Local Lipschitz-constant Functions and Maximal Subdifferentials

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\textbf{ABSTRACT.} It is shown that if $k(x)$ is an upper semicontinuous and quasi lower semi-continuous function on a Banach space $X$, then $k(x)B_X$ is the Clarke subdifferential of some locally Lipschitz function on $X$. Related results for approximate subdifferentials are also given. Moreover, on smooth Banach spaces, for every locally Lipschitz function with minimal Clarke subdifferential, one can obtain a maximal Clarke subdifferential map via its 'local Lipschitz-constant' function. Finally, some results concerning the characterization and calculus of local Lipschitz-constant functions are developed.

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1 Introduction

Throughout this paper, we will let $X$ be a Banach space with topological dual $X^*$ and dual unit ball $B_{X^*}$. In [6] it was shown that, in the category sense, most nonexpansive Lipschitz functions have Clarke subdifferentials identically equal to the dual unit ball. While this result shows that the Clarke subdifferential fails to distinguish most nonexpansive Lipschitz functions from each other, it also illustrates the importance of additional assumptions in establishing integration results for locally Lipschitz functions via Clarke subdifferentials. In this paper, we will investigate the following question which is a natural extension to the result just quoted from [6].

For which nonnegative functions $k(x)$ on a Banach space $X$, is there a locally Lipschitz function whose Clarke subdifferential is equal to $k(x)B_{X^*}$ at each $x \in X$?

Our detailed objectives are two fold. First, we extend and simplify the main results in [6] by showing that on general Banach spaces $T = k(x)B_{X^*}$ is always a Clarke subdifferential map of some locally Lipschitz function whenever $k$ is usc and quasi lsc (Theorems 4.5,5.4). Second, we connect the Clarke subdifferential more closely to the local Lipschitz-constant function, that is, on smooth Banach spaces for every local Lipschitz function with minimal Clarke subdifferential one may find a maximal Clarke subdifferential map via its local Lipschitz-constant function (Proposition 7.10). Inter alia, we examine which functions may arise as local Lipschitz-constants.

We review some notions and definitions requisite for our work:

Notations: In the Banach space $X$, we let

$$B_X := \{ x \in X : \|x\| \leq 1 \}, \quad B_\delta(x_0) := x_0 + \delta B_X, \quad B_\delta(x_0) := \{ x \in X : \|x - x_0\| < \delta \}. $$

For any set $D \subset X$ we write $k|_D$ for the restriction of $k$ to $D$. The distance function $d_D$ is given by

$$d_D(x) := \inf\{\|x - y\| : y \in D\}$$

for every $x \in X$. For a subset $C \subset X^*$, we let $\overline{C}^w$ ($\overline{C}^{w^*}$) denote the weak* closure (weak* closed convex hull) of $C$. When $C$ is weak* compact, for every $x \in X$ we write

$$\sigma_C(x) := \max\{ (x,x^*) : x^* \in C \}. $$

The dual sphere is denoted by $S_{X^*}$. A Borel set $N \subset X$ is called a Haar null set if there is a probability Radon measure $\mu$ on $X$ such that $\mu(N + x) = 0$ for all $x \in X$. A possible non-Borel set is Haar null if it is contained in a Borel Haar null set. Every Haar null subset in $X$ has empty interior; see for example [5], see also Section 6.1 of [1] for this and further background information on measures in separable Banach spaces. In any finite dimensional space, the Haar null sets are precisely the Lebesgue null sets.

Robust and topologically robust upper semicontinuity: The following two stronger notions of upper semicontinuity will play a crucial role in our study of Clarke and approximate subdifferentials. An upper semicontinuous function $k : X \to \mathbb{R}$ is called topologically robust upper semicontinuous on $X$ if $k(x) = \limsup_{y \to x, y \notin N} k(y)$ for every $x \in X$, where $D$ is the set of points at which $k$ is continuous. As in [3], we shall say $k : X \to \mathbb{R}$ is robust upper semicontinuous on $X$ if for every Haar null set $N \subset X$ we have

$$k(x) = \limsup_{y \to x, y \notin N} k(y) \quad \text{for every } x \in X.$$
In [3], the notion of robust upper semicontinuity was used to characterize Clarke subdifferentials of locally Lipschitz functions on the real line. The stronger notion of topologically robust upper semicontinuity will be useful in our constructions of functions with large Clarke and approximate subdifferentials on Banach spaces.

**Upper envelope (lower envelope) of a function:** For a real-valued function \( k : X \to \mathbb{R} \), for each \( x \) we write

\[
m_\delta(x) := \inf\{k(y) : \|y - x\| \leq \delta\}, \quad M_\delta(x) := \sup\{k(y) : \|y - x\| \leq \delta\},
\]

and define

\[
lsc(k)(x) := \lim_{\delta \to 0} m_\delta(x), \quad usc(k)(x) := \lim_{\delta \to 0} M_\delta(x).
\]

The functions \( lsc(k) \) and \( usc(k) \) are called lower and upper envelopes of \( k \). Moreover, \( lsc(k) \) is lower semicontinuous, \( usc(k) \) is upper semicontinuous on \( X \), and \( lsc(k) \leq k \leq usc(k) \). The function \( k \) is lsc (resp. usc) if and only if \( lsc(k) = k \) (resp. \( usc(k) = k \)).

