GENERALIZATIONS OF MÜNTZ’S THEOREM VIA A REMEZ-TYPE INEQUALITY FOR MÜNTZ SPACES

PETER BORWEIN AND TAMÁS ERDÉLYI

ABSTRACT. The principal result of this paper is a Remetz-type inequality for Münz polynomials:

\[ p(x) := \sum_{i=0}^{n} a_i x^{\lambda_i} \]

or equivalently for Dirichlet sums:

\[ P(t) := \sum_{i=0}^{n} a_i e^{-\lambda_i t} \]

where \( 0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots \). The most useful form of this inequality states that for every sequence \( \{ \lambda_i \}_{i=0}^{\infty} \) satisfying \( \sum_{i=1}^{\infty} 1/\lambda_i < \infty \) there is a constant \( c \) depending only on \( A := \{ \lambda_i \}_{i=0}^{\infty} \) and \( s \) (and not on \( n \), \( g \), or \( A \)) so that

\[ \|p\|_{[0,1]} \leq c \|p\|_{A} \]

for every Münz polynomial \( p \), as above, associated with \( \{ \lambda_i \}_{i=0}^{\infty} \), and for every set \( A \subset [0,1] \) of Lebesgue measure at least \( s > 0 \). Here \( \|\cdot\|_A \) denotes the supremum norm on \( A \). This Remetz-type inequality allows us to resolve two reasonably long standing conjectures.

The first, due to D. J. Newman and dating from 1978, asserts that if \( \sum_{i=1}^{\infty} 1/\lambda_i < \infty \), then the set of products \( \{ p_1 p_2 : p_1, p_2 \in \text{span}\{x^{\lambda_0}, x^{\lambda_1}, \ldots \} \} \) is not dense in \( C[0,1] \).

The second is a complete extension of Münz’s classical theorem on the denseness of Münz spaces in \( C[0,1] \) to denseness in \( C[A] \), where \( A \subset [0,\infty) \) is an arbitrary compact set with positive Lebesgue measure. That is, for an arbitrary compact set \( A \subset [0,\infty) \) with positive Lebesgue measure, \( \text{span}\{x^{\lambda_0}, x^{\lambda_1}, \ldots \} \) is dense in \( C[A] \) if and only if \( \sum_{i=1}^{\infty} 1/\lambda_i = \infty \).

Several other interesting consequences are also presented.

1991 Mathematics Subject Classification, Primary: 41A17, Secondary: 30B10, 26D15.
Key words and phrases. Remetz inequality, Münz’s Theorem, Münz spaces, Dirichlet sums, density.

Research of the first author supported, in part, by NSERC of Canada, Research of the second author supported, in part, by NSF under Grant No. DMS-9024901 and conducted while an NSERC International Fellow at Simon Fraser University.

Typeset by \\LaTeX
1. Introduction

Müntz’s beautiful, classical theorem characterizes sequences \( \Lambda := \{ \lambda_i \}_{i=0}^{\infty} \) with
\[
0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots
\]
for which the Müntz space \( M(\Lambda) := \text{span}\{x^{\lambda_0}, x^{\lambda_1}, \ldots \} \) is dense in \( C[0,1] \). Here, and in what follows, \( \text{span}\{x^{\lambda_0}, x^{\lambda_1}, \ldots \} \) denotes the collection of finite linear combinations of the functions \( x^{\lambda_0}, x^{\lambda_1}, \ldots \) with real coefficients and \( C[A] \) is the space of all real-valued continuous functions on \( A \subset [0,\infty) \) equipped with the uniform norm. Throughout this paper \( \Lambda := \{ \lambda_i \}_{i=0}^{\infty} \) denotes a sequence satisfying (1.1). Müntz’s Theorem [11, 18, 27, 30] states the following.

**Theorem.** \( M(\Lambda) \) is dense in \( C[0,1] \) if and only if \( \sum_{i=1}^{\infty} 1/\lambda_i = \infty \).

The original Müntz Theorem proved by Müntz [16] in 1914, by Szász [27] in 1916, and anticipated by Bernstein [3] was only for sequences of exponents tending to infinity. The point 0 is special in the study of Müntz spaces. Even replacing \( [0,1] \) by an interval \( [a,b] \subset [0,\infty) \) in Müntz’s Theorem is a non-trivial issue. This is, in large measure, due to Clarkson and Erdös [12] and Schwartz [24] whose works include the result that if \( \sum_{i=1}^{\infty} 1/\lambda_i < \infty \), then every function belonging to the uniform closure of \( M(\Lambda) \) on \( [a,b] \) can be extended analytically throughout the region
\[
\{ z \in \mathbb{C} \mid (-\infty,0) : |z| < b \}.
\]

There are many generalizations and variations of Müntz’s Theorem [1, 4, 5, 6, 7, 8, 9, 16, 17, 19, 24, 26, 28, 29]. There are also still many open problems. The proper generalizations to many variables are still open.

In Section 6 of this paper we show that the interval \([0,1]\) in Müntz’s Theorem can be replaced by an arbitrary compact set \( A \subset [0,\infty) \) of positive Lebesgue measure. That is, if \( A \subset [0,\infty) \) is a compact set of positive Lebesgue measure, then \( M(\Lambda) \) is dense in \( C[A] \) if and only if \( \sum_{i=1}^{\infty} 1/\lambda_i = \infty \).

If \( A \) contains an interval, then this follows from the already mentioned results of Clarkson, Erdös, and Schwartz. However, their results and methods cannot handle the case that, for example, \( A \subset [0,1] \) is a Cantor-type set of positive measure. Note also that the scaling \( x \rightarrow ax \) reduces the case that \( A \subset [0,\infty) \) is compact to the case that \( A \subset [0,1] \) is closed, so working on \([0,1]\) is perfectly general.

In the case that \( \sum_{i=1}^{\infty} 1/\lambda_i < \infty \), analyticity properties of the functions belonging to the uniform closure of \( M(\Lambda) \) on \( A \) are also established.

An analogue of the above result is also proved in \( L^q_w[A] \), where \( w \) is a nonnegative integrable weight function on \( A \) with \( \int_A w > 0 \), and \( q \in (0,\infty) \).

Speculations about the above extensions of Müntz’s Theorem are probably as old as Müntz’s Theorem itself.
Somorjai [26] and Bak and Newman [2, 20] proved that

\[ R(\Lambda) := \{ p/q : p, q \in M(\Lambda) \} \]

is always dense in \( C[0, 1] \) in the uniform norm on \([0, 1]\). (Though (1.1) is assumed throughout this paper, the above result holds for an arbitrary \( \Lambda := \{ \lambda_i \}_{i=0}^{\infty} \) containing infinitely many distinct real numbers.) This surprising result says that while the set \( M(\Lambda) \) of Müntz polynomials may be far from dense, the set \( R(\Lambda) \) of Müntz rationals is always dense in \( C[0, 1] \), no matter what the underlying sequence \( \Lambda \). In light of this result, Newman, in 1978 [20, p. 50] raises “the very same, if very prosaic question.” Are the functions

\[ \prod_{j=1}^{k} \left( \sum_{i=0}^{n_j} a_{i,j} x^{i} \right), \quad a_{i,j} \in \mathbb{R}, \quad n_j \in \mathbb{N} \]

dense in \( C[0, 1] \) for some fixed \( k \geq 2 \)? In other words does the “extra multiplication” have the same power as the “extra division” has in the Bak-Newman-Somorjai result? Newman speculated that it did not.

Denote the set of the above products by \( H_k \). Since every natural number is the sum of four squares, \( H_4 \) contains all the monomials \( x^n, n = 0, 1, 2, \ldots \). However, \( H_k \) is not a linear space, so Müntz’s Theorem itself cannot be applied to resolve the denseness or non-denseness of \( H_4 \) in \( C[0,1] \).

