Further arguments for slice convergence
in nonreflexive spaces

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Abstract. It is shown that no notion of set convergence at least as strong as Wijsman
convergence but not as strong as slice convergence can be preserved in superspaces. We
also show that such intermediate notions of convergence do not always admit representa-
tions analogous to those given by Attouch and Beer for slice convergence, and provide a
valid reformulation. Some connections between bornologies and the relationships between
certain gap convergences for nonconvex sets are also observed.

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vergence, Wijsman convergence.
1. Introduction. Extensive research has been done concerning applications of various types of set convergence. Seminal papers in this area include those of Wijsman ([17]) and Mosco ([16]). While Wijsman convergence has limitations outside of finite dimensional spaces, Mosco convergence is a very useful tool in reflexive spaces; see [1]. However, it was shown in [7] that Mosco convergence has serious limitations in nonreflexive spaces. On the other hand, Beer ([3]) has shown, that slice convergence—which agrees with Mosco convergence in reflexive spaces—is well-suited for applications in nonreflexive spaces; see [2] for results concerning convergence of subdifferentials. The purpose of this note is to provide further illustrations of the unavoidable defects of set convergence notions that are weaker than slice convergence.

We will work in real Banach spaces and consider the following forms of convergence and, except for the last section, we will deal only with closed convex sets. For sets $A_n, A \subset X$, the sequence $\langle A_n \rangle$ is said to converge Mosco to $A$ (written $A_n \xrightarrow{M} A$), if $d(x, A_n) \to 0$ whenever $x \in A$ and if $x \in A$ whenever $x_k \in A_{n_k}$ are such that $\langle x_k \rangle \xrightarrow{w} x$. For sets $A_\alpha, A \subset X$, the net $\langle A_\alpha \rangle$ is said to converge slice (resp. weak compact gap, Wijsman) to $A$ with respect to the norm $\| \cdot \|$ if $d(W, A_\alpha) \to d(W, A)$ for each closed bounded convex (resp. weakly compact convex, singleton or compact) $W \subset X$ where $d$ is measured with respect to $\| \cdot \|$; we will write $A_\alpha \xrightarrow{s} A$ (resp. $A_\alpha \xrightarrow{wg} A, A_\alpha \xrightarrow{W} A$). For $A_\alpha, A \subset X^*$, the net $\langle A_\alpha \rangle$ is said to converge $w^*$-slice to $A$ (written $A_\alpha \xrightarrow{s^*} A$), if $d(W, A_\alpha) \to d(W, A)$ for each $w^*$-compact convex $W$ where the distance is measured with respect to the dual norm. See [5, 6] for further properties of these notions.

There are several connections between set convergence and renorming. Of particular importance is Beer’s renorming invariance result ([4, Theorem 3.1]) which shows a net of closed convex sets converges slice if and only if it converges Wijsman with respect to every equivalent renorm. In particular, for Wijsman and weak compact gap convergence it is crucial to stipulate which norm is being used. We will also consider dual Kadec-Klee norms which we call Kadec norms for short. A norm on $X^*$ is said to be $\tau$-Kadec (resp. sequentially $\tau$-Kadec) if the Mackey and norm topologies agree (resp. sequentially) on its sphere (where the Mackey topology is that of uniform convergence on weakly compact sets); $w^*$-Kadec and sequentially $w^*$-Kadec norms are defined analogously. An easy example of
a dual norm that is $\tau$-Kadec but not $w^*$-Kadec is any norm on a reflexive space for which weak and norm convergence do not coincide on its sphere. We will use Beer’s renorming invariance result as well as dual Kadec norms to exploit the erratic behavior of some convergence notions weaker than slice.

In the second section, we show that every notion of convergence weaker than slice is not well-behaved in superspaces. In particular, using Beer’s result ([4, Theorem 3.1]) mentioned above, we show a sequence of closed convex sets cannot be expected to converge Wijsman in superspaces of $X$ if does not converge slice in $X$.

The third section examines alternative representations of various gap convergences. The prototypical result is a characterization of slice convergence given by Attouch and Beer in [2, Theorem 3.1]. In [13], we obtained a variant of this for Wijsman convergence in separable spaces which enabled us to show Wijsman and slice convergence coincide for closed convex sets if the dual norm is $w^*$-Kadec. Unfortunately, we show that Wijsman and weak compact gap convergence do not, in general, admit natural analogs the Attouch-Beer slice representation. In the process of building counterexamples for the weak compact gap case, we obtained an example of a sequence of closed convex sets which converges weak compact gap but not slice even though the dual norm is sequentially $\tau$-Kadec (Proposition 3.2). In sharp contrast to this, Corollary 3.4 shows weak compact gap convergence and slice convergence coincide for all nets of closed convex sets provided the dual norm is $\tau$-Kadec. This result relies on an iterated limsup characterization of weak compact gap convergence that also has variants for Wijsman and slice convergence (Theorem 3.3).

Finally, in the fourth section, we show that the relationships between Wijsman, weak compact gap and slice convergence for (nonconvex) weakly closed sets are determined precisely by the relationships between the corresponding bornologies on the space. Consequently, one cannot replace closed convex sets with weakly closed sets in our results concerning the coincidence of Wijsman (resp weak compact gap) convergence with slice convergence when the dual norm is $w^*$-Kadec (resp. $\tau$-Kadec).

As for notation, we will denote the closed convex subsets of $X$ by $C(X)$ and the $w^*$-closed convex subsets of a dual space $X^*$ will be denoted by $C^*(X)$. We will also use: $B_X = \{ x : \| x \| \leq 1 \}$, $S_X = \{ x : \| x \| = 1 \}$ and $B_\delta = \{ x : \| x \| \leq \delta \}$. Sometimes we will write e.g. $B_X(\| \cdot \|)$ to emphasize which norm is being used.
2. Preservation of set convergence in superspaces. We first observe that empty sets will not cause a problem in this section.

Remark 2.1. When \( C = \emptyset \), we use the convention \( d(S, C) = \infty \) for any set \( S \). Suppose \( \langle C_\alpha \rangle \xrightarrow{W} C \) where \( C = \emptyset \), then it is easy to see that \( \langle C_\alpha \rangle \xrightarrow{s} C \). Thus, one can check that all the results in this section are trivially true if the limit set \( C \) is empty. On the other hand, if \( C \neq \emptyset \) and \( \langle C_\alpha \rangle \xrightarrow{W} C \), then \( C_\alpha \neq \emptyset \) eventually. Because of these facts, in all the proofs of this section we implicitly assume that all sets are nonempty.