**Usco hull (cusco hull) of a set-valued map:** Let \( T \) be a set-valued mapping from a topological space \( A \) into the dual of a normed linear space \( X \). We say that \( T \) is weak* upper semi continuous on \( A \) if for each weak* open subset \( W \) of \( X^* \), \( \{x \in A : T(x) \subseteq W\} \) is open in \( A \). When the images of \( T \) are non-empty and compact we call \( T \) a weak* usco and if, in addition, the images of \( T \) are also convex then we call \( T \) a weak* cusco. We call \( T \) a minimal weak* usco (cusco) if its graph does not properly contain the graph of any other weak* usco (cusco) on \( A \). By the graph of \( T \) we mean the set \( \text{Gr}(T) := \{(x, x^*) : x^* \in T(x)\} \), which is closed whenever \( T \) is an usco. Let \( T \) be a densely defined set-valued mapping that maps a topological space \( A \) into \( X^* \). If \( T \) is locally bounded on \( A \) then there exists a unique smallest weak* usco (weak* cusco) containing \( T \) [2], denoted \( \text{USC}(T) \) (\( \text{CSC}(T) \)) and given by:

\[
\text{USC}(T)(x) := \bigcap \{\overline{T(V)}^{w^*} : V \text{ is an open neighborhood of } x\},
\]

\[
\text{CSC}(T)(x) := \bigcap \{\overline{\text{co}w^*T(V)} : V \text{ is an open neighborhood of } x\} = \overline{\text{co}w^*[\text{USC}(T)]}(x). \tag{1}
\]

**Subderivatives and subdifferentials:** Let \( A \) be a non-empty open subset of a Banach space \( X \) and let \( f : A \to \mathbb{R} \) be a locally Lipschitz function. The Clarke derivative and subdifferential of \( f \) at \( x \in A \) ([7]) are given by:

\[
f^0(x; v) := \limsup_{t \downarrow 0, y \to x} \frac{f(y + tv) - f(y)}{t} \quad \text{and},
\]

\[
\partial_c f(x) := \{x^* \in X^* : x^*(v) \leq f^0(x; v) \text{ for all } v \in X\}.
\]

The upper and lower Dini-derivatives of \( f \) at \( x \) are given by:

\[
f^+(x; v) := \limsup_{t \downarrow 0} \frac{f(x + tv) - f(x)}{t} \quad \text{and} \quad f^-(x; v) := \liminf_{t \downarrow 0} \frac{f(x + tv) - f(x)}{t}.
\]

The Dini subdifferential of \( f \) at \( x \) is defined as:

\[
\partial_- f(x) := \{x^* \in X^* : x^*(v) \leq f^-(x; v) \text{ for all } v \in X\}.
\]

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When $X$ is a Gâteaux smooth Banach space (i.e., has an equivalent Gâteaux differentiable renorm), the domain of $\partial f$ is dense in $A$ [14]. The approximate (or limiting) subdifferential, which is defined on arbitrary Banach spaces [11], can be given by

$$\partial \alpha f(x) := \text{USC}(\partial f)(x),$$

for locally Lipschitz functions $f$ defined on smooth Banach spaces [4]. Moreover, $\partial \alpha f(x) = \text{co}^{\ast} \partial \alpha f(x)$ for $x \in A$. While $\partial \alpha f$ is a weak* usco on $A$, the approximate subdifferential $\partial \alpha f$ is a weak* usco on $A$ [12, 11]. By saying that a locally Lipschitz function $f$ has a minimal Clarke's subdifferential (resp. approximate subdifferential), we mean $\partial \alpha f$ (resp. $\partial \alpha f$) is a minimal weak* usco (resp. usco) on $A$.

**Setup:** As in [6], we will be working in the space of locally Lipschitz functions with controlled local Lipschitz constant or with subdifferential maps controlled by a cusco map. More precisely, for a cusco map $T : X \to 2^X$, the complete metric space $(\mathcal{X}_T, \rho)$ is defined as

$$\mathcal{X}_T := \{ f \in \mathbb{R}^X : f \text{ is locally Lipschitz and } \partial c f(x) \subseteq T(x) \text{ for every } x \in X \},$$

with metric $\rho(f, g) := \min\{ \sup_{x \in X} |f(x) - g(x)|, 1 \}$.

The structure of the paper is as follows. In section 2, we lay out some basic properties of robust and topologically robust usc functions that will be used in the sequel. In section 3 we give a transparent proof to show that $kB_{X^\ast}$ is a Clarke subdifferential map when $k$ is a bounded and Lipschitz function on a Banach space $X$. Although more general results are presented later, we have chosen to included important special case in section 3 because it highlights many of the underlying ideas while it doesn’t rely results from [6]. Our main results begin in Section 4 where it is shown that on separable Banach spaces when $k$ is a topologically robust usc function, the set-valued map $kB_{X^\ast}$ is not only a Clarke subdifferential but also an approximate subdifferential. Then in section 5 we extend the results of sections 3 and 4 to our most general case where $k$ is a topologically robust usc function on a general Banach space. As a further application, in section 6 we show that the sum of the dual ball and the cusco generated by a countable family of minimal Clarke subdifferential maps is still a Clarke subdifferential map. Properties of local Lipschitz-constant functions are gathered up in section 7. In particular, on Gâteaux smoothable Banach spaces we observe that a locally Lipschitz function has a topologically robust usc local Lipschitz-constant provided its Clarke subdifferential map is minimal. Finally, in section 8 we show that one can always find a maximal $\beta$-subgradient whenever the lower Dini directional derivative equals the local Lipschitz-constant in some direction.

