) provided that there is a \( V \subset U \) which has a limit point in \( U \).

Proof. (a) This is the content of Theorem 2.4.

(b) This is obvious from the definition of \( I_n(z) \).

(c) Denote the open disk centered at \( z \) with radius \( \epsilon \) as \( D(z, \epsilon) \), and the corresponding closed disk as \( \overline{D}(z, \epsilon) \). Now, pick a \( z_0 \in U \), then there is an \( i_0 \), such that

\[ |Q_{z_0}(t_{i_0}(z_0))| > |Q_{z_0}(t_i(z_0))|, \quad \text{for} \quad i \neq i_0. \]

Let

\[ d = \min_{i \neq i_0} \{|Q_{z_0}(t_{i_0}(z_0))| - |Q_{z_0}(t_i(z_0))|\} > 0, \]

then since \( t_i(z) \ (i = 1, \ldots, N - 1) \) is a continuous function of \( z \) and \( Q_z(t_i(z)) \) is analytic, there is a \( \epsilon_1 \) such that

\[ \|Q_z(t_i(z))\|_{C(\overline{D}(z_0, \epsilon_1))} \geq V(z_0) - \frac{d}{4} \]

and

\[ \|Q_z(t_i(z))\|_{C(\overline{D}(z_0, \epsilon_1))} \leq V(z_0) - \frac{3}{4} d \]

for \( i \neq i_0 \) where

\[ V(z_0) = |Q_{z_0}(t_{i_0}(z_0))|. \]

According to the proof of Theorem 2.4, there is a contour \( A_{i_0}(z_0) \) from \( \gamma_{i_0}(z_0) \) to \( \delta_{i_0}(z_0) \) with constant argument, which passes through \( t_{i_0}(z_0) \). In addition, we choose \( \gamma_{i_0}(z_0) \) and \( \delta_{i_0}(z_0) \) such that

\[ |Q_{z_0}(\gamma_{i_0}(z_0))] \leq V(z_0) - \frac{3}{4} d \]

\[ |Q_{z_0}(\delta_{i_0}(z_0))] \leq V(z_0) - \frac{3}{4} d \]
and
\[|Q_{z_0}(\xi_{i_0}(z_0))| \leq V(z_0) - \frac{3}{4}d.\]

Denote the whole contour from 0 to 1 through \(t_{i_0}(z_0)\) by \(\Gamma(z_0)\) and the length of \(\Gamma(z_0)\) by \(\ell(z_0)\). And suppose the length of the part of \(A_{i_0}(z_0)\) such that \(|Q_{z_0}(t)| \geq V(z_0) - \frac{d}{2}\) is \(r\).

Now, for \(z \in D(z_0, \epsilon_1)\). We construct the contour \(\Gamma(z)\) in the same fashion, that is,
\[|Q_z(\gamma_{i_0}(z))| \leq V(z_0) - \frac{3}{4}d\]
and
\[|Q_z(\xi_{i_0}(z))| \leq V(z_0) - \frac{3}{4}d.\]

Choose \(\epsilon_2 > 0\), such that
\[\ell(z) \leq \ell(z_0) + 1, \quad \text{for } z \in D(z_0, \epsilon_2),\]
and \(\epsilon_3 > 0\), such that the length of the part of \(A_{i_0}(z)\) with \(|Q_z(t)| \geq V(z_0) - \frac{d}{2}\) is larger than or equal to \(r/2\) for \(z \in D(z_0, \epsilon_3)\).

Let \(\epsilon = \min\{\epsilon_1, \epsilon_2, \epsilon_3\}\), then for \(z \in D(z_0, \epsilon)\), we have
\[
|I_n(z)|^{1/n} = \left| \int_0^1 [Q_z(t)]^n dt \right|^{1/n} = \left| \int_{\Gamma(z)} [Q_z(t)]^n dt \right|^{1/n} = \left| \int_{A_{i_0}(z)} [Q_z(t)]^n dt + \int_{\Gamma'(z)} [Q_z(t)]^n dt \right|^{1/n}
\]
where \(\Gamma'(z)\) is the part of \(\Gamma(z)\) without \(A_{i_0}(z)\).