Section 7 of this paper deals with products of Müntz spaces and, in particular, answers the above question of Newman in the negative. For

\[ \Lambda_j := \{ \lambda_{i,j} \}_{i=0}^{\infty}, \quad 0 = \lambda_{0,j} < \lambda_{1,j} < \lambda_{2,j} < \cdots, \quad j = 1, 2, \ldots \]

we define the sets

\[ M(\Lambda_1, \Lambda_2, \ldots, \Lambda_k) := \left\{ p = \prod_{j=1}^{k} p_j : p_j \in M(\Lambda_j) \right\}. \]

Bounded Remez, Bernstein, and Nikol’skii type inequalities are established for \( M(\Lambda_1, \Lambda_2, \ldots, \Lambda_k) \) in the case that

\[ \sum_{i=1}^{\infty} \frac{1}{\lambda_{i,j}} < \infty, \quad j = 1, 2, \ldots, k. \]

This obviously implies that if (1.2) and (1.3) hold and \( A \subset [0, \infty) \) is a compact set of positive Lebesgue measure, then \( M(\Lambda_1, \Lambda_2, \ldots, \Lambda_k) \) is not dense in \( C[A] \). In particular, \( H_4 \) is not dense in \( C[0,1] \), which answers Newman’s problem negatively. In addition, under the assumptions (1.2) and (1.3), our methods give an “almost characterization” of the uniform closure of \( M(\Lambda_1, \Lambda_2, \ldots, \Lambda_k) \) on \( A \) in terms of analyticity properties. This will likely be discussed in a later publication of the authors.
The results of Sections 6 and 7 can be proved fairly simply, once one has established the bounded Remez-type inequality of Section 5 for non-dense Müntz spaces $M(\Lambda)$. This is the central result of the paper, and is a result we believe should be a basic tool for dealing with problems about Müntz spaces, in addition to those discussed in Sections 6 and 7.

Let $\mathcal{P}_n$ denote the set of all algebraic polynomials of degree at most $n$ with real coefficients. For a fixed $s \in (0, 1)$ let $$\mathcal{P}_n(s) := \{ p \in \mathcal{P}_n : m(\{ x \in [0, 1] : |p(x)| \leq 1 \}) \geq s \}$$ where $m(\cdot)$ denotes linear Lebesgue measure. The classical Remez inequality concerns the problem of bounding the uniform norm of a polynomial $p \in \mathcal{P}_n$ on $[0, 1]$ given that its modulus is bounded by 1 on a subset of $[0, 1]$ of Lebesgue measure at least $s$. That is, how large can $\| p \|_{[0,1]}$ (the uniform norm of $p$ on $[0,1]$) be if $p \in \mathcal{P}_n(s)$? The answer is given in terms of the Chebyshev polynomials. The extremal polynomials for the above problem are the Chebyshev polynomials $\pm T_n(x) := \pm \cos(n \arccos h(x))$, where $h$ is a linear function which scales $[0, s]$ or $[1-s, 1]$ onto $[-1, 1]$.

For various proofs, extensions, and applications, see [13, 14, 15, 22, 23].

We generalize the Remez inequality in the following way. Let $$M_n(\Lambda) := \operatorname{span}\{ x^{\lambda_0}, x^{\lambda_1}, \ldots, x^{\lambda_n} \}.$$ That is, $M_n(\Lambda)$ is the collection of Müntz polynomials

$$p(x) := \sum_{i=0}^{n} a_i x^{\lambda_i}, \quad a_i \in \mathbb{R},$$

We seek to find

$$(1) \quad \max \left\{ \frac{|p(0)|}{\| p \|_A} : 0 \neq p \in M_n(\Lambda), \quad A \subset [0, 1], \quad m(A) \geq s \right\}$$

and

$$(2) \quad \max \left\{ \frac{|p(1)|}{\| p \|_A} : 0 \neq p \in M_n(\Lambda), \quad A \subset [0, 1], \quad m(A) \geq s \right\}.$$ 

These two problems are no longer equivalent as they are in the polynomial case (since $x \to 1-x$ does not preserve membership in $M_n(\Lambda)$) and they have different answers. However, these two problems can be handled in essentially the same way. In Section 5 we concentrate on problem (1). Lemma 5.4 shows that an extremal function for problem (1) is the (generalized) Chebyshev polynomial

$$T_n := T_n\{ \lambda_0, \lambda_1, \ldots, \lambda_n ; [1-s, 1] \}$$
for $M_n(\Lambda)$ on $[1 - s, 1]$ defined in Section 2. This reduces problem (1) to the
interval case, $A = [1 - s, 1]$. The interval case can then be handled by a bounded
Chebyshev-type inequality

$$\|p\|_{[0,1]} \leq c\|p\|_{[1-s,1]}$$

$\|p\|_{[0,1]} \leq c\|p\|_{[1-s,1]}$, \quad $p \in M(\Lambda) = \text{span}\{x^{\lambda_0}, x^{\lambda_1}, \ldots\}$

established in Section 3 for every $\Lambda := \{\lambda_i\}_{i=0}^\infty$ satisfying $\sum_{i=0}^\infty 1/\lambda_i < \infty$, where
the constant $c$ depends only on $\Lambda$ and $s$ (and not on the “length” of $p$). This we
first prove under the gap condition

$$\inf\{\lambda_{i+1} - \lambda_i : i \in \mathbb{N}\} > 0.$$  

However, some comparison lemmas of Section 4 will allow us to drop this condition.
This leads to the central result of the paper, a bounded Remez-type inequality for
non-dense Müntz spaces (see Theorem 5.1) which states the following. For every
$\Lambda := \{\lambda_i\}_{i=0}^\infty$ with $\sum_{i=1}^\infty 1/\lambda_i < \infty$, and for every $s > 0$ there exists a constant $c$
depending only on $\Lambda$ and $s$ (and not on $p$, $A$, and the “length” of $p$) so that

$$\|p\|_{[0,s]} \leq c\|p\|_{A}$$

for every $p \in M(\Lambda) = \text{span}\{x^{\lambda_0}, x^{\lambda_1}, \ldots\}$ and for every set $A \subset [a, 1]$ of Lebesgue
measure at least $s$.

One might note that the existence of such a bounded Remez-type inequality for
a Müntz space $M(\Lambda)$ is equivalent to the non-denseness of $M(\Lambda)$ in $C[0,1]$.

In [8] the above result is established for lacunary Müntz spaces, that is when

$$\inf\{\lambda_{i+1}/\lambda_i : i \in \mathbb{N}\} > 0.$$  

Yet another remarkable consequence of the bounded Remez-type inequality of
Theorem 5.1 is that the pointwise and locally uniform convergence of a sequence
$\{p_i\}_{i=1}^\infty \subset M(\Lambda)$ on $(0,b)$ are equivalent whenever $\sum_{i=1}^\infty 1/\lambda_i < \infty$. See Theorem
6.3. In fact, one can characterize the non-dense Müntz spaces within the Müntz
spaces $M(\Lambda)$ as exactly those in which locally uniform and pointwise convergence
on $(0,b)$ are equivalent.

2. Notation

The notations

$$\|p\|_{A} := \sup_{x \in A} |p(x)|$$

$$\|p\|_{L^q[\Lambda]} := \left( \int_{A} |p|^q w \right)^{1/q}$$

and

$$\|p\|_{L^q[\Lambda]} := \left( \int_{A} |p|^q \right)^{1/q}$$
are used throughout this paper for measurable functions $p$ defined on a measurable set $A \subset [0, \infty)$, for nonnegative measurable weight functions $w$ defined on $A$, and for $q \in (0, \infty)$. The space of all real-valued continuous functions on a set $A \subset [0, \infty)$ equipped with the uniform norm is denoted by $C[A]$.