## 2 Properties of robust or topologically robust usc functions

Assume that $X$ is a Banach space. Our first result connects topologically robust properties to weakened lower semicontinuity properties. For this, recall that $k : X \to \mathbb{R}$ is called quasi (pseudo) lower semicontinuous on a Banach space $X$ if for each $x \in X$, $\varepsilon > 0$ and open neighborhood $U$ of $x$ there exists a non-empty open (non-Haar null) subset $V \subset U$ such that $k(y) > k(x) - \varepsilon$ for all $y \in V$.

**Proposition 2.1.** For a real-valued function $k : X \to \mathbb{R}$, we have:

(i) $k$ is topologically robust usc if and only if $k$ is usc and quasi lsc.
(ii) $k$ is topologically robust usc if and only if $k = \text{usc}(\text{lsc}(k))$.

(iii) $k$ is robust usc if and only if $k$ is usc and pseudo lsc. In particular, if $k$ is topologically robust usc, then $k$ is robust usc.

**Proof.** (i) Suppose $k$ is topologically robust usc. Then for every neighborhood $U$ of $x$ and $\epsilon > 0$, there exists $y \in U$ such that $k$ is continuous at $y$ and $k(y) > k(x) - \epsilon$. Because $k$ is continuous at $y$, there exists a neighborhood $V \subset U$ of $y$ such that $k(z) > k(x) - \epsilon$ for every $z \in V$; thus $k$ is quasi lsc at $x$.

Conversely, assume that $k$ is usc and quasi lsc on $X$. For every neighborhood $U$ of $x$ and $\epsilon > 0$, there exists $V \subset U$ such that $k(z) > k(x) - \epsilon$ for every $z \in V$. Since $k$ is usc, there exists $y \in V$ such that $k$ is continuous at $y$ [10, page 109]. Thus

$$k(x) \leq \limsup_{y \to x, y \in D} k(y) \leq \limsup_{y \to x} k(y) \leq k(x),$$

where $D$ is the set of points at which $k$ is continuous, so $k$ is topologically robust usc.

(ii) Assume $k = \text{usc}(g)$ with $g = \text{lsc}(k)$. Fix $x$ and $\epsilon > 0$. For every neighborhood $W$ of $x$, there exists $y \in W$ such that $g(y) > \text{usc}(g)(x) - \epsilon$. As $g$ is lsc, there exists a neighborhood $U \subset W$ of $y$ such that $g(z) > \text{usc}(g)(x) - \epsilon$ for every $z \in U$. Then

$$\text{usc}(g)(z) \geq g(z) > \text{usc}(g)(x) - \epsilon$$

for every $z \in U$.

Hence $\text{usc}(g)$ is quasi lsc, and so it is topologically robust usc by (i).

Conversely, assume that $k$ is topologically robust usc. We let $g = \text{lsc}(k)$. Whenever $k$ is continuous at $x$, we have $\text{lsc}(k)(x) = k(x)$, thus $\text{lsc}(k)|_D = k|_D$. Then

$$k = \text{usc}(k) \geq \text{usc}(\text{lsc}(k)) \geq \text{usc}(k|_D) = k,$$

as required, where the last equation follows because $k$ is topologically robust usc.

(iii) If $k$ is robust usc, for every open neighborhood $U$ of $x$ and $\epsilon > 0$, the set

$$U \cap \{y : k(y) > k(x) - \epsilon\}$$

is not Haar null.

If this is not the case for some $U$ and $\epsilon > 0$, then we let $N := U \cap \{y : k(y) > k(x) - \epsilon\}$ to get

$$k(x) = \limsup_{y \to x, y \notin N} k(y) \leq k(x) - \epsilon,$$

a contradiction.

Conversely, assume that $k$ is usc and pseudo lsc. Fix a Haar null subset $N \subset X$ and $\epsilon > 0$. Since for every open neighborhood $U$ of $x$, the set $U \cap \{y : k(y) > k(x) - \epsilon\}$ is not Haar null, we have $[U \cap \{y : k(y) > k(x) - \epsilon\}] \setminus N \neq \emptyset$, so

$$k(x) - \epsilon \leq \limsup_{y \to x, y \notin N} k(y) \leq \limsup_{y \to x} k(y) \leq k(x).$$

As $\epsilon$ is arbitrary, we have $k(x) = \limsup_{y \to x, y \notin N} k(y)$.
We next observe some stability properties of (topologically) robust usc functions.

**Proposition 2.2.** Consider any collection \( \{k_i : i \in I\} \) of equi-locally bounded functions \( k_i : X \to \mathbb{R} \), and let \( k := \text{usc}(\sup_i k_i) \). If each \( k_i \) is topologically robust usc (resp. robust usc), then \( k \) is topologically robust usc (resp. robust usc).