In [13, Example 4.2] a sequence of sets in a hyperplane of a space \( X \) isomorphic to \( c_0 \) was constructed so that it converges Wijsman in the hyperplane but not in \( X \). The following theorem completely determines when Wijsman convergence is preserved in superspaces.

Theorem 2.2. (Superspace Invariance) For a Banach space \( (X, \| \cdot \|) \) and \( C_\alpha, C \in C(X) \), the following are equivalent.

(a) \( \langle C_\alpha \rangle \xrightarrow{W} C \) with respect to every norm on \( X \times \mathbb{R} \) extending \( \| \cdot \| \).

(b) \( \langle C_\alpha \rangle \xrightarrow{W} C \) with respect to every norm on \( X \).

(c) \( \langle C_\alpha \rangle \xrightarrow{s} C \) in \( X \).

(d) \( \langle C_\alpha \rangle \xrightarrow{s} C \) in every superspace of \( X \).

Proof. (a) \( \Rightarrow \) (b): Suppose (b) fails to hold. Let \( \| \cdot \| \) be an equivalent norm on \( X \) for which \( \langle C_\alpha \rangle \xrightarrow{W} C \). Since \( \langle C_\alpha \rangle \xrightarrow{W} C \) with respect \( \| \cdot \| \), there are \( \delta > 0 \) and \( x_0 \in X \) such that

\[
3\delta + \liminf_{\alpha} d_{\| \cdot \|}(x_0, C_\alpha) < d_{\| \cdot \|}(x_0, C)
\]

where \( d_{\| \cdot \|} \) denotes distance measured with respect to \( \| \cdot \| \). By translating and passing to a subnet we may suppose

\[
3\delta + d_{\| \cdot \|}(0, C_\alpha) < d_{\| \cdot \|}(0, C)
\]

for all \( \alpha \).

Now let \( r = d_{\| \cdot \|}(0, C) - 2\delta \). Then \( B_r(\| \cdot \|) \cap C_\alpha \neq \emptyset \) for all \( \alpha \) and \( B_{r+\delta}(\| \cdot \|) \cap C = \emptyset \). By the separation theorem there is a \( \Lambda \in S_{X^*}(\| \cdot \|) \) such that

\[
\sup_{B_r(\| \cdot \|)} \Lambda + \delta \leq \inf_{C} \Lambda.
\]
Let \( a = \sup \{ \Lambda(x) : x \in B_r(|\cdot|) \} \) and choose \( x_\alpha \in C_\alpha \cap B_r(|\cdot|) \). Then we have \( |\Lambda(x_\alpha)| \leq a \) while \( \Lambda(z) \geq a + \delta \) for any \( z \in C \). Since \( \langle x_\alpha \rangle \subset B_r(|\cdot|) \) we can choose \( \beta > 0 \) so that \( \beta \|x_\alpha\| \leq 1 \) for all \( \alpha \). Set \( \phi = \tilde{\beta} \Lambda \) where \( \tilde{\beta} > 0 \) is chosen such that \( \tilde{\beta} a \leq \frac{1}{2} \) and \( |\phi(x)| \leq 1 \) provided \( \beta \|x\| \leq 1 \). Now let \( \bar{a} = \tilde{\beta} a \) and \( \bar{\delta} = \tilde{\beta} \delta \). Of course we have

\[
|\phi(x_\alpha)| \leq \bar{a} \leq \frac{1}{2} \quad \text{for all } \alpha \quad \text{and} \quad \phi(z) \geq \bar{a} + \bar{\delta} \quad \text{for } z \in C.
\]

Define the norm \( \| \cdot \| \) on \( X \times \mathbb{R} \) as the Minkowski functional of

\[
B = \{(x, t) : \beta \|x\| \leq 1, |\phi(x)| + |t| \leq 1\}.
\]

Notice that if \( \beta \|x\| \leq 1 \), then \( |\phi(x)| \leq 1 \) and so \( (x, 0) \in B \). If \( \beta \|x\| > 1 \), then \( (x, 0) \notin B \) and consequently \( \| \cdot \| \) agrees with \( \beta \| \cdot \| \) on \( X \) (identified as a subspace of \( X \times \mathbb{R} \)). Consider \( (0, 1 - \bar{a}) \in X \times \mathbb{R} \). Then for \( z \in C \), we have \( (z, 0) - (0, 1 - \bar{a}) = (z, \bar{a} - 1) \) and

\[
|\phi(z)| + |\bar{a} - 1| \geq \bar{a} + \bar{\delta} + 1 - \bar{a} = 1 + \bar{\delta}.
\]

Therefore \( \|(z, \bar{a} - 1)\| \geq 1 + \bar{\delta} \). Thus \( d_{\| \cdot \|}((0, 1 - \bar{a}), C) \geq 1 + \bar{\delta} \). On the other hand, for \( x_\alpha \in C_\alpha \) as chosen before, we have \( (x_\alpha, 0) - (0, 1 - \bar{a}) = (x_\alpha, \bar{a} - 1) \). Recall that

\[
\beta \|x_\alpha\| \leq 1 \quad \text{and} \quad |\bar{a} - 1| + |\phi(x_\alpha)| \leq 1 - \bar{a} + \bar{a} = 1.
\]

Whence \( d_{\| \cdot \|}((0, 1 - \bar{a}), C_\alpha) \leq 1 \) and so Wijsman convergence fails with respect to the norm \( \| \cdot \| \) and hence with respect to the norm \( \frac{1}{\beta} \| \cdot \| \) which extends \( \| \cdot \| \).

For the other implications, recall [4, Theorem 3.1] shows (b) implies (c) while (c) implies (d) follows from [13, Proposition 4.1]. The implication (d) implies (a) is clear from the definitions.

It follows directly from its definition that Mosco convergence is preserved in superspaces. However, in nonreflexive spaces Mosco convergence is properly weaker than Wijsman convergence (see e.g. [12]) and so it does not force Wijsman convergence in \( X \times \mathbb{R} \). In fact a lot more can be said as the next corollary shows.
Corollary 2.3. If $C_n, C \in C(X)$ and $\langle C_n \rangle \xrightarrow{W} C$, then $\langle C_n \rangle \xrightarrow{M} C$ with respect to some norm on $X \times \mathbb{R}$ extending the original norm on $X$.

Proof. If $\langle C_n \rangle \xrightarrow{W} C$ in $X$, we are done; otherwise we apply Theorem 2.2. \hfill \square

In particular, Corollary 2.3 applies when $X$ is reflexive and $\langle C_n \rangle \xrightarrow{M} C$. Regarding $w^*$-slice convergence we have the following results.