Throughout this paper $\Lambda := \{\lambda_i\}_{i=0}^{\infty}$ denotes a sequence of real numbers satisfying

$$0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots.$$ 

The nonnegative-valued functions $x^{\lambda_i}$ are well-defined on $[0, \infty)$. The collection

$$\{x^{\lambda_0}, x^{\lambda_1}, \ldots, x^{\lambda_n}\}$$

is called a (finite) Müntz system. The linear space

$$M_n(\Lambda) := \text{span}\{x^{\lambda_0}, x^{\lambda_1}, \ldots, x^{\lambda_n}\}$$

over $\mathbb{R}$ is called a (finite) Müntz space. That is, the Müntz space $M_n(\Lambda)$ is the collection of Müntz polynomials

$$p(x) = \sum_{i=0}^{n} a_i x^{\lambda_i}, \quad a_i \in \mathbb{R}.$$ 

The set

$$M(\Lambda) := \bigcup_{n=0}^{\infty} M_n(\Lambda) = \text{span}\{x^{\lambda_0}, x^{\lambda_1}, \ldots\}$$

is called the (infinite) Müntz space associated with $\Lambda$.

One of the most basic properties of a Müntz space $M_n(\Lambda)$ is the fact that it is a Chebyshev space on every $A \subset [0, \infty)$ containing at least $n+1$ points. That is, $M(\Lambda) \subset C[A]$ and every $p \in M_n(\Lambda)$ having at least $n+1$ (distinct) zeros in $A$ is identically 0. In fact, Müntz spaces are the “canonical” examples for Chebyshev spaces and the following properties of Müntz spaces $M_n(\Lambda)$ are well known (see, for example, [9, 11, 21]).

**Theorem 2.1 (Unique Interpolation Property).** For every

$$0 \leq x_0 < x_1 < \ldots < x_n \quad \text{and} \quad y_0, y_1, \ldots, y_n \in \mathbb{R}$$

there exists a unique $p \in M_n(\Lambda)$ so that

$$p(x_j) = y_j, \quad j = 0, 1, \ldots, n.$$ 

**Theorem 2.2 (Existence of Chebyshev Polynomials).** Let $A$ be a compact subset of $[0, \infty)$ containing at least $n+1$ points. Then there exists a unique (extended) Chebyshev polynomial

$$T_n := T_n\{\lambda_0, \lambda_1, \ldots, \lambda_n; A\}$$
for $M_n(\Lambda)$ on $A$ defined by

$$T_n(x) = c \left( x^{\lambda_n} - \sum_{i=0}^{n-1} a_i x^{\lambda_i} \right)$$

where the numbers $a_0, a_1, \ldots, a_{n-1} \in \mathbb{R}$ are chosen to minimize

$$\left\| x^{\lambda_n} - \sum_{i=0}^{n-1} a_i x^{\lambda_i} \right\|_A$$

and where $c \in \mathbb{R}$ is a normalization constant chosen so that

$$\| T_n \|_A = 1$$

and the sign of $c$ is determined by

$$T_n(\max A) > 0.$$

**Theorem 2.3 (Alternation Characterization).** The Chebyshev polynomial

$$T_n := T_n\{\lambda_0, \lambda_1, \ldots, \lambda_n; A\} \in M_n(\Lambda)$$

is uniquely characterized by the existence of an alternation set

$$\{x_0 < x_1 < \cdots < x_n\} \subset A$$

for which

$$T_n(x_j) = (-1)^{n-j} = (-1)^{n-j}\|T_n\|_A, \quad j = 0, 1, \ldots, n.$$  

3. **Bounded Chebyshev and Bernstein Type Inequalities for $M(\Lambda)$**

The main results of this section are the following two theorems.

**Theorem 3.1.** Suppose $\sum_{i=1}^{\infty} 1/\lambda_i < \infty$. Let $s \in (0, 1)$. Then there exists a constant $c$ depending only on $\Lambda := \{\lambda_i\}_{i=0}^{\infty}$ and $s$ (and not on the “length” of $p$) so that

$$\|p\|_{[0, 1]} \leq c \|p\|_{[0, s]}$$

for every $p \in M(\Lambda) := \text{span}\{x^{\lambda_0}, x^{\lambda_1}, \ldots\}$. 
Theorem 3.2. Suppose \( \sum_{i=1}^{\infty} 1/\lambda_i < \infty \) and \( \lambda_i \geq 1 \). Let \( s \in (0, 1) \). Then there exists a constant \( c \) depending only on \( \Lambda := \{\lambda_i\}_{i=0}^{\infty} \) and \( \varepsilon \) (and not on the “length” of \( p \)) so that

\[
\|p'\|_{[0,1-\varepsilon]} \leq c \|p\|_{[0,1]}
\]

for every \( p \in M(\Lambda) := \text{span} \{x^{\lambda_0}, x^{\lambda_1}, \ldots\} \).

To prove the above two theorems we need three lemmas. Lemmas 3.3 and 3.4 establish the conclusion of Theorems 3.2 and 3.1, respectively, under the gap condition

(3.1) \( \inf \{\lambda_{i+1} - \lambda_i : i \in \mathbb{N}\} > 0 \)

which is then dropped with the aid of Lemma 4.5.

Lemma 3.3. Suppose \( \sum_{i=1}^{\infty} 1/\lambda_i < \infty \), \( \lambda_i \geq 1 \), and the gap condition (3.1) holds. Let \( \varepsilon \in (0, 1) \). Then there exists a constant \( c_1(\varepsilon) \) depending only on \( \Lambda := \{\lambda_i\}_{i=0}^{\infty} \) and \( \varepsilon \in (0, 1) \) (and not on \( i \) and \( n \)) so that

\[
|a_i| \leq c_1(\varepsilon)(1 + \varepsilon)^{\lambda_i} \|p\|_{L^n([0,1])} \leq c_1(\varepsilon)(1 + \varepsilon)^{\lambda_i} \|p\|_{[0,1]}
\]

for every \( p \in M(\Lambda) := \text{span} \{x^{\lambda_0}, x^{\lambda_1}, \ldots\} \).

Proof. Clarkson and Erdős [12] observed that under the conditions of the lemma, there exists a constant \( c_1(\varepsilon) \) depending only on \( \Lambda = \{\lambda_i\}_{i=0}^{\infty} \) and \( \varepsilon \in (0, 1) \) (and not on \( i \) and \( n \)) so that

\[
p(x) = \sum_{i=0}^{n} a_i x^{\lambda_i}, \quad a_i \in \mathbb{R}.
\]

Therefore, if \( y \in [0,1-\varepsilon] \) and \( \|p\|_{[0,1]} \leq 1 \), then

\[
|p'(y)| = \left| \sum_{i=1}^{n} \lambda_i a_i y^{\lambda_i-1} \right| \leq \sum_{i=1}^{n} |a_i| |\lambda_i| y^{\lambda_i-1}
\]

\[
\leq \sum_{i=1}^{\infty} c_1(\varepsilon) \lambda_i (1 + \varepsilon)^{\lambda_i} (1 - \varepsilon)^{\lambda_i-1}
\]

\[
= \frac{c_1(\varepsilon)}{1 - \varepsilon} \sum_{i=1}^{\infty} \lambda_i (1 - \varepsilon^2)^{\lambda_i}
\]

\[
\leq \frac{c_1(\varepsilon)}{1 - \varepsilon} \sum_{i=1}^{k} \lambda_i (1 - \varepsilon^2)^{\lambda_i} + \sum_{i=k+1}^{\infty} i^{-2} =: c
\]

where \( k \in \mathbb{N} \) is chosen so that

\[
(1 - \varepsilon^2)^{\lambda_i} \leq \frac{1}{\lambda_i k^2}, \quad i = k + 1, k + 2, \ldots
\]

This proves the lemma. \( \square \)
Lemma 3.4. Suppose $\sum_{i=1}^{\infty} 1/\lambda_i < \infty$ and the gap condition (3.1) holds. Let $s \in (0,1)$. Then there exists a constant $c$ depending only on $\Lambda := \{\lambda_i\}_{i=0}^{\infty}$ and $s$ (and not on the “length” of $p$) so that

$$|p(0)| \leq c\|p\|_{[1-s,1]}$$

for every $p \in M(\Lambda) := \text{span}\{x^{\lambda_0}, x^{\lambda_1}, \ldots\}$.