**Proof.** Let \( \epsilon > 0 \) and let \( N(x) \) be an open neighborhood of \( x \). Then there exists \( z \in N(x) \) with \( \sup_i k_i(z) > k(x) - \epsilon \), so we may choose an \( i \) such that \( k_i(z) > k(x) - \epsilon \). In the topologically robust usc case, \( k_i \) is quasi lsc. Hence, there exists an open set \( U \subset N(x) \) such that \( k_i(y) > k_i(z) - \epsilon \) for \( y \in U \). For \( y \in U \) we have

\[
k(y) = \text{usc}(\sup_i k_i)(y) \geq (\sup_i k_i)(y) > k_i(z) - \epsilon > k(x) - 2\epsilon.
\]

This implies \( k \) is quasi lsc at \( x \), and so \( k \) is topologically robust usc by Proposition 2.1.

In the robust usc case, let \( N \) be a Haar null set. Because \( k_i \) is robust usc, there exists \( y \in N(x) \setminus N \) such that \( k_i(y) > k_i(z) - \epsilon \), then, as above, (2) holds for this \( y \). Consequently, \( \limsup_{y \to x, y \notin N} k(y) \geq k(x) \).

Because \( k \) is usc, we have

\[
\limsup_{y \to x, y \notin N} k(y) \leq \limsup_{y \to x} k(y) = k(x).
\]

Hence \( k \) is robust usc. \( \square \)

In particular, the maximum of a finite number of (topologically) robust usc functions is (topologically) robust usc. However, these classes are not closed under summation or minimum operations (see Example 7.5).

## 3 Maximal Clarke subdifferentials in Banach spaces

Let \( X \) be a Banach space. Let us write

\[
\mathcal{X}_l(x)_{B_X} := \{f : X \to \mathbb{R} : \partial_c f(x) \subset l(x)B_{X}\text{ for all } x \in X, \text{ and } f \text{ is bounded on } X\},
\]

with metric \( \rho(f,g) := \sup_{x \in X} |f(x) - g(x)| \). In this section, our goal is to give a relatively simple and direct proof of an extension of Corollary 9 in [6]; in later sections we will extend this result further in a less transparent manner by building on results and techniques from [6]. For a Banach space \( X \), we call a set \( S \subset X \) an \( \epsilon \)-net if \( (a) \|x - y\| \geq \epsilon \) for any two distinct points \( x, y \in S \) and \( (b) S \) is maximal with respect to \( (a) \). Zorn’s lemma yields that \( \epsilon \)-nets exist for every \( \epsilon > 0 \).

**Theorem 3.1.** Suppose \( l : X \to [0,\infty) \) is Lipschitz and bounded. Then \( \{f \in \mathcal{X}_l(x)_{B_X} : \partial_c f(x) = l(x)B_{X}\} \) is residual in \( (\mathcal{X}_l(x)_{B_X}, \rho) \).

**Proof.** Let \( \{x^\alpha_n : \alpha \in \Gamma_n\} \) be a \( \frac{1}{n} \)-net in \( \{x : l(x) \geq \frac{1}{n}\} \). Define

\[
O_{n,k} := \left\{ f \in \mathcal{X}_l(x)_{B_X} : \inf_{\alpha \in \Gamma_n, v \in S_X} \left( \frac{f(x^\alpha_n + tkv) - f(x^\alpha_n)}{tk} - l(x^\alpha_n) \right) > -\frac{1}{k} \text{ for some } 0 < t_k < \frac{1}{k} \right\}.
\]

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We first show that \( O_{n,k} \) is open. Let \( f_0 \in O_{n,k} \). Then for some \( 0 < t_k < \frac{1}{k} \) that depends on \( f_0 \) and \( k \), we have
\[
\inf_{\alpha \in \Gamma_n, v \in S_X} \left( \frac{f_0(x_n^\alpha + t_k v) - f_0(x_n^\alpha)}{t_k} - l(x_n^\alpha) \right) > -\frac{1}{k}.
\] (3)

If \( \rho(f, f_0) < \epsilon \), we have
\[
\inf_{\alpha \in \Gamma_n, v \in S_X} \left( \frac{f(x_n^\alpha + t_k v) - f(x_n^\alpha)}{t_k} - l(x_n^\alpha) \right) \geq \inf_{\alpha \in \Gamma_n, v \in S_X} \left( \frac{(f - f_0)(x_n^\alpha + t_k v) - (f - f_0)(x_n^\alpha)}{t_k} + \inf_{\alpha \in \Gamma_n, v \in S_X} \left( \frac{f_0(x_n^\alpha + t_k v) - f_0(x_n^\alpha)}{t_k} - l(x_n^\alpha) \right) \right) \geq \frac{-2\epsilon}{t_k} + \inf_{\alpha \in \Gamma_n, v \in S_X} \left( \frac{f_0(x_n^\alpha + t_k v) - f_0(x_n^\alpha)}{t_k} - l(x_n^\alpha) \right).
\] (4)

By (3), we can choose \( \epsilon \) sufficiently small such that the expression in (4) is larger than \(-\frac{1}{k}\), thus \( B_k(f_0) \subset O_{n,k} \), and \( O_{n,k} \) is open.