From the choice of \(\epsilon\) and the construction of \(\Gamma(z)\), we have
\[
|I_n(z)|^{1/n} \geq \left| \int_{A_{i_0}(z)} [Q_z(t)]^n dt \right|^{1/n} \times \left[ 1 - \frac{\int_{\Gamma'(z)} [Q_z(t)]^n dt}{\int_{A_{i_0}(z)} [Q_z(t)]^n dt} \right]^{1/n} \geq \left[ \frac{r}{2} \left( V(z_0) - \frac{d}{2} \right) \right]^{1/n} \left[ 1 - \frac{(\ell(z_0) + 1)(V(z_0) - \frac{d}{2})}{\frac{r}{2}(V(z_0) - \frac{d}{2})} \right]^{1/n}.
\]
Therefore, there is an \(n_0\) which depends only on \(z_0\), such that
\[|I_n(z)|^{1/n} > 0 \quad \text{for } z \in D(z_0, \epsilon), \quad n \geq n_0.\]

Now, consider any compact subset \(K\) of \(U\). For any \(z_0 \in K\), by the above argument, there are \(\epsilon(z_0)\) and \(n(z_0)\), such that
\[|I_n(z)|^{1/n} > 0, \quad \text{for } z \in D(z_0, \epsilon(z_0)), \quad n \geq n_0.\]
Thus, we can pick up finitely many z in K, say \( z_i, \ i = 1, 2, \ldots, M \), such that

\[ K \subset \bigcup_{i=1}^{M} D(z_i, \epsilon(z_i)). \]

Let \( n_0 = \max_{1 \leq i \leq M} \{ n(z_i) \} \), then

\[ |I_n(z)|^{1/n} > 0 \quad \text{for } z \in k, \ n \geq n_0. \]

That is, \( I_n(z)^{1/n} \) is analytic on \( K \) for \( n \geq n_0 \) (in the sense that there is a well defined analytic nth root).

From the above arguments, (a) and (b), and applying Vitali theorem we know that \( I_n(z)^{1/n} \) converges uniformly on compact subsets of \( U \) to a analytic function \( Q(z)(t_i(z)) \). Now, we can apply the uniqueness theorem, which implies that \( I_n(z)^{1/n} \) must converges to the same \( Q(z)(t_i(z)) \) on all compact subsets of \( U \). And from Hurwitz’s Theorem, we see that

\[ Q(z)(t_i(z)) \neq 0 \quad \text{for all } z \in U. \]

\[ \square \]

From Theorem 2.4 and 2.5, we have knowledge of the limit function of \( I_n(z)^{1/n}, \ n = 1, 2, \ldots \). In fact, the limit function tells us more.

**Corollary 2.6.** Let \( I_n(z), f_z(t) \) and \( Q_z(t) \) be as in Theorem 2.5. Suppose that for each \( z, Q_z(t) \) is a polynomial of degree \( N \) in \( t \), and further that \( Q_z(t) \) is analytic in \( z \). Then, the limit points of the zeros of \( I_n(z) \) can only cluster on the curve

\[ \{ z : |Q_z(t_i(z))| = |Q_z(t_j(z))|, \quad \text{for some } i \neq j \} \]

or at points where \( Q_z(t_i(z)) = 0 \), or at points where \( Q_z(t_i(z)) \) is not analytic. (Note \( t_i := t_i(z) \).)

**Proof.** Let \( U \) be an open and connected set which is disjoint from the curves and points stated in this corollary. Suppose \( S \) is a compact subset of \( U \) where \( I_n(z)^{1/n} \) is analytic, which will be whenever the n-th root is well defined and non-zero. Then by Theorem 2.4 and 2.5

\[ I_n(z)^{1/n} \to Q_z(t_i) \neq 0, \quad \text{as } n \to \infty. \]

pointwise on \( S \). Therefore, applying Vitali Theorem, \( I_n(z)^{1/n} \) converges uniformly on any such compact subset of \( U \) to the non-zero analytic limit \( Q_z(t_i(z)) \). \( \square \)

From now on, we will call the curve

\[ \{ z : |Q_z(t_i(z))| = |Q_z(t_j(z))|, \quad \text{for some } i \neq j \} \]

the critical curve of \( I_n(z) \).
§3. Padé approximation to \((1 + z)^{\alpha n+1}\).

In this section, we apply the results of Theorem 2.4 and 2.5, and their Corollary to the Padé approximation to \((1 + z)^{\alpha n+1}\) at 0, and analyze the limiting location of the zeros of \(p_{cn}(z), q_{n}(z)\) and \(e_{(\alpha - c)n}(z)\). In fact, all the procedures we discuss in this section can be executed automatically by computer. (This we did using Maple.)

First let’s note the following facts. For \(c = 1\), from Corollary 2.3, if we have knowledge of the distribution of the zeros of \(q_{n}(z)\), then we know the distribution of the zeros of \(p_{n}(z)\). Similarly, when \(\alpha - c = 1\), we know the distribution of the zeros of \(e_{(\alpha - c)n}(z)\) from that of \(q_{n}(z)\). Since \(I_{n}(x)^{1/n}\) converges uniformly on any compact subset \(S \subset U\) to the non-zero analytic limit \(Q_{z}(t_{i}(z))\), where \(I_{n}(z)^{1/n}\) is analytic, in order to see which root of \(\frac{d}{dz}(Q_{z}(t)) I_{n}(z)^{1/n}\) goes to, it is sufficient, by analytic continuation to check which root it will approach on a segment \(A\) of the real axis provided that \(A \subset U\).