Proof. Using the scaling $x \to x^{1/\lambda_1}$, without loss of generality we may assume that $\lambda_1 = 1$. Suppose there exist

$$p_i \in M(\Lambda), \quad i = 1, 2, \ldots$$

so that

$$A_i := \|p_i\|_{[0,1]} \to \infty$$

while

$$\|p_i\|_{[1-s,1]} = 1, \quad i = 1, 2, \ldots.$$ 

Let $q_i := p_i/A_i$. Note that $\|q_i\|_{[0,1]} = 1$ and

$$\lim_{i \to \infty} \|q_i\|_{[1-s,1]} = 0. \quad (3.2)$$

By Lemma 3.3, there exists a constant $c$ depending only on $\Lambda := \{\lambda_i\}_{i=0}^{\infty}$ and $\varepsilon$ so that

$$\|q_i\|_{[0,1-\varepsilon]} \leq c\|q_i\|_{[0,1]} = c$$

for every $\varepsilon \in (0,1)$. Hence $\{q_i\}_{i=0}^{\infty}$ is a sequence of uniformly bounded and equicontinuous functions on every closed subinterval of $[0,1)$. So, by the Arzela-Ascoli Theorem, we may extract a uniformly convergent subsequence on $[0,1-s/2]$. By a theorem of Clarkson and Erdős [12], this subsequence converges uniformly to a function $F$ analytic on $(0,1-s/2)$. Combining this with (3.2) and the Unicity Theorem, we can deduce that $F$ is identically zero. This is a contradiction since $\|q_i\|_{[0,1]} = 1$ and

$$\|q_i\|_{[0,1-\varepsilon]} = \|q_i\|_{[0,1]}$$

for every sufficiently large $i$. The lemma is now proved. □

Proof of Theorem 3.1. Observe that $\lim_{i \to \infty} \lambda_i / i = \infty$. Therefore, there is an $m \in \mathbb{N}$ so that $\lambda_i > 3i$ whenever $i > m$. Let $\Gamma := \{\gamma_i\}_{i=1}^{\infty}$ be defined by

$$\gamma_i := \begin{cases} 
\min\{\lambda_{i+1}, i\} & \text{if } i = 0, 1, \ldots, m \\
\lambda_i - i & \text{if } i = m + 1, m + 2, \ldots
\end{cases}$$

Then $0 = \gamma_0 < \gamma_1 < \gamma_2 < \ldots$, $\sum_{i=1}^{\infty} 1/\gamma_i < \infty$, $\gamma_i \leq \lambda_i$, $i = 0, 1, 2, \ldots$ and

$$\inf\{\gamma_{i+1} - \gamma_i : i \in \mathbb{N}\} > 0.$$ 

Now Lemma 3.4 and the first part of Lemma 4.5 will yield the theorem. □

Proof of Theorem 3.2. Without loss of generality we may assume that $\lambda_1 = 1$. Let $\Gamma := \{\gamma_i\}_{i=0}^{\infty}$ be defined as in the proof of Theorem 2.1. Now Lemma 3.3, Theorem 3.1, and the second part of Lemma 4.5 yield the theorem. □
4. Comparison Lemmas

One of the basic properties of a Müntz system
\[ \{x^{\lambda_0}, x^{\lambda_1}, \ldots, x^{\lambda_n}\}, \quad 0 = \lambda_0 < \lambda_1 < \ldots < \lambda_n \]
is that it is a Descartes system on every interval \([a, b] \subset (0, \infty)\), see [21]. The following comparison lemma due to Smith [25] is valid for every Descartes system.

Lemma 4.1. Suppose \(\{f_0, f_1, \ldots, f_n\}\) is a Descartes system on \([a, b]\). Suppose
\[
p = f_0 + \sum_{i=1}^{k} a_i f_\lambda, \quad a_i \in \mathbb{R}
\]
\[
q = f_0 + \sum_{i=1}^{k} b_i f_\gamma, \quad b_i \in \mathbb{R}
\]
where \(0 \leq \lambda_1 < \lambda_2 < \ldots < \lambda_k \leq n, 0 \leq \gamma_1 < \gamma_2 < \ldots < \gamma_k \leq n,
\[
0 \leq \gamma_i \leq \lambda_i < \alpha, \quad i = 1, 2, \ldots, m
\]
and \(\alpha < \lambda_i \leq \gamma_i \leq n, \quad i = m+1, m+2, \ldots, k\)

with strict inequality for at least one index \(i = 1, 2, \ldots, k\). Then
\[
p(x_i) = q(x_i) = 0, \quad i = 1, 2, \ldots, k
\]
with distinct \(x_i \in [a, b]\) implies
\[
|p(x)| \leq |q(x)|
\]
for every \(x \in [a, b]\) with strict inequality for every
\[
x \in [a, b] \setminus \{x_1, x_2, \ldots, x_k\}.
\]

To formulate the next lemmas we introduce the following notation. Let
\[
0 = \lambda_0 < \lambda_1 < \ldots < \lambda_n, \quad 0 = \gamma_0 < \gamma_1 < \ldots < \gamma_n
\]
and
\[
\lambda_i \leq \gamma_i, \quad i = 1, 2, \ldots, n.
\]
Let
\[
M_n(\Lambda) := \text{span}\{x^{\lambda_0}, x^{\lambda_1}, \ldots, x^{\lambda_n}\}
\]
and
\[
M_n(\Gamma) := \text{span}\{x^{\gamma_0}, x^{\gamma_1}, \ldots, x^{\gamma_n}\}.
\]
Let \(s \in (0, 1)\) be fixed. Let
\[
T_{n, \lambda} := T_n\{\lambda_0, \lambda_1, \ldots, \lambda_n; [1 - s, 1]\}
\]
and
\[
T_{n, \gamma} := T_n\{\gamma_0, \gamma_1, \ldots, \gamma_n; [1 - s, 1]\}
\]
denote the Chebyshev polynomials on \([1 - s, 1]\) for \(M_n(\Lambda)\) and \(M_n(\Gamma)\), respectively.
(See Theorem 2.2).
Lemma 4.2. Let \( y \in [0, 1 - s) \) be fixed. Then both
\[
\max_{0 \neq p \in M_n(\Lambda)} \frac{|p(y)|}{\|p\|_{1-s,1}}
\]
and
\[
\max_{0 \neq p \in M_n(\Lambda)} \frac{|p'(y)|}{\|p\|_{1-s,1}}
\]
are attained by \( p = T_{n,\lambda} \). In the second case we assume \( \lambda_1 \geq 1 \) if \( y = 0 \).

Proof. A simple compactness argument shows that the maxima in the lemma are attained by some \( p^* \in M_n(\Lambda) \) and \( q^* \in M_n(\Lambda) \), which can be identified as \( T_{n,\lambda} \) by a standard variational method. See [9], for example. □

Lemma 4.3. We have
\[
[T_{n,\gamma}(0)] \leq [T_{n,\lambda}(0)].
\]
Further, if \( \lambda_1 = \gamma_1 = 1 \), then
\[
[T'_{n,\gamma}(0)] \leq [T'_{n,\lambda}(0)].
\]

Proof. Let \( p \in M_n(\Lambda) \) interpolate \( T_{n,\gamma} \) at the \( n \) zeros of \( T_{n,\gamma} \) in \([1 - s, 1]\) and at 0.
It follows from Lemma 4.1 that
\[
|p(x)| \leq |T_{n,\gamma}(x)|, \quad x \in [0, 1].
\]
In particular,
\[
\|p\|_{1-s,1} \leq \|T_{n,\gamma}\|_{1-s,1} = 1
\]
which, together with \( p(0) = T_{n,\gamma}(0) \) and Lemma 4.2 gives
\[
[T_{n,\gamma}(0)] = |p(0)| \leq \frac{|p(0)|}{\|p\|_{1-s,1}} \leq \frac{[T_{n,\lambda}(0)]}{[T_{n,\lambda}(0)]} = [T_{n,\lambda}(0)].
\]
This proves the first part of the lemma.