We next show that \( O_{n,k} \) is dense. Fix \( M > 1 \) such that \( l \) is \( M \)-Lipschitz and bounded by \( M \). Fix \( f \in \mathcal{A}(x)B_X \), and \( 0 < \epsilon < 1 \). Let \( f_1 := (1 - \frac{\epsilon}{3N})f \) where \( N \) is chosen so that \( \|f\|_\infty < N \). We have
\[
\rho(f_1, f) = \sup_{x \in X} |f - (1 - \frac{\epsilon}{3N})f| = \frac{\epsilon}{3N} \|f\|_\infty < \frac{\epsilon}{3}.
\] (5)

Because \( l \) is \( M \)-Lipschitz and \( l(x_n^\alpha) \geq \frac{1}{n} \) for all \( \alpha \in \Gamma_n \), we can find \( \delta > 0 \) and \( \epsilon' > 0 \) so that
\[
(1 - \frac{\epsilon}{3N})l(x) \leq l(x_n^\alpha) - 2\epsilon' \leq l(x_n^\alpha) - \epsilon' \leq l(x) \quad \text{whenever} \quad \|x - x_n^\alpha\| \leq \delta.
\] (6)

By replacing \( \epsilon' \) and \( \delta \) with smaller numbers as necessary, we may and do assume that \( 0 < \epsilon' < \min\{\frac{1}{2}, \frac{\epsilon}{3M} \} \) and \( 0 < \delta < \min\{\frac{\epsilon}{6M}, \frac{1}{n}\} \). Now let \( f_{n,\alpha}(x) = f_1(x_n^\alpha) - \delta \epsilon' + (l(x_n^\alpha) - \epsilon')\|x - x_n^\alpha\| \). Because \( l(x) \) is bounded by \( M \), it follows that \( f_{n,\alpha} \) and \( f_1 \) are \( M \)-Lipschitz, and so
\[
f_{n,\alpha}(x) \leq f_1(x) \quad \text{if} \quad \|x - x_n^\alpha\| \leq \frac{\delta \epsilon'}{2M}, \quad \text{and} \quad (7)
\]
\[
|f_{n,\alpha}(x) - f_1(x)| \leq \epsilon' + \frac{\epsilon}{3} \quad \text{if} \quad \|x - x_n^\alpha\| \leq \delta.
\] (8)

Let \( g \) be defined by
\[
g(x) := \begin{cases} 
\min\{f_{n,\alpha}(x), f_1(x)\} & \text{if} \quad \|x - x_n^\alpha\| \leq \delta \\
f_1(x) & \text{otherwise.}
\end{cases}
\]

The definition of \( g \) together with (5) and (8) imply \( \rho(f_1, g) < \frac{\epsilon}{3} \), and ultimately \( \rho(f, g) < \epsilon \).

Now \( \partial_x f_1(x) \subset l(x)B_X \) for all \( x \), and \( \partial_x f_{n,\alpha}(x) \subset l(x)B_X \) for \( \|x - x_{n,\alpha}\| \leq \delta \) (by (6)). Thus it will follow from the definition of \( g \), that \( g \in \mathcal{A}(x)B_X \) as long as \( f_1(x) \leq f_{n,\alpha}(x) \) whenever \( \|x - x_n^\alpha\| = \delta \). Indeed, for \( \|x - x_n^\alpha\| = \delta \), by Lebourg’s mean value theorem [7, page 41] there exists \( \xi \in [x, x_n^\alpha] \) such that
\[
f_1(x) \leq f_1(x_n^\alpha) + (1 - \frac{\epsilon}{3N})l(\xi)\|x - x_n^\alpha\|
\leq f_1(x_n^\alpha) + (l(x_n^\alpha) - 2\epsilon')\delta
= f_1(x_n^\alpha) - \delta \epsilon' + (l(x_n^\alpha) - \epsilon')\|x - x_n^\alpha\| = f_{n,\alpha}(x).
\]

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Therefore, \( g(x) = f_1(x) \) if \( \|x - x^\alpha_n\| = \delta \) and so \( g \in \mathcal{X}_{l(x)}B_X \).

To complete the proof that \( O_{n,k} \) is dense, it remains to show that \( g \in O_{n,k} \). Indeed, we fix \( 0 < t_k < \min\{\delta/2M, 1/k\} \). By (7) we have \( g(x) = f_1(x^\alpha_n) - \delta e' + l(x^\alpha_n) - e'\|x - x^\alpha_n\| \) when \( \|x - x^\alpha_n\| \leq t_k \). Therefore

\[
g(x^\alpha_n + t_kv) - g(x^\alpha_n) = l(x^\alpha_n) - e' > l(x^\alpha_n) - \frac{1}{k}.
\]

Hence \( g \in O_{n,k} \).