Corollary 2.4. In any nonreflexive dual space $X^*$, there is a sequence of $w^*$-closed convex sets $C_n$ converging $w^*$-slice to $C$ in $X^*$ endowed with a dual norm $\| \cdot \|^*$, which does not converge Wijsman in $X^* \times \mathbb{R}$ with respect to some norm extending $\| \cdot \|^*$.

Proof. Because $X^*$ is nonreflexive, Beer ([5]) shows there is a sequence in $C^*(X^*)$ converging $w^*$-slice but not slice; see also Proposition 2.6 below. \hfill \square

The extension norm on $X \times \mathbb{R}$ in Corollary 2.4 cannot be a dual norm. Indeed:

Theorem 2.5. (Dual Superspace Invariance) For a dual space $X^*$ equipped with dual norm $\| \cdot \|^*$ and $C_\alpha, C \in C^*(X^*)$, the following are equivalent.

(a) $\langle C_\alpha \rangle \xrightarrow{W} C$ with respect to every dual norm on $X^* \times \mathbb{R}$ which extends $\| \cdot \|^*$.

(b) $\langle C_\alpha \rangle \xrightarrow{W} C$ with respect to every dual norm on $X^*$.

(c) $\langle C_\alpha \rangle \xrightarrow{w^*} C$ in $X^*$.

(d) $\langle C_\alpha \rangle \xrightarrow{w^*} C$ in every dual superspace of $X^*$.

Proof. For (a) $\Rightarrow$ (b), one follows the proof of Theorem 2.2 and chooses the separating functional $\Lambda$ from $X$. Thus the ball $B$ will be $w^*$-closed which means it generates a dual norm. The other implications follow from $w^*$-slice versions of [4, Theorem 3.1] and [13, Proposition 4.1]. \hfill \square

Because Beer’s $w^*$-slice result ([5], see also [6, Section 2.4, Exercise 10]) will be crucial in subsequent examples of this note, we are including a self-contained proof for completeness. Additionally our sets are decreasing subspaces which is of interest in light of [12].
Proposition 2.6. In every nonreflexive dual space, there is a decreasing sequence of \(w^\ast\)-closed subspaces \(\langle Y_n \rangle\) that converges \(w^\ast\)-slice but not slice to its intersection.

Proof. Let \(X\) be a nonreflexive space and let \(Z\) be a separable nonreflexive subspace of \(X\). As in [12], let \(\{z_k\}_{k=1}^\infty\) be dense in \(Z\) and define

\[ Y = Z^\perp \text{ and } Y_n = \{ x^* \in X^*: x^*(z_k) = 0 \text{ for } k = 1, 2, \ldots, n \}. \]

Since \(\{z_k\}_{k=1}^\infty\) is dense in \(Z\), it follows that \(Y = \bigcap_{n=1}^\infty Y_n\). Now \(\langle Y_n \rangle \overset{s^*}{\longrightarrow} Y\) since it is decreasing. Indeed, \(d(K, Y_n) \leq d(K, Y)\) for each \(n\) and \(w^\ast\)-compact \(K \in C^\ast(X^\ast)\). For the reverse inequality, if \(d(K, Y) = r > 0\) then \((K + B_{r-\delta}) \cap Y = \emptyset\) for any \(\delta > 0\). Since \(K + B_{r-\delta}\) is \(w^\ast\)-compact, the finite intersection property implies \((K + B_{r-\delta}) \cap Y_n = \emptyset\) for all large \(n\). Thus \(\liminf_n d(K, Y_n) \geq d(K, Y)\).

To show that slice convergence fails, observe that \(Z \cap B_X\) is not weakly compact and so there is a \(\Phi \in Z \cap B_X^{w^\ast} \setminus Z\). Replacing \(\Phi\) with \(\Phi/\|\Phi\|\), we have \(\|\Phi\| = 1\), \(\Phi \in Y^\perp\) and \(d(\Phi, Z) > 0\). Choose \(\Lambda \in B_X^{w^\ast}\)-such that \(\Lambda(\Phi) = \delta\) for some \(\delta > 0\) and \(\Lambda(Z) = \{0\}\). If \(C = 2B_X \cap \Phi^{-1}(1)\), then \(d(C, Y) \geq 1\) because \(\Phi \in Y^\perp\) and \(\|\Phi\| = 1\). To conclude the proof we show that \(d(C, Y_n) < 1 - \frac{\delta}{3}\) for each \(n\). For \(n\) fixed, let \(\{b_1, \ldots, b_m\} \subset B_X\) be a basis of \(\text{span}\{z_1, \ldots, z_n\}\), let \(\{\phi_1, \ldots, \phi_m\} \subset X^\ast\) satisfy \(\phi_i(b_j) = \delta_{ij}\), and let \(\bar{c} = \sup_{j \leq m} \|\phi_j\|\). By Goldstine’s theorem, we can find \(x^* \in B_X^{w^\ast}\) such that

\[ \Phi(x^*) > \frac{2}{3}\delta \quad \text{and} \quad |x^*(b_j)| < \frac{\delta}{3m\bar{c}} \quad \text{for } j = 1, \ldots, m. \]

Let \(z^* = x^* - \sum_{j=1}^m x^*(b_j)\phi_j\), then \(z^* \in Y_n\), \(\Phi(z^*) > \frac{\delta}{3}\) and \(\|z^*\| < 1 - \frac{\delta}{3}\). Moreover, \(d(\Phi^{-1}(1), z^*) < 1 + \frac{\delta}{3}\) and so there exists \(x^* \in \Phi^{-1}(1)\) with \(\|z^* - x^*\| < 1 - \frac{\delta}{3}\) and hence \(\|x^*\| \leq 2\). Consequently \(x^* \in C\) and \(d(Y_n, C) < 1 - \frac{\delta}{3}\) as desired.

If \(X\) is separable and nonreflexive, then Proposition 2.6 also follows from results in [4] and [12]. Indeed, suppose \(\text{span}\{x_k\}_{k=1}^\infty = X\) and let \(Y_n = \{x^* \in X^*: x^*(x_k) = 0\text{ for } k = 1, \ldots, n\}\). Since \(X\) is 1-norming with respect to all dual norms on \(X^\ast\), [12, Theorem 6] shows that \(\langle Y_n \rangle \overset{w}{\longrightarrow} \{0\}\) with respect to all dual norms on \(X^\ast\). Therefore the \(w^\ast\)-slice variant of [4, Theorem 3.1] shows that \(\langle Y_n \rangle \overset{s^*}{\longrightarrow} \{0\}\). Moreover, because \(X \neq X^{**}\), there
are nondual norms on $X^*$ for which $X$ is not 1-norming, and so [12, Theorem 6] shows that with respect to such a norm $\langle Y_n \rangle \overset{w}{\to} \{0\}$.