The second part of the lemma can be proved in essentially the same way. Let \( p \in M_n(\Lambda) \) interpolate \( T_{n,\gamma} \) at the \( n \) zeros of \( T_{n,\gamma} \) in \([1 - s, 1]\). Note that \( p'(0) \neq 0 \), otherwise
\[
p \in \text{span}\{x^j : j = 0, 1, \ldots, n, \quad j \neq 1\}
\]
cannot have \( n \) zeros in \([1 - s, 1]\). Similarly \( T'_{n,\gamma}(0) \neq 0 \). Normalize \( p \) so that
\[
p'(0) = T'_{n,\gamma}(0).
\]
It follows from Lemma 4.1 that
\[
|p(x)| \leq |T_{n,\gamma}(x)|, \quad x \in [0, 1].
\]
In particular
\[
\|p\|_{1-s,1} \leq \|T_{n,\gamma}\|_{1-s,1} = 1
\]
which, together with \( p'(0) = T'_{n,\gamma}(0) \) and Lemma 4.2, yields
\[
[T'_{n,\gamma}(0)] = |p'(0)| \leq \frac{|p'(0)|}{\|p\|_{1-s,1}} \leq \frac{[T'_{n,\lambda}(0)]}{[T_{n,\lambda}(0)]} = [T'_{n,\lambda}(0)].
\]
This proves the second part of the lemma. □
Lemma 4.4. The functions $|T_{n,\lambda}|$ and $|T_{n,\gamma}|$ are decreasing on $[0, 1 - s]$. Further, if $\lambda_1 = \gamma_1 = 1$, then $|T_{n,\lambda}'|$ and $|T_{n,\gamma}'|$ are also decreasing on $[0, 1 - s]$.

Proof. Suppose $|T_{n,\lambda}|$ is not monotone decreasing on $[0, 1 - s]$. Then
$$T_{n,\lambda}' \in \text{span}\{x^{\lambda_1-1}, x^{\lambda_2-1}, \ldots, x^{\lambda_n-1}\}$$
must have at least $n$ zeros in $(0, 1)$, which is impossible. This proves the first statement of the lemma.

Suppose now $\lambda_1 = 1$ and $|T_{n,\lambda}'|$ is not monotone decreasing on $[0, 1 - s]$. Then
$$T_{n,\lambda}'' \in \text{span}\{x^{\lambda_2-2}, x^{\lambda_3-2}, \ldots, x^{\lambda_n-2}\}$$
must have at least $n - 1$ zeros in $(0, 1)$, which is impossible. This proves the second statement of the lemma. $\square$

The main result of this section is the following lemma which plays a crucial role in the proof of Theorems 3.1 and 3.2 of the previous section.

Lemma 4.5. We have
$$\max_{0 \neq p \in M_{\lambda}(\Gamma)} \frac{\|p\|_{[0,1]}}{\|p\|_{[0,s]}} \leq \max_{0 \neq p \in M_{\lambda}(\lambda)} \frac{\|p\|_{[0,1]}}{\|p\|_{[0,1-s]}}.$$ Further, if $\lambda_1 = \gamma_1 = 1$, then
$$\max_{0 \neq p \in M_{\lambda}(\Gamma)} \frac{\|p'\|_{[0,1-s]}}{\|p\|_{[0,s]}} \leq \max_{0 \neq p \in M_{\lambda}(\lambda)} \frac{\|p'\|_{[0,1-s]}}{\|p\|_{[1,1-s]}}.$$

Proof. Combine Lemmas 4.2, 4.3 and 4.4. $\square$

5. BOUNDED REMEZ-TYPE INEQUALITY FOR NON-DENSE MUTZ SPACES

The central result of this paper is the following.

Theorem 5.1. Suppose $\sum_{i=1}^{\infty} \frac{1}{\lambda_i} < \infty$. Let $s > 0$. Then there exists a constant $c$ depending only on $\Lambda := \{\lambda_i\}_{i=0}^{\infty}$ and $s$ (and not on $\rho$, $\lambda$), or the “length” of $p$ so that
$$\|p\|_{[0,s]} \leq c\|p\|_{\Lambda}$$
for every $p \in M(\Lambda) := \text{span}\{x^{\lambda_0}, x^{\lambda_1}, \ldots\}$ and for every set $A \subset [0, 1]$ of Lebesgue measure at least $s$.

The proof of Theorem 5.1 is based on interpolation. By the Unique Interpolation Property of Chebyshev spaces, associated with
$$0 \leq x_0 < x_1 < \ldots < x_n$$
we can uniquely define
\[ l_k := l_k(x_0, x_1, \ldots, x_n) \in M_n(\Lambda), \quad k = 0, 1, \ldots, n \]
so that
\[ l_k(x_0, x_1, \ldots, x_n)(x_j) = \delta_{j,k} := \begin{cases} 1 & \text{if } j = k, \\ 0 & \text{if } j \neq k. \end{cases} \]

**Lemma 5.2.** Let
\[ 0 < x_0 < x_1 < \ldots < x_n \quad \text{and} \quad 0 \leq \hat{x}_0 < \hat{x}_1 < \ldots < \hat{x}_n. \]
Suppose \( 0 \leq k \leq n \) and
\[
\begin{align*}
  x_j &\leq \hat{x}_j & \text{if } j = 0, 1, \ldots, k - 1 \\
  x_j &\geq \hat{x}_j & \text{if } j = k \\
  x_j &\geq \hat{x}_j & \text{if } j = k + 1, k + 2, \ldots, n.
\end{align*}
\]
For the sake of brevity let
\[ l_k := l_k(x_0, x_1, \ldots, x_n) \]
and
\[ \tilde{l}_k := l_k(\hat{x}_0, \hat{x}_1, \ldots, \hat{x}_n). \]
Then
\[ |l_k(0)| \leq |\tilde{l}_k(0)| \]

**Proof.** It is sufficient to prove the lemma in the case that there is an index \( m \) so that \( 1 \leq m \leq n, \; m \neq k, \) and
\[
\begin{align*}
  x_j &\leq \hat{x}_j & \text{if } j = 0, 1, \ldots, n, \; j \neq n \\
  x_m &< \hat{x}_m & \text{if } m < k \\
  x_m &> \hat{x}_m & \text{if } m > k.
\end{align*}
\]
The general case of the lemma then follows from repeated applications of the above special cases. Note that in the above special cases
\[ l_k - \tilde{l}_k \in M_n(\Lambda) \]
has a zero at each of the points
\[ x_j, \quad j = 0, 1, \ldots, n, \; j \neq m \]
hence it changes sign at each of these points, and does not have any other zero in \([0, \infty)\). It is also obvious that
\[ \text{sign}(l_k(x)) = \text{sign}(\tilde{l}_k(x)), \quad x \in [0, x_1] \]
which, together with the previous observation and the inequality \( x_1 \leq \hat{x}_1 \), yields that
\[ |l_k(0)| \leq |\tilde{l}_k(0)| \]
which finishes the proof. \( \square \)