To conclude the proof, let \( f \in O_n := \bigcap_{k=1}^{\infty} O_{n,k} \). Then for every \( k \) there exists \( 0 < t_k < 1/k \) such that for every \( \alpha \in \Gamma_n \) and \( \|v\| = 1 \) we have

\[
\frac{f(x^\alpha_n + t_kv) - f(x^\alpha_n)}{t_k} - l(x^\alpha_n) > - \frac{1}{k}.
\]

Letting \( k \to \infty \), we obtain

\[
l(x^\alpha_n) \geq f^0(x^\alpha_n; v) \geq \limsup_{t_k \downarrow 0} \frac{f(x^\alpha_n + t_kv) - f(x^\alpha_n)}{t_k} \geq l(x^\alpha_n).
\]

That is, \( f^0(x^\alpha_n; v) = l(x^\alpha_n) \) for every \( \alpha \in \Gamma_n \) and \( \|v\| = 1 \). If \( f \in G := \bigcap_{n=1}^{\infty} O_n \), then for every \( \alpha \in \Gamma_n \), \( n \in \mathbb{N} \), and \( \|v\| = 1 \) we have \( f^0(x^\alpha_n; v) = l(x^\alpha_n) \). Since \( \{x^\alpha_n : n \in \mathbb{N}, \alpha \in \Gamma_n\} \) is dense in \( \{x : l(x) \neq 0\} \), when \( l(x) \neq 0 \) we have

\[
l(x) \geq f^0(x; v) \geq \limsup_{x^\alpha_n \to x} f^0(x^\alpha_n; v) \geq l(x).
\]

When \( l(x) = 0 \), as \( f \in \mathcal{X}_{l(x)}B_X \), we also have \( f^0(x; v) = 0 = l(x) \). That is, \( f^0(x; v) = l(x) \) for every \( x \) and \( \|v\| = 1 \), dually \( \partial f(x) = l(x)B_X \) for all \( x \in X \).

\[
\square
\]

4 Subdifferentials of the form \( k(x)B_X^* \) on separable spaces

We begin this section with two lemmas that will enable us to give a characterization of the approximate and Clarke subdifferential on separable spaces.

**Lemma 4.1.** Let \( A \) be a non-empty open subset of a separable Banach space \( X \) and \( \Omega : A \to X^* \) be a densely defined locally bounded set-valued mapping. Then there exists a countable set \( C \subset \text{Gr}(\Omega) \) such that \( \text{Gr}(\text{USC}(\Omega)) = \text{cl} C \) where the closure is taken in the product topology of \( X \times X^* \) with \( X^* \) endowed with weak* topology.

**Proof.** Note that every subspace of a separable metric space is separable. Define \( A_m := \{x \in A : \Omega(x) \subset mB_X\} \). As \( B_X \) is weak*-compact and metrizable in the weak*-topology, \( A_m \times mB_X^* \) is separable in the product topology, thus \( \text{Gr}(\Omega) \cap (A_m \times mB_X^*) \) is separable. Then there exists a countable set \( C_m \subset \text{Gr}(\Omega) \cap (A_m \times mB_X^*) \) with \( \text{Gr}(\Omega) \cap (A_m \times mB_X^*) \subseteq \text{cl} C_m \). Let \( C := \bigcup_{m=1}^{\infty} C_m \). Then \( C \) is countable and

\[
\text{Gr}(\Omega) = \text{Gr}(\Omega) \cap \bigcup_{m=1}^{\infty} (A_m \times mB_X^*) = \bigcup_{m=1}^{\infty} \text{Gr}(\Omega) \cap (A_m \times mB_X^*) \subseteq \bigcup_{m=1}^{\infty} \text{cl} C_m \subseteq \text{cl} C.
\]

Then \( \text{Gr}(\text{USC}(\Omega)) = \text{cl} \text{Gr}(\Omega) \subseteq \text{cl} C \subseteq \text{cl} \text{Gr}(\text{USC}(\Omega)) = \text{Gr}(\text{USC}(\Omega)) \). \( \square \)
Lemma 4.2. Assume $F := \{ f_\alpha : \alpha \in I \}$ is a family of equi-locally Lipschitz functions on $A$, in which $I$ may be countable or uncountable. Define $\Omega : A \to 2^{X^*}$ by $\Omega(x) := \bigcup_{\alpha \in I} \partial_{f_\alpha}(x)$ for $x \in A$. Then there is a countable subset $\{ \alpha_i : i = 1, 2, \ldots \}$ of $I$ so that $\text{USC}(\Omega) = \text{USC}(\tilde{\Omega})$ where $\tilde{\Omega}(x) := \bigcup_{i=1}^{\infty} \partial_{f_{\alpha_i}}(x)$ for each $x \in A$.

Proof. By Lemma 4.1 there exists a countable set $C \subseteq \text{Gr}(\Omega)$ such that $\text{Gr}(\text{USC})(\Omega) = \text{cl}C$. Write

\[ C = \{(x_i, x_i^*) : (x_i, x_i^*) \in \text{Gr}(\Omega)\} = \{(x_i, x_i^*) : x_i^* \in \partial_{f_{\alpha_i}}(x_i) \text{ for some } \alpha_i \in I\} \]

Define $\tilde{\Omega} : A \to 2^{X^*}$ by $\tilde{\Omega}(x) := \bigcup_{i=1}^{\infty} \partial_{f_{\alpha_i}}(x)$ for $x \in A$. Then $\tilde{\Omega}(x) \subseteq \Omega(x)$ for every $x \in A$ and $x_i^* \in \tilde{\Omega}(x_i)$ for all $i \in \mathbb{N}$. Then

\[ \text{Gr}(\text{USC}(\Omega)) = \text{cl}C \subseteq \text{clGr}(\tilde{\Omega}) \subseteq \text{Gr}(\text{USC}(\tilde{\Omega})) \subseteq \text{Gr}(\text{USC}(\Omega)) \]

This yields $\text{USC}(\Omega) = \text{USC}(\tilde{\Omega})$. \qed

The following result which is Theorem 1 from [6] will be useful for our purposes.