It is amusing to observe that the critical curves for \(p_{cn}(z), q_{n}(z)\) and \(e_{(\alpha - c)n}(z)\) are all the same, essentially since we can write

\[
(3.1) \quad p_{cn}(z) = (1 + t)^{\alpha n+1} \int_{0}^{z} [(1 - t)^{\alpha}(t(1 + z) - 1)]^{n} dt,
\]

\[
(3.2) \quad q_{n}(z) = \int_{0}^{1} [(1 - t)^{\alpha}(t(1 + z) - 1)]^{n} dt,
\]

and

\[
(3.3) \quad e_{(\alpha - c)n}(z) = \frac{(1 + z)^{\alpha n+1}}{z^{\alpha n+1}} \int_{1/z}^{1} [(1 - t)^{\alpha}(t(1 + z) - 1)]^{n} dt
\]

from the proof of Theorem 2.1. And notice that

\[
g_{z}(0) = g_{z}(1) = g_{z}\left(\frac{1}{1 + z}\right) = 0
\]

where \(g_{z}(t) = (1 - t)^{\alpha}(t(1 + z) - 1)\). However, \(p_{cn}(z), q_{n}(z)\) and \(e_{(\alpha - c)}(z)\) may pick up different branches of that critical curve. We shall see that later.

To illustrate the procedures, we consider the case \(c = 1\). In this case, we have

\[
(3.4) \quad p_{n}(z) = \int_{0}^{1} [(t - 1)^{\alpha}(1 - t + z)]^{n} dt,
\]

\[
(3.5) \quad q_{n}(z) = \int_{0}^{1} [(1 - t)^{\alpha}(t(1 + z) - 1)]^{n} dt,
\]

and

\[
(3.6) \quad e_{(\alpha - 1)n}(z) = \int_{0}^{1} [(1 - t)(1 + tz)^{\alpha - 1}]^{n} dt.
\]
Let $Q_z(t) = (1 - t)t^{\alpha-1}(t(1 + z) - 1)$, then

$$Q_z(0) = Q_z(1) = Q_z\left(\frac{1}{1 + z}\right) = 0,$$

and

$$\frac{d}{dt}Q_z(t) \bigg|_{t = t_{1,2}(z)} = 0$$

where

$$t_{1,2}(z) = \frac{\alpha(z + 2) \pm \mu}{2(z + 1)(1 + \alpha)},$$

$$\mu = (\alpha^2 z^2 + 4z + 4)^{1/2}.$$

Therefore, from Corollary 2.6 and the above observation, the critical curve for $p_n(z)$, $q_n(z)$ and $e_{(\alpha - 1)n}(z)$ is

$$\{z : |Q_z(t_1(z))| = |Q_z(t_2(z))|\},$$

which is

$$\left\{ z : \begin{vmatrix} \frac{\alpha z + 2z + 2 + \mu}{\alpha z + 2z + 2 - \mu} & \frac{\alpha z - 2 - \mu}{\alpha z - 2 + \mu} & \frac{\alpha z + 2\alpha - \mu}{\alpha z + 2\alpha + \mu} \end{vmatrix}^{\alpha - 1} = 1 \right\}$$

where

$$\mu = (\alpha^2 z^2 + 4z + 4)^{1/2}.$$

From (3.9), one can see that the critical curve is always symmetric about the real axis for any $\alpha$. The critical curves for $\alpha = 2$, $\alpha = 3$, $\alpha = 5$ and $\alpha = 8$ are shown in Figures 1, 2, 3 and 4 respectively. In Figures 5 and 6, we plot the zeros of $p_n(z)$ and $q_n(z)$ for $\alpha = 2$, $n = 20$ and $\alpha = 3$, $n = 10$ respectively. We also plot the zeros of $e_{(\alpha - 1)n}(z)$ for $\alpha = 3$, $n = 15$ in Figure 7. These pictures indicate that the zeros of $p_n(z)$, $q_n(z)$ and $e_{(\alpha - 1)n}(z)$, $n = 1, 2, \ldots$, are dense on the three different branches of the critical curve (3.8). Indeed, one can prove this fact.