3. **Attouch-Beer type representations of set convergence.** Let $\langle C_\alpha \rangle \subset C(X)$, $C \in C(X)$ with $C \neq \emptyset$ and consider the following two conditions.

(AB$_1$) If $x \in C$, then $d(x, C_\alpha) \to 0$.

(AB$_2$) If $\phi \in S_{X^*}$ attains its supremum on $C$, then there exists $\langle \phi_\alpha \rangle \subset S_{X^*}$ such that $\langle \phi_\alpha \rangle$ converges to $\phi$ in a topology $T$ on $X^*$ and

$$\limsup_{\alpha} \sup_{C_\alpha} \phi_\alpha \leq \sup_C \phi.$$ 

We will refer to the second condition as (AB$_2$) (resp. (AB$_2^\circ$), (AB$_2^*$)) if $T$ is the norm (resp. Mackey, $w^*$) topology. Observe that if $C = \emptyset$, then (AB$_2$) is always vacuously satisfied. Thus when (AB$_1$) and (AB$_2$) are used to represent set convergence it is necessary to assume $C \neq \emptyset$; and when $C \neq \emptyset$, (AB$_1$) ensures $C_\alpha \neq \emptyset$ eventually, thus we will often implicitly assume all sets are nonempty in our proofs. For sequences of closed convex sets, Attouch and Beer showed (AB$_1$) and (AB$_2$) are an equivalent and very useful formulation of slice convergence; see [2]. Unfortunately, its analogs for Wijsman and weak compact convergence are not always valid, as we now demonstrate.

**Theorem 3.1.** Let $C_n, C \in C(X)$ with $C \neq \emptyset$.

(a) $\langle C_n \rangle \overset{s}{\to} C$ if and only if (AB$_1$) and (AB$_2$) are satisfied.

(b) $\langle C_n \rangle \overset{w}{\to} C$ provided (AB$_1$) and (AB$_2^\circ$) are satisfied; there are separable spaces in which the converse fails.

(c) $\langle C_n \rangle \overset{w}{\to} C$ if (AB$_1$) and (AB$_2^*$) are satisfied; the converse is valid in separable spaces but there are nonseparable spaces in which it fails.

**Proof.** Recall that (a) follows from [2, Theorem 3.1]. In [13, Proposition 1.2] it was observed that (AB$_1$) and (AB$_2$) (resp. (AB$_2^\circ$), (AB$_2^*$)) imply slice (resp. weak compact gap Wijsman) convergence even for nets of sets in $C(X)$. The converse implication in (c) for separable spaces was established in [13, Theorem 2.1 and Remark 2.2].

Let $X$ be any nonreflexive space such that $X^{**}$ is separable (e.g. the James space). Then Proposition 3.2 shows that the converse of (b) can fail in the separable space $X^*$. 

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A space in which the converse of (c) can fail is \( \ell_1(\Gamma) \) endowed with its usual norm \( \| \cdot \|_1 \) where \( \Gamma \) is uncountable. To see this, let \( C_n = \{ x \in \ell_1(\Gamma) : |x(\gamma)| \leq \frac{1}{n} \text{ for all } \gamma \in \Gamma \} \). To show \( C_n \overset{w}{\to} \{0\} \), we need only observe that \( \liminf_{n \to \infty} d(x, C_n) \geq \|x\|_1 \text{ for each } x \in \ell_1(\Gamma) \).

For this, fix \( x \in \ell_1(\Gamma) \) and let \( \epsilon > 0 \). Choose \( \gamma_1, \ldots, \gamma_m \) such that \( \sum_{i=1}^{m} |x(\gamma_i)| \geq \|x\|_1 - \frac{\epsilon}{2} \).

For \( n \) such that \( \frac{m}{n} < \frac{\epsilon}{2} \), we have

\[
\|x - z\|_1 \geq \sum_{i=1}^{m} |(x - z)(\gamma_i)| \geq \sum_{i=1}^{m} |x(\gamma_i)| - \frac{1}{n} > \|x\|_1 - \epsilon \text{ for each } z \in C_n.
\]

Thus \( C_n \) converges Wijsman to \( \{0\} \). Nevertheless, we now show that \( \langle AB_2^* \rangle \) cannot hold for \( \phi \in \ell_\infty(\Gamma) \) defined by \( \phi(\gamma) = 1 \) for all \( \gamma \in \Gamma \). Indeed, suppose \( \langle \phi_n \rangle \subset B_{\ell_\infty(\Gamma)} \) and \( \langle \phi_n \rangle \overset{w^*}{\to} \phi \). Let \( \Gamma_n = \{ \gamma \in \Gamma : \phi_k(\gamma) \geq \frac{1}{2} \text{ for } k \geq n \} \). Then \( \bigcup_{n=1}^{\infty} \Gamma_n = \Gamma \) and so for some \( n_0, \Gamma_{n_0} \) is infinite. Fix \( \{\gamma_1, \gamma_2, \ldots\} \subset \Gamma_{n_0} \) and fix \( x_n \in C_n \) with \( x_n(\gamma_i) = \frac{1}{n} \) for \( 1 \leq i \leq n^2 \), and \( x_n(\gamma) = 0 \) otherwise. Thus for \( n \geq n_0 \), \( \phi_n(x_n) \geq \frac{1}{n} \sum_{i=1}^{n^2} \phi_n(\gamma_i) \geq \frac{n}{2} \) and so

\[
\limsup_{n \to \infty} \sup_{C_n} \phi_n = \infty
\]

while \( \sup C = 0 \). Hence \( \langle AB_2^* \rangle \) does not hold.

\[\square\]

One can think of epigraphs of lsc convex functions as closed convex sets in the above theorem. In fact, these conditions can be reformulated to characterize epi-convergence of functions—this was the original approach of Attouch and Beer’s (see [2]). However, for convergences weaker than slice one must be careful to specify which norm on \( X \times \mathbb{R} \) is being used in the functional formulation; since a complete discussion of this is in [14] we will restrict our attention to convex sets in \( X \). The next proposition shows that \( \langle AB_2 \rangle \) is intermediate to weak compact gap and slice convergence.