Proposition 4.3. Let $A$ be a non-empty open subset of a separable Banach space $X$ and let $T : A \to 2^{X^*}$ be a weak* cusco on $A$. Then for each $f \in X_T$, $\{ g \in X_T : \partial_a f(x) \subseteq \partial_a g(x) \text{ for all } x \in A \}$ is residual in $(X_T, \rho)$.

These results provide us with the following characterization.

Theorem 4.4. Let $A$ be a non-empty open subset of a separable Banach space $X$ and let $T : A \to 2^{X^*}$ be a weak* cusco. Then the following are equivalent:

(i) $T$ is an approximate as well as Clarke subdifferential map of some locally Lipschitz function on $A$;

(ii) $T = \text{USC}(\Omega)$ where $\Omega(x) := \bigcup_{\alpha \in I} \partial_{f_\alpha}(x)$ for each $x \in A$ for some family $\{ f_\alpha : \alpha \in I \}$ of equi-locally Lipschitz functions on $A$;

(iii) $T = \text{USC}(\Omega)$ where $\Omega(x) := \bigcup_{i=1}^{\infty} \partial_{f_i}(x)$ for each $x \in A$ for some countable family $\{ f_i : i \in \mathbb{N} \}$ of equi-locally Lipschitz functions on $A$.

Proof. The equivalence (ii)$\Leftrightarrow$ (iii) follows directly from Lemma 4.2. Because (i) $\Rightarrow$ (iii) is trivial, it remains to prove (iii) $\Rightarrow$ (i). By the hypothesis, $T$ is a weak* cusco and so the set

\[ S_n = \{ g \in X_T : \partial_a f_n(x) \subseteq \partial_a g(x) \text{ for all } x \in X \}, \]

is residual according to Proposition 4.3. Now let $f \in \bigcap_{n=1}^{\infty} S_n$. Then $\Omega(x) \subseteq \partial_a f(x) \subseteq T(x)$ for all $x \in A$. Moreover, $\partial_a f$ is a weak* usco map, and so $\text{USC}(\Omega)(x) \subseteq \partial_a f(x)$; consequently $\partial_a f(x) = T(x)$ for all $x \in A$. Now, $\text{conv}^* \partial_a f(x) = \partial_c f(x)$. Because $T$ is a weak* cusco, we have $\partial_c f(x) = \partial_a f(x)$ for all $x \in A$. \qed

The previous result enables us to show that a broad cross-section of cusco maps arise as approximate as well as Clarke subdifferential maps.
Theorem 4.5. Let $X$ be a separable Banach space and $k : X \to [0, +\infty)$ be topologically robust usc. Then the cusco map $T : X \to 2^{X^*}$ given by $T(x) := k(x)B_{X^*}$ is an approximate as well as Clarke subdifferential map.

Proof. When $k \equiv 0$, the result is obvious. Let us assume $k \neq 0$. By Proposition 2.1, there exists a lsc $g : X \to \mathbb{R}$ such that $k = \text{usc}(g)$. For each $u \in X$ and $n \in \mathbb{N}$ with $g(u) > \frac{1}{n}$, choose $\delta_n(u) > 0$ such that $g(x) > g(u) - \frac{1}{n} > 0$ whenever $\|x - u\| < \delta_n(u)$. Now choose $\eta_n(u) > 0$ so that $(g(u) - \frac{1}{n})\|x - u\| - \eta_n(u) \geq 0$ if $\|x - u\| \geq \delta_n(u)$. Then, for such $u$ and $n$, define $f_{n,u}$ by

$$f_{n,u}(x) := \min\{(g(u) - \frac{1}{n})\|x - u\| - \eta_n(u), 0\}.$$ 

Because $f_{n,u}(x) = (g(u) - 1/n)\|x - u\| - \eta_n(u)$ for $x$ sufficiently near by $u$, we have $\partial_a f_{n,u}(u) = (g(u) - \frac{1}{n})B_{X^*}$ for all $u$ and $n$ such that $g(u) > \frac{1}{n}$. Define $\Omega(x)$ by

$$\Omega(x) := \bigcup \{\partial_a f_{n,u}(x) : g(u) > \frac{1}{n}\}.$$ 

Then $g(x)B_{X^*} \subseteq \overline{\Omega(x)} \subseteq k(x)B_{X^*}$ for each $x \in X$. Consequently, USC($\Omega$)$(x) = k(x)B_{X^*}$ for all $x \in X$ because $k = \text{usc}(g)$. Thus, an application of Theorem 4.4 completes the proof. □

A direct consequence of Proposition 2.2 and Theorem 4.5 is: Let $X$ be a separable Banach space and $k_i : X \to [0, +\infty)$ be topologically robust usc for $i \in I$. If $\{k_i : i \in I\}$ are locally equi-bounded on $X$, then the set-valued map $T : X \to 2^{X^*}$ given by $T(x) := \text{usc}(\text{sup}_{i \in I} k_i)B_{X^*}$ is simultaneously an approximate and Clarke subdifferential map.