By a simple scaling we can extend Lemma 5.2 as follows. We use the notation introduced in Lemma 5.2.
Lemma 5.3. Let
\[ 0 \leq x_0 < x_1 < \ldots < x_n \quad \text{and} \quad 0 \leq \tilde{x}_0 < \tilde{x}_1 < \ldots < \tilde{x}_n. \]
Suppose \( 0 \leq k \leq n, \alpha \geq 0, \) and
\[
\begin{align*}
x_j & \leq \tilde{x}_j - \alpha \quad \text{if } j = 0, 1, \ldots, k - 1, \\
x_j & = \tilde{x}_j - \alpha \quad \text{if } j = k, \\
x_j & \geq \tilde{x}_j - \alpha \quad \text{if } j = k + 1, k + 2, \ldots, n.
\end{align*}
\]
Then
\[ |\bar{l}_k(0)| \leq |\tilde{l}_k(0)|. \]

Proof. If \( \alpha = 0, \) then Lemma 5.2 yields the lemma. So we may suppose that \( \alpha > 0. \)
Let
\[
\beta := \frac{x_k}{\tilde{x}_k} = \frac{\tilde{x}_k - \alpha}{\tilde{x}_k},
\]
\[
x_j^* := \beta \tilde{x}_j, \quad j = 0, 1, \ldots, n
\]
and
\[ l_k^* := l_k\{x_0^*, x_1^*, \ldots, x_n^*\}, \quad k = 0, 1, \ldots, n. \]
Obviously
\[ l_k(\beta x) = l_k^*(x), \quad x \in [0, \infty) \]
and
\[
\begin{align*}
x_j & \leq x_j^* \quad \text{if } j = 0, 1, \ldots, k - 1, \\
x_j & = x_j^* \quad \text{if } j = k, \\
x_j & \geq x_j^* \quad \text{if } j = k + 1, k + 2, \ldots, n.
\end{align*}
\]
Hence Lemma 5.2 implies that
\[ |l_k(0)| \leq |l_k^*(0)| = |\tilde{l}_k(0)| \]
which finishes the proof. \( \square \)

The next two lemmas are interesting in their own right. They show that the appropriately placed Chebyshev polynomial is always extremal for the the Remez-type inequality we are considering.

Lemma 5.4. Let \( A \) be a closed subset of \([0, 1]\) of Lebesgue measure at least \( s \in (0, 1). \) Then
\[
|p(0)| \leq |T_n(\lambda_0, \lambda_1, \ldots, \lambda_n; [1 - s, 1])(0)||p||_A
\]
for every \( p \in M_n(A). \)

Proof. Let
\[
\tilde{x}_0 < \tilde{x}_1 < \ldots < \tilde{x}_n
\]
denote the extreme points of

\[ T_n := T_n\{\lambda_0, \lambda_1, \ldots, \lambda_n; [1 - s, 1]\} \]

in \([1 - s, 1]\), that is

\[ T_n(\tilde{x}_j) = (-1)^{n-j}, \quad j = 0, 1, \ldots, n. \]

Let \(x_j \in A, j = 0, 1, \ldots, n\), be defined by

\[ m([x_j, 1] \cap A) = m([\tilde{x}_j, \tilde{x}_n]) = \tilde{x}_n - \tilde{x}_j. \]

Since \(A\) is a closed subset of \([0, 1]\) with \(m(A) \geq s\), such points \(x_j \in A\) exist. Let \(p \in M_n(A)\). Then, using Lemma 5.3, we obtain

\[
\begin{align*}
|p(0)| &= \left| \sum_{k=0}^{n} p(x_k) I_k(0) \right| \\
&\leq \left( \sum_{k=0}^{n} |I_k(0)| \right) \|p\|_A \\
&\leq \left( \sum_{k=0}^{n} |I_k(0)| \right) \|p\|_A \\
&= \left| \sum_{k=0}^{n} (-1)^{n-k} I_k(0) \right| \|p\|_A \\
&= \left| \sum_{k=0}^{n} T_n(\tilde{x}_k) I_k(0) \right| \|p\|_A \\
&= |T_n(0)\|p\|_A
\end{align*}
\]

which proves the lemma. \(\square\)

**Lemma 5.5.** Let \(A\) be a closed subset of \([0, 1]\) with Lebesgue measure at least \(s \in (0, 1)\). Then

\[
|p(y)| \leq |T_n\{\lambda_0, \lambda_1, \ldots, \lambda_n; [1 - s, 1]\}(0)|\|p\|_A
\]

for every \(p \in M_n(A)\) and \(y \in [0, \inf A]\).

**Proof.** Let \(y \in [0, \inf A]\) be fixed. Simple compactness and perturbation arguments show that

\[
\max_{0 \neq p \in M_n(A)} \frac{|p(y)|}{\|p\|_A}
\]

is attained by

\[ T_{n,A} := T_n\{\lambda_0, \lambda_1, \ldots, \lambda_n; A\}. \]
Note that $\lambda_0 = 0$ implies that $T_{n,A}$ is decreasing on $[0, \infty]$, otherwise

$$T_{n,A}' \in \text{span} \{ x^{\lambda_1-1}, x^{\lambda_2-1}, \ldots, x^{\lambda_n-1} \}$$

must have at least $n + 1$ zeros in $(0, 1]$, which is impossible. Hence, by Lemma 5.4,

$$\frac{|p(y)|}{\|p\|_A} \leq \frac{|T_{n,A}(y)|}{\|T_{n,A}\|_A} = |T_{n,A}(y)| \leq |T_{n,A}(0)|$$

for every $0 \neq p \in M_0(A)$. This finishes the proof. $\square$

**Proof of Theorem 5.1.** Without loss of generality we may assume that $A$ is closed. Therefore, the theorem follows from Lemma 5.5 and Theorem 3.1. $\square$

The next theorem establishes an $L^q$ version of Theorem 5.1.

**Theorem 5.6.** Suppose $\sum_{i=1}^{\infty} 1/\lambda_i < \infty$. Let $s > 0$ and $q \in (0, \infty)$. Then there exists a constant $c$ depending only on $A := \{ \lambda_i \}_{i=0}^{\infty}$, $s$, and $q$ (and not on $g$, $A$, or the “length” of $p$) so that

$$\|p\|_{[0,q]} \leq c\|p\|_{L^q[A]}$$

for every $p \in M(A) := \text{span} \{ x^{\lambda_1}, x^{\lambda_2}, \ldots \}$ and for every set $A \subset [g, 1]$ of Lebesgue measure at least $s$.

**Proof.** Note that $m(A) \geq s$ implies that

$$m \left( \left\{ x \in A : |p(x)| \geq \left( \frac{2}{s} \right)^{1/q} \|p\|_{L^q[A]} \right\} \right) \leq \frac{s}{2}$$

hence

$$m \left( \left\{ x \in A : |p(x)| < \left( \frac{2}{s} \right)^{1/q} \|p\|_{L^q[A]} \right\} \right) \geq \frac{s}{2}.$$

The theorem now follows from Theorem 5.1. $\square$

6. **Müntz’s Theorem on Compact Sets of Positive Measure**

The results of this section are straightforward consequences of the Remez-type inequality of Theorem 5.1.

**Theorem 6.1.** Suppose $\sum_{i=1}^{\infty} 1/\lambda_i < \infty$ and $A \subset [0, \infty)$ is a set of positive Lebesgue measure. Then $M(A)$ is not dense in $C[A]$.

Moreover, if the gap condition

$$\inf \{ \lambda_{i+1} - \lambda_i : i \in \mathbb{N} \} > 0$$

(6.1)
holds, then every function $f \in C[A]$ from the uniform closure of $M(\Lambda)$ on $A$ is of the form

$$f(x) = \sum_{i=0}^{\infty} a_i x^{\lambda_i}, \quad x \in A \cap [0, r_A)$$

where

$$r_A := \sup\{x \in [0, \infty) : m(A \cap (x, \infty)) > 0\}.$$

is the essential supremum of $A$.