**Proposition 3.2.** If \( X \) is not reflexive, \( X^* \not\cong \ell_1 \) and \( \| \cdot \|_1^* \) is a dual norm on \( X^* \), then:

(a) the dual norm of \( \| \cdot \|_1^* \) is sequentially \( \tau \)-Kadec on \( X^{**} \), but there exist \( C_n, C \in C^*(X^*) \) such that \( \langle C_n \rangle \overset{w^*}{\to} C \) but \( \langle C_n \rangle \overset{\mu^*}{\not\to} C \);

(b) there are weak compact gap convergence sequences in \( C^*(X^*) \) for which \( \langle AB_2 \rangle \) fails.

**Proof.** (a) Since \( X^* \not\cong \ell_1 \) it follows that every Mackey convergent sequence in \( X^{**} \) is norm convergent; see e.g. [10, Theorem 5]. In particular every norm on \( X^{**} \) is \( \tau \)-Kadec. According to Proposition 1.6, there are \( C_n, C \in C^*(X^*) \) such that \( \langle C_n \rangle \overset{\tau^*}{\to} C \) but
\( \langle C_n \rangle \not\rightarrow C \). Since weakly compact sets in \( X^\ast \) are \( w^\ast \)-compact, it follows that \( \langle C_n \rangle \overset{wg}{\rightarrow} C \)
with respect to \( \| \cdot \|^\ast \).

(b) Wijsman convergence ensures \( (AB_1) \) holds. If \( (AB_2^\ast) \) holds for a given weak compact gap convergent sequence, then so does \( (AB_2) \) because the dual norm on \( X^{**} \) is sequentially \( \tau \)-Kadec. Thus it would follow from Theorem 3.1(a) that weak compact gap convergence implies slice convergence for sequences in \( C(X) \) which contradicts (a). \( \square \)

In contrast to Proposition 3.2(a) we will show every weak compact gap convergent net in \( C(X) \) is slice convergent provided the dual norm is \( \tau \)-Kadec. This follows directly from the next theorem which provides formulations for Wijsman, weak compact gap or slice convergence in the spirit of the Attouch-Beer result ([2, Theorem 3.1]).

**Theorem 3.3.** (Iterated Limsup Representation) If \( C_\alpha, C \in C(X) \) and \( C \neq \emptyset \), then \( \langle C_\alpha \rangle \) converges slice (resp. weak compact gap, Wijsman) to \( C \) if and only if (i) and (ii) hold.

(i) For every \( x \in C \), \( d(x, C_\alpha) \to 0 \).

(ii) If \( \phi \in S_{X^\ast} \) attains its supremum on \( C \), then there is a net \( \langle \phi_{\alpha, \beta} \rangle \subset S_{X^\ast} \) such that
\[
\langle \phi_{\alpha, \beta} \rangle \text{ converges to } \phi \text{ in the norm (resp. Mackey, } w^\ast \text{) topology on } X^\ast \text{ and }
\]
\[
\lim_{\beta} \left( \lim_{\alpha} \sup \left\{ \sup_{C_\alpha} \phi_{\alpha, \beta} \right\} \right) \leq \sup_{C} \phi.
\]

Before proving Theorem 3.3, we record the following corollary.

**Corollary 3.4.** If the norm on \( X \) has \( \tau \)-Kadec dual norm, then for \( C_\alpha, C \in C(X) \),
\( \langle C_\alpha \rangle \overset{a}{\rightarrow} C \) provided \( \langle C_\alpha \rangle \overset{wg}{\rightarrow} C \).

**Proof.** If \( C = \emptyset \) then this is trivial. If \( C \neq \emptyset \) and \( \langle C_\alpha \rangle \overset{wg}{\rightarrow} C \), then Theorem 3.3(ii) is satisfied with the Mackey topology; because the dual norm is \( \tau \)-Kadec, it is also satisfied with the norm topology and hence \( \langle C_\alpha \rangle \overset{a}{\rightarrow} C \). \( \square \)

One can also use Theorem 3.3 to show Wijsman and slice (resp. weak compact gap) convergence coincide for sets in \( C(X) \) when the \( w^\ast \) and norm (resp. Mackey) topologies agree on the dual sphere. In [13], these results were derived using separation arguments and a separable sequential reduction result. However, Proposition 3.2 shows that the relationship between weak compact gap and slice convergence is not separably sequentially
determined, thus Corollary 3.4 does not follow from methods of [12] alone. The strategy we employ in the upcoming proof of Theorem 3.3 is a bornology-based indexing scheme along with separation arguments as used in [13, Theorem 2.1].

Proof of Theorem 3.3. ⇒: We will prove this for the weak compact gap case because it is the key ingredient required in Corollary 3.4. Moreover, the proofs for the Wijsman and slice cases can be obtained by considering the bornologies of compact and bounded sets respectively in place of the weakly compact sets used below; cf. Theorem 3.5.

It is clear that (i) holds so we establish (ii). Let \( L = \{ x \in X : \phi(x) = 0 \} \) and let \( \mathcal{W} \) be the collection of weakly compact subsets of \( L \). Set \( \mathcal{B} = \mathcal{W} \times \mathbb{N} \) and direct \( \mathcal{B} \) as follows: for \( \beta_1 = (W_1, n_1) \) and \( \beta_2 = (W_2, n_2) \), we have \( \beta_1 \leq \beta_2 \) if \( n_1 \leq n_2 \) and \( W_1 \subseteq W_2 \). Fix \( x_0 \in C \) such that \( \phi(x_0) = \sup C \phi \). Denote the hyperplane \( \{ x \in X : \phi(x) = \phi(x_0) + 1 \} \) by \( H \) and choose \( x_1, x_2, \ldots \in H \) such that \( \| x_n - x_0 \| \leq 1 + \frac{1}{n} \). For \( \beta = (W, n) \) fixed, using the Krein-Šmulian theorem and weak compact gap convergence (since \( d(L, C) = 1 \)) we fix \( \alpha(\beta) \) such that

\[
(\overline{\text{conv}}(W + \{x_1, \ldots, x_n\}) + B_{1 - \frac{1}{n}}) \cap C_\alpha = \emptyset \quad \text{for} \quad \alpha \geq \alpha(\beta).
\]

We now construct the net \( \langle \phi_{\alpha, \beta} \rangle \) as follows. For \( \beta = (W, n) \), let \( \phi_{\alpha, \beta} = \phi \) if \( \alpha \not\leq \alpha(\beta) \); when \( \alpha \geq \alpha(\beta) \) by the separation theorem choose \( \phi_{\alpha, \beta} \in S_{X^*} \) such that

\[
\inf \{ \phi_{\alpha, \beta}(x) : x \in \overline{\text{conv}}(W + \{x_1, \ldots, x_n\}) + B_{1 - \frac{1}{n}} \} \geq \sup \{ \phi_{\alpha, \beta}(x) : x \in C_\alpha \}. \quad (3.1)
\]

