THE ARC LENGTH OF THE LEMNISCATE \( \{ |p(z)| = 1 \} \)

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Abstract. We show that the length of the set
\[
\{ z \in \mathbb{C} : |\Pi_{i=1}^n (z - \alpha_i)| = 1 \}
\]
is at most \(8\pi en\). This gives the correct rate of growth in a long standing open problem of Erdős, Herzog and Piranian and improves the previous bound of \(74n^2\)
due to Pommerenke.

In 1958 Erdős, Herzog and Piranian [2] raised a number of problems concerned with the lemniscate

\[
E_n := E_n(p) := \{ z \in \mathbb{C} : |p(z)| = 1 \}
\]

where \(p\) is a monic polynomial of degree \(n\), so

\[
p(z) := \Pi_{i=1}^n (z - \alpha_i) \quad \alpha_i \in \mathbb{C}.
\]

One in particular, Problem 12, conjectures that the maximum length of \(E_n\) is achieved for \(p(z) := z^n - 1\). (Which is of length \(2n + o(1)\).) The best partial to date is due to Pommerenke [7] who shows that the maximum length is at most \(74n^2\). This problem has been re-posed by Erdős several times, including recently at a Budapest meeting honouring his 80th birthday. (See also [3].) It now carries with it a cash prize from Erdős of $250.

This note derives an upper bound of \(8\pi en\), which gives the correct rate of growth.

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**Theorem.** Let $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{C}$. Then the length of

$$E_n := \{z \in \mathbb{C} : |\prod_{i=1}^{n} (z - \alpha_i)| = 1\}$$

is at most $8\pi n (\leq 69n)$.

The proof relies on two classical theorems. One due to Cartan and one due to Poincaré.

**Cartan’s Lemma.** ([1, p174]) If $p(z) := \prod_{i=1}^{n} (z - \alpha_i)$ then the inequality

$$|p(z)| > 1$$

holds outside at most $n$ circular discs, the sum of whose radii is at most $2e$.

**Poincaré’s Formula.** [8, 9] Let $\Gamma$ be a rectifiable curve contained in $S$ (the Riemann sphere). Let $v(\Gamma, x)$ denote the number of times that a great circle consisting of points equidistant from the antipodes $\pm x$ intersects $\Gamma$. (If this is infinite set $v(\Gamma, x) = 0$.) Then the length of $\Gamma$, $L_S(\Gamma)$, is given by

$$L_S(\Gamma) = \frac{1}{4} \int_{\mathbb{S}} v(\Gamma, x) dx$$

where $dx$ is area measure on $S$.

We need the following corollary of this result.

**Corollary.** Suppose $\Gamma$ is an algebraic curve in $\mathbb{R}^2$ of degree at most $N$ and $D$ is a disc of radius $R$. Then the length of $\Gamma \cap D$ is at most $2\pi NR$.

**Proof.** By an affine scaling it suffices to prove this for $D$ a disc of radius 1 about the origin. Now any conic intersects $\Gamma$ in at most $2N$ points by Bezout’s theorem. It follows that the projection of $\Gamma$ in the Riemann sphere is intersected by any great circle in $S$ in at most $2N$ points. Thus the length of the projection of $\Gamma$ in $S$ is at most $2\pi N$ by Poincaré formula. Since the projection back to the unit disc doesn’t increase arclength the result is proved.

We can now prove the theorem.

**Proof of Theorem.** Fix $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$. By Cartan’s Lemma there exist circles $D_1, \ldots, D_m$ with radius $r_1, \ldots, r_m$ so that

$$E_n \subset \bigcup_{i=1}^{m} D_i$$

and

$$\sum_{i=1}^{m} r_i \leq 2e.$$
Observe that $E_n$ is an algebraic curve in $\mathbb{R}^2$ of degree at most $2n$ in $x$ and $y$ where $z = x + iy$. So by the Corollary each disc $D_i$ contains a portion of $E_n$ of length at most $4\pi r_i n$. On summing over $i$ we deduce that the length of $E_n$ is at most $8\pi n$.

The constant $2$ in the Corollary can be removed with some effort, so a sharpening to $4\pi n$ is possible. The constant $e$ in Cartan’s Lemma is probably unnecessary, but this is open. Even with these improvements we would only get a bound of $4\pi n$ which still isn’t sharp. Indeed it seems likely that this type of method is too blunt to yield an exact result.

There are a number of interesting related results. See for example Pommerenke [4,5,6,7]. In Pommerenke [6] it is shown that if the roots in the Theorem are all real then the length is at most $4\pi$.

In Pommerenke [5] it is shown that if the set $E_n$ is connected then the length is at least $2\pi$, with equality only for $z^n$. When $E_n$ is connected one can find a disc of radius $2$ that contains it [5]. So in this case the length of $E_n$ is at most $4\pi n$.

References