If the gap condition (6.1) does not hold, then every function $f \in C[A]$ from the uniform closure of $M(\Lambda)$ on $A$ can still be extended analytically throughout the region

$$\{z \in \mathbb{C} \setminus (-\infty, 0] : |z| < r_A\}.$$

Proof. Suppose $f \in C[A]$ and suppose there is a sequence $\{p_i\}_{i=1}^{\infty} \subset M(\Lambda)$ so that

$$\lim_{i \to \infty} \|p_i - f\|_A = 0.$$

Then the sequence $\{p_i\}_{i=1}^{\infty}$ is uniformly Cauchy on $A$. Therefore, Theorem 5.1 and the definition of $r_A$ yields that $\{p_i\}_{i=1}^{\infty}$ is uniformly Cauchy on every closed subinterval of $[0, r_A)$. If the gap condition (6.1) holds, then the characterization of the uniform closure of $M(\Lambda)$ on $A$ follows from the results of Clarkson and Erd"os [12]. If the gap condition (6.1) does not hold, then results of Schwartz [24] yield the theorem. \qed

**Theorem 6.2.** Suppose $A \subset [0, \infty)$ is a compact set of positive Lebesgue measure. Then $M(\Lambda)$ is dense in $C[A]$ if and only if $\sum_{i=1}^{\infty} 1/\lambda_i = \infty$.

Proof. Suppose $\sum_{i=1}^{\infty} 1/\lambda_i = \infty$. Let $f \in C[A]$. By Tietze's Theorem there exists an $\hat{f} \in C[0, 1]$ so that $\hat{f}(x) = f(x)$ for every $x \in A$. By M"untz's Theorem there is a sequence $\{p_i\}_{i=1}^{\infty} \subset M(\Lambda)$ so that

$$\lim_{i \to \infty} \|\hat{f} - p_i\|_{[0,1]} = 0.$$

Therefore

$$\lim_{i \to \infty} \|f - p_i\|_A = 0$$

which finishes the trivial part of the theorem. Suppose now that $\sum_{i=1}^{\infty} 1/\lambda_i < \infty$. Then Theorem 6.1 yields that $M(\Lambda)$ is not dense in $C[A]$. \qed

The following surprising theorem shows that if $\sum_{i=1}^{\infty} 1/\lambda_i < \infty$, then the pointwise and locally uniform convergence of a sequence $\{p_i\}_{i=1}^{\infty} \subset M(\Lambda)$ on $[0, b)$ are equivalent. An amusing consequence of this is that if $\sum_{i=1}^{\infty} 1/\lambda_i < \infty$, then the set

$$\left\{ f(x) = \sum_{i=0}^{\infty} a_i x^{\lambda_i}, \quad x \in [0, b) \right\}$$

is closed under pointwise convergence.
Theorem 6.3. Suppose $\sum_{i=1}^{\infty} 1/\lambda_i < \infty$. Let $A \subset [0, \infty)$ be a set of positive Lebesgue measure, and let $r_A$ be the essential supremum of $A$ defined as in Theorem 6.1. Assume $\{p_i\}_{i=1}^{\infty} \subset M(\Lambda)$ and
\[ p_i(x) \to f(x), \quad x \in A. \]
Then $\{p_i\}_{i=1}^{\infty}$ converges uniformly on every closed subinterval of $[0, r_A)$.

This characterizes non-dense Müntz spaces within Müntz spaces, since in any Müntz space $M(\Lambda)$ with $\sum_{i=1}^{\infty} 1/\lambda_i = \infty$ and for any $A \subset [0, \infty)$ of positive Lebesgue measure, there exists a sequence $\{p_i\}_{i=1}^{\infty} \subset M(\Lambda)$ that converges pointwise on $[0, \infty)$ but not locally uniformly on $A$. This follows easily from Müntz’s Theorem.

Proof of Theorem 6.3. Let $\delta \in (0, r_A)$ be fixed. Egoroff’s Theorem and the definition of $r_A$ imply the existence of a set $B \subset A \cap (\delta, \infty)$ of positive Lebesgue measure so that $\{p_i\}_{i=1}^{\infty}$ converges uniformly on $B$, hence it is uniformly Cauchy on $B$. Now Theorem 5.1 yields that $\{p_i\}_{i=1}^{\infty}$ is uniformly Cauchy on $[0, \delta]$, which proves the theorem. \qed

Theorem 6.4. Suppose $\sum_{i=1}^{\infty} 1/\lambda_i < \infty$, $A \subset [0, \infty)$ is a set of positive Lebesgue measure, $w$ is a nonnegative-valued, integrable weight function on $A$ with $\int_A w > 0$, and $q \in (0, \infty)$. Then $M(\Lambda)$ is not dense in $L^q_w[A]$.

Moreover, if the gap condition (6.1) holds, then every function $f \in L^q_w[A]$ belonging to the $L^q_w[A]$ closure of $M(\Lambda)$ can be represented as
\[ f(x) = \sum_{i=0}^{\infty} a_i x^{\lambda_i}, \quad x \in A \cap [0, r_w) \]
where
\[ r_w := \sup \left\{ x \in [0, \infty) : \int_{A \cap [x, \infty)} w(x) dx > 0 \right\}. \]

If the gap condition (6.1) does not hold, then every function $f \in L^q_w[A]$ belonging to the $L^q_w[A]$ closure of $M(\Lambda)$ can still be represented as an analytic function on
\[ \{ z \in \mathbb{C} \setminus (-\infty, 0] : |z| < r_w \} \]
restricted to $A$.

Proof. Suppose $f \in L^q_w[A]$ and suppose there is a sequence $\{p_i\}_{i=1}^{\infty} \subset M(\Lambda)$ so that
\[ \lim_{i \to \infty} ||f - p_i||_{L^q_w[A]} = 0. \]
Minkowski’s Inequality (if $q \in (0, 1)$, then a multiplicative factor $2^{1/q - 1}$ is needed) yields that $\{p_i\}_{i=1}^{\infty}$ is a Cauchy sequence in $L^q_w[A]$. The assumptions on $w$ imply that for every $\delta \in (0, r_w)$ there exists an $\alpha > 0$ so that
\[ B := \{ x \in A \cap (\delta, \infty) : w(x) > \alpha \} \]
is of positive Lebesgue measure. Note that
\[ \|p\|_{L^q(B)} \leq \frac{1}{\alpha} \|p\|_{L^q[A]} \leq \frac{1}{\alpha} \|p\|_{L^q[A]} \]
for every \( p \in L^q[B] \). Therefore, \( \{p_i\}_{i=1}^{\infty} \) is a Cauchy sequence in \( L^q[B] \). So, by Theorem 5.2, \( \{p_i\}_{i=1}^{\infty} \) is uniformly Cauchy on \([0, \delta]\). If the gap condition (6.1) holds, then the theorem now follows from results of Clarkson and Erdös [12]. If the gap condition (6.1) does not hold, then a result of Schwartz [24] yields the theorem. \( \square \)

**Theorem 6.5.** Suppose \( A \subset [0,1] \) is a set of positive Lebesgue measure, \( w \) is a nonnegative-valued integrable weight function on \( A \) with \( \int_A w > 0 \), and \( q \in (0, \infty) \). Then \( M(\Lambda) \) is dense in \( L^q_w[A] \) if and only if \( \sum_{i=1}^{\infty} 1/\lambda_i = \infty \).

**Proof.** Suppose \( \sum_{i=1}^{\infty} 1/\lambda_i = \infty \). Let \( f \in L^q_w[A] \). It is standard measure theory to show that for every \( \varepsilon > 0 \) there exists a \( g \in C[0,1] \) so that
\[ \|f - g\|_{L^q_w[A]} < \frac{\varepsilon}{2} \]
Now Müntz’s Theorem implies that there exists a \( p \in M(\Lambda) \) so that
\[ \|g - p\|_{L^q_w[A]} \leq \|g - p\|_A \left( \int_A w \right)^{1/q} < \frac{\varepsilon}{2} \]
Therefore \( M(\Lambda) \) is dense in \( L^q_w[A] \).