To prove \( \langle \phi_{\alpha, \beta} \rangle \rightharpoonup \phi \), first let \( W \) be an arbitrary weakly compact subset of \( L \). Let \( W_0 = \overline{\text{conv}}(-W \cup W) \), then \( W_0 \) is weakly compact by the Krein-Šmulian theorem. Let \( \epsilon > 0 \) and choose \( n_0 \) such that \( \frac{4}{n_0} < \epsilon \) and choose \( \alpha_0 \) such that

\[
d(x_0, C_\alpha) < \frac{1}{n_0} \quad \text{for} \quad \alpha \geq \alpha_0. \quad (3.2)
\]

We now show \( \sup_{W_0} |\phi_{\alpha, \beta}| < \epsilon \) for \( \alpha_0, \beta_0 \leq (\alpha, \beta) \) where \( \beta_0 = (W_0, n_0) \). In the case \( \phi_{\alpha, \beta} = \phi \) there is nothing further to do; in the other case (3.1) and (3.2) imply:

\[
\inf_{W_0 + \{x_1, \ldots, x_n\}} \phi_{\alpha, \beta} \geq \sup_{C_\alpha} \phi_{\alpha, \beta} + (1 - \frac{1}{n}) \geq \phi_{\alpha, \beta}(x_0) + (1 - \frac{2}{n_0}) \quad \text{for} \quad \alpha \geq \alpha(\beta). \quad (3.3)
\]
Thus for \( w \in W_0 \), one has
\[
\phi_{\alpha, \beta}(w + x_n) \geq \phi_{\alpha, \beta}(x_0) + \left(1 - \frac{2}{n_0}\right),
\]
and because \( \|x_0 - x_n\| \leq 1 + \frac{1}{n_0} \), this shows
\[
\phi_{\alpha, \beta}(w) \geq \phi_{\alpha, \beta}(x_0 - x_n) + \left(1 - \frac{2}{n_0}\right) \geq -\left(1 + \frac{1}{n_0}\right) + \left(1 - \frac{2}{n_0}\right) > -\varepsilon.
\]
Since \( W_0 \) is symmetric, \( |\phi_{\alpha, \beta}| < \varepsilon \) on \( W_0 \).

It is now a routine argument to show \( (\phi_{\alpha, \beta}) \overset{\tau}{\to} \phi \). Indeed, let \( W \) be an arbitrary weakly compact subset of \( B_X \). Choose \( n_0 \) such that \( \frac{1}{n_0} < \varepsilon \), let \( y = x_{n_0} - x_0 \) and define the projection \( P \) by \( Px = \phi(x)y \). Now \( W' = (I - P)(W) \) is a weakly compact subset of \( L \), so we choose \((\tilde{\alpha}, \tilde{\beta})\) with \( \tilde{\beta} \) having integer coordinate \( \tilde{n} \) such that \( \tilde{n} \geq n_0 \) and
\[
\sup_{W'} |\phi_{\alpha, \beta}| \leq \frac{\varepsilon}{2} \text{ for all } (\alpha, \beta) \geq (\tilde{\alpha}, \tilde{\beta}). \tag{3.4}
\]
If \( \alpha \not\geq \alpha(\beta) \), then \( \phi_{\alpha, \beta} = \phi \); if \( \alpha \geq \alpha(\beta) \), then (3.3) shows \( \phi_{\alpha, \beta}(y) \geq 1 - \frac{2}{n_0} \). Combining this with \( \|y\| \leq 1 + \frac{1}{n_0} \) and \( \frac{1}{n_0} < \varepsilon \), we see that
\[
|\phi_{\alpha, \beta}(y) - \phi(y)| < \frac{\varepsilon}{2} \text{ for all } (\alpha, \beta) \geq (\tilde{\alpha}, \tilde{\beta}). \tag{3.5}
\]
For \( w \in W \), one has \((I - P)(w) \in W'\), hence for \((\alpha, \beta) \geq (\tilde{\alpha}, \tilde{\beta})\), (3.4) and (3.5) show:
\[
|\phi_{\alpha, \beta}(w) - \phi(w)| \leq |\phi_{\alpha, \beta} - \phi)(w - Pw)| + |(\phi_{\alpha, \beta} - \phi)(Pw)|
= |\phi_{\alpha, \beta}(w - Pw)| + |(\phi_{\alpha, \beta} - \phi)(\phi(w)y)|
< \frac{\varepsilon}{2} + |\phi(w)||\phi_{\alpha, \beta} - \phi)(y)| < \varepsilon.
\]
Thus \( (\phi_{\alpha, \beta}) \overset{\tau}{\to} \phi \) as promised.

To prove the inequality on the limsups, fix \( k_0 \) such that \( \frac{3}{k_0} < \varepsilon \). Using \( w^*\)-convergence, choose \( \alpha_0 \) and \( \beta_0 = (W_0, n_0) \) with \( n_0 \geq k_0 \) such that
\[
|\phi_{\alpha, \beta}(x_{k_0}) - \phi(x_{k_0})| < \frac{1}{k_0} \text{ for all } (\alpha, \beta) \geq (\alpha_0, \beta_0).
\]
According to (3.3) for \( \beta \geq \beta_0 \) and \( \alpha \geq \alpha(\beta) \), one has
\[
\sup_{C_\alpha} \phi_{\alpha, \beta} \leq \phi_{\alpha, \beta}(x_{k_0}) - (1 - \frac{2}{k_0}) \leq \phi(x_{k_0}) + \frac{3}{k_0} - 1 < \phi(x_0) + \varepsilon.
\]

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It follows that \( \limsup_{\beta} (\sup_{\alpha} \phi_{\alpha, \beta}) \leq \sup_{C} \phi \).