Now, we restrict our attention to the case $\alpha = 2$, $c = 1$. Then, we have

$$q_n(z) = \int_0^1 [(1 - t)t(t(1 + z) - 1)]^n dt$$

and the critical curve (3.9) is replaced by

$$\left\{ z : \begin{vmatrix} \frac{2z + 1 + \nu}{2z + 1 + \nu} & \frac{z - 1 - \nu}{z - 1 + \nu} & \frac{z + 2 - \nu}{z + 2 + \nu} \end{vmatrix} = 1 \right\}$$

where

$$\nu = (1 + z + z^2)^{1/2}.$$
To analyze which root \( q_n(z) \) will pick up on the four regions bounded by (3.11) and the branch lines where \( v \) changes its branches (see Figure 1), it is sufficient to consider the real segments contained in these four regions \( R_1, R_2, R_3 \) and \( R_4 \). We specify the four regions by \( R_1 \) contains \(-\infty, \ R_2 \) contains \(-1, \ R_3 \) contains \( 0 \) and \( R_4 \) contains \( +\infty \). From (3.10) and (3.7), we know that

\[
Q_z(t) = (1 - t)t(1 + z) - 1
\]

and

\[
t_{1,2}(z) = \frac{z + 2 \pm \nu}{3(1 + z)}.
\]

Let \( A_1 = \{x : x \text{ is real, } -5 \leq x \leq -3\} \subset R_1 \), then

\[
t_1(x) = \frac{x + 2 + \nu}{3(1 + x)} \notin [0, 1], \quad \text{for } x \in A_1,
\]

and

\[
t_2(x) = \frac{x + 2 - \nu}{3(1 + x)} \in [0, 1], \quad \text{for } x \in A_1.
\]

Then, by a saddle point argument (see [7, p.287, #198])

\[
\{q_n(x)\}^{1/n} \to Q_x(t_2(x))
\]

pointwise on \( A_1 \). Now, applying the argument we used in the proof of Theorem 2.5, we obtain that

\[
\{q_n(z)\}^{1/n} \to Q_x(t_2(z))
\]

uniformly on compact subsets of \( R_1 \).

Let \( A_2 = \{x : x \text{ is real, } -3/2 \leq x \leq -11/10\} \subset R_2 \), then

\[
t_1(x) \notin [0, 1], \quad \text{for } x \in A_2,
\]

and

\[
t_2(x) \in [0, 1], \quad \text{for } x \in A_2.
\]

Therefore, \( \{q_n(z)\}^{1/n} \) converges to \( Q_x(t_2(z)) \) uniformly on compact subsets of \( R_2 \).

Set \( A_3 = \{x : x \text{ is real, } 0 \leq x \leq 1/2\} \subset R_3 \), then for \( x \in A_3 \), we have \( t_1(x) \in [0, 1] \) and \( t_2(x) \in [0, 1] \). But

\[
|Q_x(t_1(x))| < |Q_x(t_2(x))|.
\]

Thus, \( \{q_n(z)\}^{1/n} \) converges to \( Q_x(t_2(z)) \) uniformly on compact subsets of \( R_3 \).

Set \( A_4 = \{x : x \text{ is real, } 2 \leq x \leq 4\} \subset R_4 \), then \( t_1(x), t_2(x) \in [0, 1] \), for \( x \in A_4 \), but

\[
|Q_x(t_1(x))| > |Q_x(t_2(x))|.
\]

Therefore, \( \{q_n(z)\}^{1/n} \) converges to \( Q_x(t_1(z)) \) uniformly on compact subsets of \( R_4 \).

From the above consideration, the uniqueness theorem and Montel’s theorem (see [2]) we can prove that the limit points of the zeros of \( \{q_n(z)\}^{\infty}_{n=1} \) are dense on the branch \( B_3 \), which is the boundary between \( R_3 \) and \( R_4 \).

Therefore, we have proved
Theorem 3.1. \( \{q_n(z)\}^{1/n} \) converges to \( Q_z(t_2(z)) \) uniformly on any compact subset of \( R_1, R_2 \) and \( R_3 \), and to \( Q_z(t_3(z)) \) uniformly on any compact subset of \( R_4 \). Moreover, the limit points of the zeros of \( \{q_n(z)\}^{\infty}_{n=1} \) are dense on the branch \( B_3 \), which is the boundary between \( R_3 \) and \( R_4 \).

Similarly, we can consider \( p_n(z) \) and \( e_{(\alpha-1)n}(z) \). The analogs for \( p_n(z) \) and \( e_{(\alpha-1)n}(z) \) are summarized in Theorem 3.2 and 3.3.

Theorem 3.2. \( \{p_n(z)\}^{1/n} \) converges to \((1+z)^\alpha Q_z(t_1(z))\) uniformly on any compact subset of \( R_1 \) and \( R_2 \), and to \((1+z)^\alpha Q_z(t_2(z))\) uniformly on any compact subset of \( R_3 \) and \( R_4 \). Moreover, the limit points of the zeros of \( \{p_n(z)\}^{\infty}_{n=1} \) are dense on the branch \( B_2 \), which is the boundary between \( R_2 \) and \( R_3 \).