Suppose now that \( \sum_{i=1}^{\infty} 1/\lambda_i < \infty \). Then Theorem 6.4 yields that \( M(\Lambda) \) is not dense in \( L^q_w[A] \). \( \square \)

7. **Products of Müntz Spaces**

For
\[ \Lambda_j := \{\lambda_{i,j}\}_{i=0}^{\infty}, \quad 0 = \lambda_{0,j} < \lambda_{1,j} < \lambda_{2,j} < \cdots, \quad j = 1, 2, \ldots \]
we define the sets
\[ M(\Lambda_1, \Lambda_2, \ldots, \Lambda_k) := \left\{ p = \prod_{j=1}^{k} p_j : p_j \in M(\Lambda_j) \right\}. \]

First we prove the following Remez-type inequality for \( M(\Lambda_1, \Lambda_2, \ldots, \Lambda_k) \).
Theorem 7.1. Suppose (7.1) holds and

\[ \sum_{i=1}^{\infty} \frac{1}{\Lambda_i, j} < \infty, \quad j = 1, 2, \ldots, k. \]

Let \( s > 0 \). Then there exists a constant \( c \) depending only on \( \Lambda_1, \Lambda_2, \ldots, \Lambda_k, s \), and \( k \) (and not on \( \varrho \) or \( A \)) so that

\[ \|p\|_{[0, \varrho]} \leq c \|p\|_A \]

for every \( p \in M(\Lambda_1, \Lambda_2, \ldots, \Lambda_k) \) and for every set \( A \subset [\varrho, 1] \) of Lebesgue measure at least \( s \).

Proof. Theorem 5.1 implies that there exist constants \( \alpha_j > 0 \) depending only on \( \Lambda_1, \Lambda_2, \ldots, \Lambda_k, s \), and \( k \) so that

\[ m(\{x \in [y, 1] : |p(x)| > \alpha_j^{-1}|p(y)|\}) \geq 1 - y - \frac{s}{2k} \]

for every \( p \in M(\Lambda_j) \) and \( y \in [0, 1 - s] \). Now let

\[ p = \prod_{j=1}^{k} p_j, \quad p_j \in M(\Lambda_j). \]

Then, for every \( y \in [0, 1 - s] \),

\[ m(\{x \in [y, 1] : |p(x)| > (\alpha_1 \alpha_2 \cdots \alpha_k)^{-1}|p(y)|\}) \]

\[ \geq m\left( \bigcap_{j=1}^{k} \{x \in [y, 1] : |p_j(x)| > \alpha_j^{-1}|p_j(y)|\} \right) \]

\[ \geq 1 - y - k \frac{s}{2k} = 1 - y - \frac{s}{2}. \]

Hence \( y \in [0, \inf A] \) and \( m(A) \geq s \) imply

\[ m(\{x \in A : |p(x)| > (\alpha_1 \alpha_2 \cdots \alpha_k)^{-1}|p(y)|\}) \geq \frac{s}{2} > 0 \]

and the theorem follows with \( c := \alpha_1 \alpha_2 \cdots \alpha_k. \)

Theorem 7.1 immediately solves Newman’s problem [20].

Corollary 7.2. Suppose (7.1) and (7.2) hold and \( A \subset [0, 1] \) is a set of positive Lebesgue measure. Then \( M(\Lambda_1, \Lambda_2, \ldots, \Lambda_k) \) is not dense in \( C[A] \). Moreover, if \( w \) is a non negative-valued integrable weight function on \( A \) with \( \int_A w > 0 \) and \( q \in (0, \infty) \), then \( M(\Lambda_1, \Lambda_2, \ldots, \Lambda_k) \) is not dense in \( L^q_w[A] \).

Our next theorem establishes a Bernstein-type inequality for \( M(\Lambda_1, \Lambda_2, \ldots, \Lambda_k) \).
Theorem 7.3. Suppose (7.1) and (7.2) hold and
\[ \lambda_{i,j} \geq 1, \quad j = 1, 2, \ldots, k. \]
Let \( s > 0 \). Then there exists a constant \( c \) depending only on \( \Lambda_1, \Lambda_2, \ldots, \Lambda_k \), \( s \) and \( k \) (and not on \( g \) or \( A \)) so that
\[ \| p' \|_{[0, \delta]} \leq c \| p \|_A \]
for every \( p \in M(\Lambda_1, \Lambda_2, \ldots, \Lambda_k) \) and for every set \( A \subset [\delta, 1] \) of Lebesgue measure at least \( s \).

Proof. Note that a combination of Theorems 5.1 and 3.2 implies that there exist constants \( \beta_j > 0 \) depending only on \( \Lambda_1, \Lambda_2, \ldots, \Lambda_k \), \( s \), and \( k \) so that
\[ m(\{ x \in [\delta, 1] : |p(x)| > \beta_j^{-1}|p'(y)| \}) \geq 1 - y - \frac{s}{2k} \]
for every \( p \in M(\Lambda_j) \) and \( y \in [0, 1 - s] \). Let \( \alpha_j > 0, j = 1, 2, \ldots, k \), be chosen as in the proof of Theorem 7.1. Now let
\[ p \in M(\Lambda_1, \Lambda_2, \ldots, \Lambda_k) \]
that is
\[ p = \prod_{j=1}^{k} p_j, \quad p_j \in M(\Lambda_j). \]
Then, as in the proof of Theorem 7.1, for every \( y \in [0, 1 - s] \),
\[ m(\{ x \in [\delta, 1] : |p(x)| > \gamma_j^{-1}|(p_1 \cdots p_{j-1} p_j p_{j+1} \cdots p_k)(y)| \}) \]
\[ \geq 1 - y - k \frac{s}{2k} = 1 - y - \frac{s}{2} \]
where
\[ \gamma_j := \alpha_1 \cdots \alpha_{j-1} \beta_j \alpha_{j+1} \cdots \alpha_k \]
Hence \( y \in [0, \inf A] \) and \( m(A) \geq s \) imply
\[ m(\{ x \in A : |p(x)| > \gamma_j^{-1}|(p_1 \cdots p_{j-1} p_j p_{j+1} \cdots p_k)(y)| \}) \geq \frac{s}{2} > 0 \]
and the theorem follows with \( c := \sum_{j=1}^{k} \gamma_j \)  

The following Nikolskii-type inequality is also valid for \( M(\Lambda_1, \Lambda_2, \ldots, \Lambda_k) \).

Theorem 7.4. Suppose (7.1) and (7.2) hold. Let \( s > 0 \) and \( q \in (0, \infty) \). Then there exists a constant \( c \) depending only on \( \Lambda_1, \Lambda_2, \ldots, \Lambda_k, s, k, q, \) and \( w \) (and not on \( g \) and \( A \)) so that
\[ \| p \|_{[0, \delta]} \leq c \int_A |p(x)|^q w(x)dx \]
for every \( p \in M(\Lambda_1, \Lambda_2, \ldots, \Lambda_k) \), for every set \( A \subset [\delta, 1] \) of Lebesgue measure at least \( s \), and for every function \( w \) measurable and positive a.e. on \( [0, 1] \).

Proof. This a straightforward consequence of Theorem 7.1
REFERENCES


27. O. Szász, Über die Approximation stetiger Funktionen durch lineare Aggregate von Potenzen 77 (1916), 482-496.


29. A. K. Taslaky, Some properties of Legendre quasi-polynomials with respect to a Müntz system, Mathematics 2 (1984), 179-189; Erevan University, Erevan, (Russian, Armenian Summary).


Department of Mathematics, Simon Fraser University, Burnaby B.C., Canada V5A 186