\( \Leftarrow \): Again, we only prove the weak compact gap case, as the other proofs are almost identical. We proceed as in [13, Proposition 1.2]. Let \( W \) be a weakly compact set, according to (i), \( \limsup_{\alpha} d(W, C_{\alpha}) \leq d(W, C) \). Let \( \epsilon > 0 \) and let \( r = d(W, C) - 2\epsilon \). We wish to show \( \liminf_{\alpha} d(W, C_{\alpha}) \geq r \) so we may assume \( r > 0 \). Combining the separation theorem with a general form of the Bishop-Phelps theorem ([9, Theorem 2]) shows the existence of \( \phi \in S_X \) such that \( \phi \) attains its supremum on \( C \) and

\[
\sup_{C} \phi + (r + \epsilon) \leq \inf_{W} \phi.
\]

Applying (ii), we find \( \langle \phi_{\alpha, \beta} \rangle \subset S_X \) such that \( \phi_{\alpha, \beta} \xrightarrow{\tau} \phi \) and we fix \( (\alpha_0, \beta_0) \) such that:

\[
\inf_{W} \phi_{\alpha, \beta} \geq \inf_{W} \phi - \frac{\epsilon}{2} \quad \text{for} \quad (\alpha, \beta) \geq (\alpha_0, \beta_0), \quad \text{and}
\]

\[
\limsup_{\alpha} \sup_{C_{\alpha}} \phi_{\alpha, \beta} \leq \sup_{C} \phi + \frac{\epsilon}{2}.
\]

Thus for \( \alpha \geq \alpha_0 \):

\[
d(W, C_{\alpha}) \geq \inf_{W} \phi_{\alpha, \beta_0} - \sup_{C_{\alpha}} \phi_{\alpha, \beta_0} \geq \inf_{W} \phi - \sup_{C} \phi - \epsilon \geq r.
\]

Therefore \( \liminf_{\alpha} d(W, C_{\alpha}) \geq d(W, C) \) as promised. \( \qed \)

For completeness, let us note some cases when the natural net analog of [2, Theorem 3.1] (i.e. where the index \( \beta \) is suppressed) is valid.

**Theorem 3.5.** Let \( X \) be a Banach space and let \( C_{\alpha}, C \subset C(X) \).

(a) \( \langle C_{\alpha} \rangle \xrightarrow{\alpha} C \) if and only if \( (AB_1) \) and \( (AB_2) \) hold.

(b) If \( X \) is separable, \( \langle C_{\alpha} \rangle \xrightarrow{W} C \) if and only if \( (AB_1) \) and \( (AB_2^*) \) hold.

**Proof.** (a) Since the “if” implication is known ([13, Proposition 2.1]), we outline how the converse follows from the proof of Theorem 3.3 using its notation. Let \( W_{n} = B_{n} \cap L \), then \( \bigcup_{n=1}^{\infty} W_{n} = L \). Using slice convergence for each \( n \) we fix \( \alpha(n) \geq \alpha(n - 1) \) such that

\[
\{x_1\} + W_{n} + B_{1 - \frac{1}{n}} \cap C_{\alpha} = \emptyset \quad \text{for} \quad \alpha \geq \alpha(n).
\]

If \( \alpha \geq \alpha(n) \) and \( \alpha \nless \alpha(n + 1) \), fix \( \phi_{\alpha} \in S_{X} \) such that \( \phi_{\alpha} \) separates \( \{x_1\} + W_{n} + B_{1 - \frac{1}{n}} \) and \( C_{\alpha} \). For \( \alpha \nless \alpha(1) \) we let \( \phi_{\alpha} = \phi \). If there are no \( \alpha \) such that \( \alpha \geq \alpha(n) \) for all \( n \) one
can proceed as in the proof of Theorem 3.3 to establish \((AB_2)\). In the other case observe that if \(\alpha \geq \alpha(n)\) for all \(n\), then \(d(C_\alpha, H) \geq 1\) and so \(C_\alpha \subset \{ x : \phi(x) \leq \phi(x_0) \}\). Thus it is obvious the net \(\langle \phi_\alpha \rangle\) where \(\phi_\alpha = \phi\) for all \(\alpha\) satisfies \((AB_2)\).

(b) The Wijsman case follows along the same lines by letting \(W_1 \subset W_2 \subset \ldots\) be compact sets such that \(\bigcup_{n=1}^{\infty} W_n\) is dense in \(L\); see also [13, Theorem 2.1] where the sequential case was shown.

Observe that Theorems 3.3 and 3.5 give insight as to why a norm whose dual is sequentially \(\tau\)-Kadec does not necessarily imply that weak compact gap and slice convergence coincide for sequences of closed convex sets although the corresponding result is valid if the dual norm is \(\tau\)-Kadec. Indeed, in such cases Mackey convergence cannot be determined by a prescribed countable collection of weakly compact sets and so the second index in Theorem 3.3(ii) must be uncountable. On the other hand, in separable spaces, \(w^*\)-convergence is determined by a fixed countable dense set (or a sequence of compact sets whose union is dense) and so a diagonalization argument was used to suppress the bornology-based index in Theorem 3.5(b). However, questions which arose in [13] are still open: is there a norm whose dual is sequentially \(w^*\)-Kadec, but not \(w^*\)-Kadec and if so, does such a norm ensure that Wijsman and slice convergence coincide for sequences of closed convex sets?

Recall that a space is said to have the Schar property if weakly convergent sequences are norm convergent. We conclude this section with some observations on another intermediate form of convergence. Suppose \(C_\alpha, C \in C(X)\) with \(C \neq \emptyset\). Let us say \(\langle C_\alpha \rangle\) converges \(T\) to \(C\) (written \(\langle C_\alpha \rangle \xrightarrow{T} C\)) if \((AB_1)\) and \((AB_2)\) are satisfied. Then this form of convergence is properly stronger than weak compact gap convergence in many spaces (Proposition 3.2) even though its representation is rather similar to weak compact gap convergence (Theorem 3.3). We now record a couple of its properties.

Proposition 3.6. (a) If \(C_n, C \in C(X)\) with \(C \neq \emptyset\) and if \(X \not\cong \ell_1\), then \(\langle C_n \rangle \xrightarrow{\sigma} C\) provided \(\langle C_n \rangle \xrightarrow{T} C\).

(b) If \(X\) is separable and has the Schar property and if \(C_\alpha, C \in C(X)\) with \(C \neq \emptyset\), then \(C_\alpha \xrightarrow{T} C\) if and only if \(\langle C_\alpha \rangle \xrightarrow{W} C\).

Proof. Theorem 3.1(a) and [10, Theorem 5] show (a) is true while (b) follows from Theorem 3.5(b).
In particular, for $X$ nonreflexive and $X^* \not\cong \ell_1$, $T$-convergence is properly stronger than $w^*$-slice convergence for sets in $C^*(X^*)$. On the other hand, $T$-convergence is properly weaker than $w^*$-slice convergence in $\ell_1$ because one can construct Wijsman convergent sequences in $\ell_1$ which don’t converge $w^*$-slice. Thus, while $T$-convergence has a nice representation, it does not appear to correspond to an easily verified form of gap convergence.

4. Convergence of nonconvex sets and bornologies. We now show that if one considers nonconvex sets, then the relationships between various types of convergence depend only on the bornologies and not the norm under consideration. In particular, this section shows that Corollary 3.4 and [13, Theorems 2.1 and 2.3] fail for nonconvex sets. First we provide an explicit example which illustrates the essential techniques.