Theorem 3.3. \( \{e_{(\alpha-1)n}\}^{1/n} \) converges to \((1+z)^\alpha Q_z(t_1(z))/z^2\) uniformly on any compact subset of \( R_1 \) and to \((1+z)^\alpha Q_z(t_2(z))/z^2\) uniformly on any compact subset of \( R_2 \), \( R_3 \) and \( R_4 \). Moreover, the limit points of the zeros of \( \{e_n(z)\}^{\infty}_{n=1} \) are dense on the branch \( B_1 \), which is the boundary between \( R_1 \) and \( R_2 \).

§4. Incomplete rationals.

We have established the results on the zeros and poles of Padé approximants to \((1+z)^{\alpha n+1}\) and on the zeros of the Padé remainder in Section 3. In addition, we know that \( p_n(z) \), \( q_n(z) \) and \( e_{(\alpha-1)n}(z) \) converge to some analytic functions on \( R_1, R_2, R_3 \) and \( R_4 \) respectively. In this section we apply these results to analyze the limit functions of \((1+z)^{\alpha n+1} q_n(z)/p_n(z)\) on \( R_1, R_2, R_3 \) and \( R_4 \). And then we prove that the collection \( \{(1+z)^{\alpha} r_n(z)/s_n(z)\}^{\infty}_{n=1} \) is dense on \( R_3 \) where \( r_n(z) \) and \( s_n(z) \) belong to \( \pi_n \).

First, we prove the following theorem.

Theorem 4.1. Let \( p_n(z) \), \( q_n(z) \) and \( e_{(\alpha-1)n}(z) \) be as in Corollary 2.2 in the case \( c = 1 \). Then we have that \((1+z)^{\alpha n+1} q_n(z)/p_n(z)\) converges

(a) to \( \infty \) uniformly on any compact subset of \( R_1 \) and \( R_4 \);

(b) to \( 0 \) uniformly on any compact subset of \( R_2 \);

(c) to \( 1 \) uniformly on any compact subset of \( R_3 \).

Remark. Observe that \( 1 \) cannot be approximated on any region strictly larger than \( R_3 \) by the Rouché’s Theorem, so \( R_3 \) is a natural maximal region of denseness.

Proof. (a) We consider \( R_1 \) first (similarly for \( R_4 \)). Let \( K_1 \) be a compact subset of \( R_1 \). Then from (3.1), (3.2), (3.14), and Theorem 3.1 and 3.2, we have

\[
\{p_n(z)\}^{1/n} \rightarrow (1+z)^{\alpha} Q_z(t_1(z)),
\]

and

\[
\{q_n(z)\}^{1/n} \rightarrow Q_z(t_2(z))
\]

uniformly on \( K_1 \).
Therefore,
\[
\left| (1 + z)^{\alpha n + 1} \frac{q_n(z)}{p_n(z)} \right|^{1/n} \to \frac{|Q_z(t_2(z))|}{|Q_z(t_1(z))|} > 1 + \epsilon
\]
by the definition of critical curve (see (3.8)) and nature of $R_1$, where $\epsilon \in (0, 1)$ depends on $K_1 \subset R_1$. Thus, we conclude that $(1+z)^{\alpha n + 1} q_n(z)/p_n(z)$ converges to $\infty$ uniformly on $K_1$.

Now, let $K_4$ be a compact subset of $R_4$, then
\[
\{p_n(z)\}^{1/n} \to (1 + z)^{\alpha} Q_z(t_2(z)),
\]
and
\[
\{q_n(z)\}^{1/n} \to Q_z(t_1(z))
\]
uniformly on $K_4$, which implies
\[
\left| (1 + z)^{\alpha n + 1} \frac{q_n(z)}{p_n(z)} \right|^{1/n} \to \frac{|Q_z(t_2(z))|}{|Q_z(t_1(z))|} > 1 + \epsilon, \quad \text{on } K_4.
\]
Therefore, we complete the proof of (a).

(b) Let $K_2$ be a compact subset of $R_2$, then
\[
\{p_n(z)\}^{1/n} \to (1 + z)^{\alpha} Q_z(t_1(z)),
\]
and
\[
\{q_n(z)\}^{1/n} \to Q_z(t_2(z))
\]
uniformly on $K_2$, thus we have
\[
\left| (1 + z)^{\alpha n + 1} \frac{q_n(z)}{p_n(z)} \right|^{1/n} \to \frac{|Q_z(t_2(z))|}{|Q_z(t_1(z))|} < 1 - \epsilon, \quad \text{on } K_2.
\]
(see (3.15)). Therefore, $(1 + z)^{\alpha n + 1} q_n(z)/p_n(z)$ converges to $0$ uniformly on $K_2$.