Example 4.1. Consider $F = \{x \in \ell_2 : x(1) = 1\}$ and $F_n = F \cup \{e_n\}$. Then $F_n, F$ are weakly closed and $\langle F_n \rangle \overset{W}{\longrightarrow} F$, but $\langle F_n \rangle \not\overset{f^*}{\longrightarrow} F$.

Proof. Observe $\langle F_n \rangle \overset{f^*}{\longrightarrow} F$ since for $C = \{x \in B_{\ell_2} : x(1) = 0\}$, we have $\inf\{d(C,F) = 1$ while $\inf\{d(C,F_n) = 0$ for $n \geq 2$. On the other hand, for any $x \in \ell_2$, $\liminf\|x - e_n\| \geq 1$ and thus $\limsup_{n \to \infty} d(x,F_n) = \limsup_{n \to \infty} \{\min d(x,F), 1\} = d(x,F)$. Since $F \subset F_n$ for each $n$, this shows Wijsman convergence.

We now generalize the above example. In what follows, $BW(X)$ denotes the bounded weakly closed subsets of $X$.

Theorem 4.2. (a) $X$ is finite dimensional if and only if every Wijsman convergent sequence in $BW(X)$ is slice convergent.

(b) $X$ is reflexive if and only if every weak compact gap convergent sequence in $BW(X)$ is slice convergent.

(c) $X$ has the Schur property if and only if every Wijsman convergent sequence in $BW(X)$ is weak compact gap convergent.

Proof. Since the “only if” statements are immediate from the definitions, we will only prove the converse implications. We prove (b) first and then outline how to obtain the other statements. Suppose $X$ is not reflexive, fix $\phi \in S_{X^*}$ and let $Y = \phi^{-1}(0)$. Then $Y$ is not reflexive and thus by the Eberlein-Šmulian theorem there is a sequence $\langle y_k \rangle \subset Y \cap B_X$
such that $\langle y_k \rangle$ has no weakly convergent subsequence. According to [11, Lemma 2.2] there is a subsequence $\langle y_{k_n} \rangle$ and $0 < \epsilon < 1$ such that if $z_n \in X$ satisfies $\|z_n - y_{k_n}\| < \epsilon$, then $\langle z_n \rangle$ has no weakly convergent subsequence. Now let $x_n = y_{k_n}$ and define

$$F = \{ x : |\phi(x)| \geq \frac{\epsilon}{4} \cap 4B_X \} \quad \text{and} \quad F_n = F \cup \{ x_n \}.$$ 

Let $C = B_X \cap Y$, then $d(C, F_n) = 0$ for all $n$, while $d(C, F) = \frac{\epsilon}{4}$. Thus $F_n \not\rightarrow F$, however, we now show $F_n \not\rightarrow W$. Indeed, let $W$ be an arbitrary weakly compact subset of $X$. If $W \cap 3B_X \neq \emptyset$, then $d(W, F) \leq \frac{\epsilon}{4}$. On the other hand, by the construction of the sequence $\langle x_n \rangle$ and the weak compactness of $W$, there is an $n_0$ such that

$$d(W, x_n) \geq \epsilon \quad \text{for all} \quad n \geq n_0.$$

Therefore $d(W, F_n) = \min\{d(W, F), \epsilon\} = d(W, F)$ for $n \geq n_0$ in the case $W \cap 3B_X \neq \emptyset$. If $W \cap 3B_X = \emptyset$, then it is clear $d(W, x_n) \geq d(W, F)$ for all $n$ and hence in this case $d(W, F) = d(W, F_n)$ for all $n$. This completes the proof of (b).

(a) One can argue as in (b) since if $X$ is not finite dimensional, there are a sequence $\langle x_n \rangle \subset Y \cap B_X$ and $\epsilon > 0$ such that $\|x_n - x_m\| > \epsilon$ for $m \neq n$.

(c) In space failing the Schur property, we can find $\langle x_n \rangle \subset Y \cap B_X$ and $\epsilon > 0$ such that $\|x_n - x_m\| > \epsilon$ for $m \neq n$ but $\langle x_n \rangle \not\rightarrow W$, thus we can proceed as in (b). \hfill \Box

So far, we have made no mention of the powerful Attouch-Wets convergence: $\langle A_n \rangle$ is said to converge Attouch-Wets to $A$ (written $\langle A_n \rangle \rightarrow_{AW} A$), if $d(\cdot, A_n)$ converges to $d(\cdot, A)$ uniformly on bounded sets. Let us also say $A_n$ converges nonconvex bounded gap to $A$ (written $\langle A_n \rangle \rightarrow_{nbg} A$) if $d(B, A_n) \rightarrow d(B, A)$ for each bounded set $B$. We close by recording an observation that shows these notions are significantly stronger than slice convergence—which is the main reason they did not enter our discussion heretofore. It is clear $\langle A_n \rangle \rightarrow_s A$ if $\langle A_n \rangle \rightarrow_{nbg} A$ and one can check $\langle A_n \rangle \rightarrow_{nbg} A$ if $\langle A_n \rangle \rightarrow_{AW} A$. Moreover:

**Fact 4.3.** (a) $X$ is finite dimensional if and only if $\langle C_n \rangle \rightarrow_{AW} C$ whenever $C_n, C \in C(X)$ and $\langle C_n \rangle \rightarrow_{nbg} C$.

(b) If $X$ does not have the Schur property, then there exist $C_n, C \in C(X)$ such that $\langle C_n \rangle \rightarrow_s C$ but $\langle C_n \rangle \not\rightarrow_{nbg} C$. 

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Proof. The proof of (a) is implicitly contained in [8, Lemma 2.4]. To show (b), let \( \langle x_n \rangle \subset S_X \) be such that \( \langle x_n \rangle \stackrel{w}{\rightharpoonup} \emptyset \). Let \( C_n = \text{conv}\{0, x_n\} \) and let \( C = \{0\} \). To establish slice convergence one can directly use a separation argument or observe that (\( AB_2 \)) is satisfied for any \( \phi \in S_X \) by letting \( \phi_n = \phi \) for all \( n \). On the other hand, \( d(S_X, C_n) = 0 \) for each \( n \) while \( d(S_X, C) = 1 \) and thus \( \langle C_n \rangle \rightharpoonup C \).

\[ \square \]

Note the same argument as in (b) shows that one can construct a net of closed convex sets which converges slice but not nonconvex bounded gap to \( \{0\} \) in any infinite dimensional space.

References