(c) Let $K_3$ be compact subset of $R_3$. Since
\[
(1 + z)^{\alpha n + 1} \frac{p_n(z)}{q_n(z)} = \frac{z^{\alpha n + 1} e(\alpha - 1) n(z)}{q_n(z)},
\]
and from (3.1), (3.2), Theorem 3.2 and 3.3, we have
\[
(1 + z)^{\alpha n + 1} \frac{q_n(z)}{p_n(z)} - 1 = z^{2n + 1} e(\alpha - 1) n(z).
\]
But
\[
\{p_n(z)\}^{1/n} \to (1 + z)^{\alpha} Q_z(t_2(z)),
\]
and
\[
\{e(\alpha - 1) n(z)\}^{1/n} \to (1 + z)^{\alpha} Q_z(t_1(z))/z^2
\]
uniformly on $K_3$. Thus, from (3.15), we obtain
\[
\left| (1 + z)^{\alpha n + 1} \frac{q_n(z)}{p_n(z)} - 1 \right|^{1/n} = \left| z^{\alpha n + 1} e(\alpha - 1) n(z) \right|^{1/n} \to \frac{|Q_z(t_1(z))|}{|Q_z(t_2(z))|} < 1 - \epsilon, \quad \text{on } K_3.
\]
Therefore, we obtain the desired results. \( \square \)
Theorem 4.2. \( \{ (1 + z)^{\alpha_n} r_n(z) / s_n(z) : r_n(z), s_n(z) \in \pi_n, n \in \mathbb{N} \} \) is dense in \( A(K) \) where \( K \) is an arbitrary compact subset of \( \mathbb{R}^3 \).

Proof. Note first that
\[
T = \{ f(z) = (1 + z)^{\alpha_n} r_n(z) / s_n(z) : r_n(z), s_n(z) \in \pi_n, n \in \mathbb{N} \}
\]
is closed under addition provided that we have the same degree and same denominator, and closed under multiplication.

Therefore, if \( (1 + z)^{\alpha_n} r_n(z) / s_n(z) \) can approximate 1 and \( z \) with the same \( s_n(z) \), they can approximate the linear form \( az + b \). And from the above observation we know that \( (1 + z)^{\alpha_n} r_n(z) / s_n(z) \) can approximate any polynomial \( p(z) \) since it can be written as
\[
p(z) = \Pi(a_k z + b_k).
\]

Notice that the collection of all polynomials is dense in \( A(K) \), thus
\[
\{ (1 + z)^{\alpha_n} r_n(z) / s_n(z) : r_n(z), s_n(z) \in \pi_n, n \in \mathbb{N} \} \]
is dense in \( A(K) \) provided that \( (1 + z)^{\alpha_n} r_n(z) / s_n(z) \) can approximate 1 and \( z \) with the same denominator.

Let \( K \) be an arbitrary compact subset of \( \mathbb{R}^3 \). We choose \( \delta > 0 \) small enough such that \( K \) is a subset of \( \mathbb{R}^3 \) corresponding to \( \alpha' = \alpha(1 + \delta) \). Note that from (3.8) or (3.9) we know that the critical curve is a continuous function of \( \alpha \) and \( R'_3 \subset \mathbb{R}^3 \).

From Theorem 4.1, we have \( p_n(z) \) and \( q_n(z) \) for \( \alpha' = \alpha(1 + \delta) \) such that \( (1 + z)^{\alpha(1 + \delta)n+1} q_n(z) / p_n(z) \) converges uniformly to 1 on \( K \). Now, we choose \( p[\delta n](z) \), \( q[\delta n](z) \) and \( t[\delta n](z) \) such that
\[
\frac{q[\delta n](z)}{p[\delta n](z)} \rightarrow 1 + z, \quad \frac{t[\delta n](z)}{p[\delta n](z)} \rightarrow z(1 + z)
\]
uniformly on \( K \). Therefore, we have
\[
(1 + z)^{\alpha(1 + \delta)n+1} \frac{q_n(z) q[\delta n](z)}{p_n(z) p[\delta n](z)} \rightarrow 1 + z \tag{4.3}
\]
and
\[
(1 + z)^{\alpha(1 + \delta)n+1} \frac{q_n(z) t[\delta n](z)}{p_n(z) p[\delta n](z)} \rightarrow z(1 + z) \tag{4.4}
\]
uniformly on \( K \). That is
\[
(1 + z)^{\alpha(1 + \delta)n} \frac{q_n + [\delta n](z)}{p_n + [\delta n](z)} \rightarrow 1 \tag{4.5}
\]
and
\[
(1 + z)^{\alpha(1 + \delta)n} \frac{t_n + [\delta n](z)}{p_n + [\delta n](z)} \rightarrow z \tag{4.6}
\]
uniformly on \( K \) where \( p_n + [\delta n](z) = p_n(z)p[\delta n](z) \), \( q_n + [\delta n](z) = q_n(z)q[\delta n](z) \) and \( t_n + [\delta n](z) = q_n(z)t[\delta n](z) \).

From (4.5) and (4.6) we know that
\[
(1 + z)^{\alpha(1 + \delta)n} \frac{a[\delta n](z) + b[\delta n](z)}{p_n + [\delta n](z)}
\]
converges to \( az + b \) uniformly on \( K \), which completes the proof of the theorem. \( \Box \)
§5. Incomplete polynomials.

If we let $c = 0$, then instead of incomplete rationals, we have the incomplete polynomials. (For a discussion of approximation by incomplete polynomials, applications and the relations among Padé approximants, incomplete polynomials, and orthogonal polynomials, see Lorentz [4] and Saff [8] and the references therein.)

From Corollary 2.2 and (3.3) we have

\begin{align}
\tag{5.1}
p_0(z) &= \int_0^1 [(t - 1)t^\alpha]^n dt,
\end{align}

\begin{align}
\tag{5.2}
q_n(z) &= \int_0^1 [t^\alpha(t(1 + z) - 1)]^n dt,
\end{align}

and

\begin{align}
\tag{5.3}
e_{\alpha n}(z) &= \frac{(1 + z)^{\alpha n + 1}}{z^{n + 1}} \int_0^1 [t^\alpha(t(1 + z) - 1)]^n dt.
\end{align}

Let $R_z(t) = t^\alpha(t(1 + z) - 1)$, then

\begin{align}
\tag{5.4}
R_z(0) &= R_z \left( \frac{1}{1 + z} \right) = 0.
\end{align}

Since we do not have the factor $(1 - t)$ in $R_z(t)$, we can not apply Theorem 2.4 and 2.5 to $q_n(z)$ and $e_{\alpha n}(z)$ directly. But, since $R_z(t)$ is a polynomial in both $t$ and $z$, and has exactly one non-trivial critical point $t^* = \alpha/[(1 + \alpha)(1 + z)]$, by the argument in Theorem 2.4, there is a contour $B$ from 0 to $1/(1 + z)$, and a downhill contour which starts at 1, and terminates at 0 or $1/(1 + z)$. Therefore, there are contours that connect 0 and 1 (for $q_n(z)$), or $1/(1 + z)$ and 1 (for $e_{\alpha n}(z)$).

From this observation, and modifying the proofs of Theorem 2.4, 2.5 and Corollary 2.6, we have

**Theorem 5.1.** Let $q_n(z)$, $e_{\alpha n}(z)$ as stated in (5.2) and (5.3). Then $q_n(z)$ and $e_{\alpha n}(z)$ have the same critical curve

\begin{align}
\tag{5.5}
\{z : |R_z(t^*)| = |R_z(1)|\}
\end{align}

where $R_z(t) = t^\alpha(t(1 + z) - 1)$, $t^* = \alpha/[(1 + \alpha)(1 + z)]$. That is, the limit points of the zeros of $q_n(z)$ or $e_{\alpha n}(z)$ can only cluster on the curve (5.5).

We can write (5.5) explicitly

\begin{align}
\tag{5.6}
\{z : |z(1 + z)^\alpha| = \alpha^\alpha/\alpha^{1+\alpha}\}.
\end{align}

Figure 8 is the critical curve (5.6) when $\alpha = 2$. By almost identical arguments to those in section 3, the following theorems can be proved. Note that this time we do not have any branch lines.
Theorem 5.2. \( \{q_n(z)\}^{1/n}_{n=1} \) converges to \( R_3(1) \) uniformly on any compact subset of \( R_1 \) and \( R_2 \), and to \( R_3(t^*) \) uniformly on any compact subset of \( R_3 \). Moreover, the limit points of the zeros of \( \{q_n(z)\}^{\infty}_{n=1} \) are dense on the branch \( B_2 \), which is the boundary between \( R_1 \) and \( R_2 \).

Theorem 5.3. \( \{e_{\alpha n}(z)\}^{1/n}_{n=1} \) converges to \((1+z)^{\alpha} R_3(1)/z\) uniformly on any compact subset of \( R_1 \) and \( R_3 \), and to \((1+z)^{\alpha} R_3(t^*)/z\) uniformly on any compact subset of \( R_2 \). Moreover, the limit points of the zeros of \( \{e_{\alpha n}(z)\}^{\infty}_{n=1} \) are dense on the branch \( B_1 \), which is the boundary between \( R_1 \) and \( R_2 \).

The analog of Theorem 4.2 is the following

Theorem 5.4. \( \{(1+z)^{\alpha n} p_n(z) : p_n(z) \in \pi_n\}^{\infty}_{n=1} \) is dense in \( A(K) \) where \( K \) is an arbitrary compact subset of \( R_3 \).

§6. Padé approximation to \( e^z \).

In this section, we consider Padé approximation to \( e^z \). In a sequence of papers of Saff and Varga ([9], [10], [11]), they examined the Padé approximation to \( e^z \) in detail. The purpose of this section is to observe that this is the limiting case of Padé approximation to \((1+z)^{\alpha n+1}\). We verify this as follows.

From Corollary 2.2 and using the substitution \( t = 1 - \frac{z}{\alpha} \), we can write

\[
p_{cn}(z) = \int_0^1 [(t-1)^{\alpha-c}(1+z-t)^c]^{n}dt \\
= (-1)^n \left( \frac{1}{\alpha} \right)^{c_{n+1}} \int_0^{\alpha} \left[ s \left( 1 - \frac{s}{\alpha} \right)^{\alpha-c} (\alpha z + s)^c \right]^{n}ds.
\]

And similarly, we have

\[
q_n(z) = (-1)^n \left( \frac{1}{\alpha} \right)^{c_{n+1}} \int_0^{\alpha} \left[ t^c \left( 1 - \frac{t}{\alpha} \right)^{\alpha-c} ((1+z)t - \alpha z) \right]^{n}dt.
\]

Therefore, (a) of Corollary 2.2 can be written as

\[
(1+z)^{\alpha n+1} = \frac{\int_0^{\alpha} [t(1-\frac{t}{\alpha})^{\alpha-c}((1+z)t - \alpha z)]^n dt}{\int_0^{\alpha} [t^{\alpha-c}(1+z)^{\alpha-c}(1+tz)^{\alpha-c}]^n dt} = (-1)^n(\alpha z)^{c_{n+1}} \frac{\int_0^{1} ((1-t)^c t(1+tz)^{\alpha-c})^n dt}{\int_0^{\alpha} [t^{\alpha-c}((1+z)t - \alpha z)]^n dt}.
\]

Let \( z = y/2 \), and next let \( \alpha \to \infty \), then from (6.3), we obtain

\[
e^{ny} = \frac{\int_0^{\infty} [te^{-t}(t+y)^c]^n dt}{\int_0^{\infty} [te^{-t}(t-y)]^n dt} = (-1)^n y^{c_{n+1}} \frac{\int_0^{1} [(1-t)^c te^y]^n dt}{\int_0^{\infty} [t^{\alpha-c}((1+z)t - \alpha z)]^n dt},
\]

\[
e^{wy} = \frac{\int_0^{\infty} [te^{-t}(t+y)^c]^n dt}{\int_0^{\infty} [te^{-t}(t-y)]^n dt} = (-1)^n y^{c_{n+1}} \frac{\int_0^{1} [(1-t)^c te^y]^n dt}{\int_0^{\infty} [t^{\alpha-c}((1+z)t - \alpha z)]^n dt},
\]

\[
e^{wy} = \frac{\int_0^{\infty} [te^{-t}(t+y)^c]^n dt}{\int_0^{\infty} [te^{-t}(t-y)]^n dt} = (-1)^n y^{c_{n+1}} \frac{\int_0^{1} [(1-t)^c te^y]^n dt}{\int_0^{\infty} [t^{\alpha-c}((1+z)t - \alpha z)]^n dt},
\]
which is exactly the Padé approximations to $e^z$ (see [11]).

From (3.8) and (3.9), we have the critical curve for Padé approximants to $(1 + z)^{\alpha n + 1}$, which is

(6.5) \[
\left\{ z : \frac{\alpha z + 2z + 2 - \mu}{\alpha z + 2z + 2 + \mu} = 1, \quad \frac{\alpha z - 2 + \mu}{\alpha z - 2 - \mu} = 1, \quad \frac{\alpha z + 2\alpha + \mu}{\alpha z + 2\alpha - \mu} = 1 \right\}
\]

where $\mu = (\alpha^2 z^2 + 4z + 4)^{1/2}$.

Set $\alpha z = y$, and then let $\alpha \to \infty$, from (6.5) we have

(6.6) \[
\left\{ y : \frac{2 - \sqrt{y^2 + 4}}{2 + \sqrt{y^2 + 4}} = 1, \quad e^{\sqrt{y^2 + 4}} = 1 \right\}.
\]

Let $y = 2x$, then we can rewrite (6.6) as

\[
\left\{ x : \frac{x e^{\sqrt{x^2 + 1}}}{1 + \sqrt{x^2 + 1}} = 1 \right\},
\]

which is exactly the critical curve for Padé approximates to $e^z$ with $\sigma = 1$ in [11]. This limiting argument, however, requires some careful justification.

References