5 Subdifferentials equal to $k(x)B_{X^*}$ in arbitrary Banach spaces

Theorems 5.1 and 5.2 below are proved in [6]. In this section we supply a unified proof which relies on either the Borwein-Preiss smooth variational principle or Ekeland’s variational principle and the sum rule for subdifferentials [7, 11]. These results essentially say that for every $f \in \mathcal{X}_T$ one may find a residual subset $G \subseteq \mathcal{X}_T$ such that $f$ and $g \in G$ have at least one common subgradient (either Clarke’s or the approximate) at every point. Note that the proof given here for the approximate subdifferential applies even when the Banach spaces only admit smooth Lipschitz bumps [8, Theorem I.2.3].

Theorem 5.1. Let $A$ be a nonempty open subset of a Banach space $X$ with a Gâteaux smooth renorm. Let $T : A \to 2^{X^*}$ be a weak* cusco on $A$. If $f \in \mathcal{X}_T$, then the set

$$\{g \in \mathcal{X}_T : \partial_a g(x) \cap (-\partial_a (-f)(x)) \neq \emptyset \text{ for all } x \in A\}$$

is residual in $(\mathcal{X}_T, \rho)$.

In particular, the set $\{g \in \mathcal{X}_T : \partial_a f(x) \subseteq \partial_a g(x) \text{ for all } x \in A\}$ is residual in $(\mathcal{X}_T, \rho)$ when $\partial_a f$ is a minimal weak* usco on $A$; the set $\{g \in \mathcal{X}_T : -\partial_a (-f)(x) \subseteq \partial_a g(x) \text{ for all } x \in A\}$ is residual in $(\mathcal{X}_T, \rho)$ when $\partial_a (-f)$ is a minimal weak* usco on $A$.

Proof. For each $m \in \mathbb{N}$, let $A_m := \text{int}\{t \in A : T(t) \subseteq mB_{X^*}\}$. Since $T$ is a norm to weak* cusco, $T$ is locally norm bounded [6]; moreover, we have $A = \bigcup_{m \in \mathbb{N}} A_m$ and each $g \in \mathcal{X}_T$ is $m$-Lipschitz on each
convex subset of $A_m$. Let $J := \{J_n : n \in \mathbb{N}\}$ be an enumeration of all the open intervals in $\mathbb{R}$ with rational end-points. For each $(m, n, \epsilon) \in \mathbb{N}^2 \times (0, \infty)$ we consider the set,

$$O_{(m,n,\epsilon)} := \{ g \in \mathcal{X}_T : \text{for each connected open set } U \text{ with } U + \epsilon B_X \subseteq A_m$$

and $J_n \subseteq (f - g)(U)$ there exist $z_0 \in U$ and $0 < r_0 < \epsilon$

so that $\inf_{B_0[z_0]} \{ g-f\}(z) + \epsilon r_0 \geq (g-f)(z_0)\}.$

(a) We claim that for each $(m, n, \epsilon) \in \mathbb{N}^2 \times (0, \infty)$, $\text{int } O_{(m,n,\epsilon)}$ is dense in $(\mathcal{X}_T, \rho)$.

Let $g_0 \in \mathcal{X}$ and $\delta \in (0, 1)$. We need to verify that $B_\delta(g_0) \cap \text{int } O_{(m,n,\epsilon)} \neq \emptyset$. So, suppose $J_n := (r_n, s_n)$ and $0 < \delta' := \min\{(s_n-r_n)/5, \delta\}$. Now let us choose a dense open subset $E$ of $\mathbb{R}$ such that $\mu(E) < \delta'$ and define $h : A \rightarrow \mathbb{R}$ by

$$h(x) := \int_0^{(f-g_0)(x)} \chi_E(s)ds + g_0(x),$$

where $\chi_E(s) := 1$ if $s \in E$ and 0 otherwise. Then $h \in \mathcal{X}_T$ (see [6]) and $\rho(g_0, h) < \delta' \leq \delta$. We claim that $h \in \text{int } O_{(m,n,\epsilon)}$. To this end, choose $0 < r < 2m\epsilon$ and $t \in \mathbb{R}$ so that $[t-r, t+r] \subseteq (r_n+2\delta', s_n-2\delta') \cap E$ and let $0 < d < \min\{(er)/4m, \delta'\}$. We will show that $B_d(h) \subseteq O_{(m,n,\epsilon)}$. Take $g \in B_d(h)$ and let $U$ be any connected open subset of $A_m$ with $U + \epsilon B_X \subseteq A_m$ and $J_n \subseteq (f - g)(U)$. Then,

$$\|f - g_0\| \leq \|g - g_0\| \leq \|\text{int } O_{(m,n,\epsilon)}\| < \delta' + \delta \leq 2\delta'.$$

Choose $z_0 \in U$ so that $(f - g_0)(z_0) = t$. Then for any $z \in B_{r_0}[z_0]$ with $r_0 = r/2m < \epsilon$, we have $(f - g_0)(z) \in [t-r, t+r] \subseteq E$ and so $h(z) - h(z_0) = f(z) - f(z_0).$ Therefore by our choice of $d$,

$$(g-f)(z) - (g-f)(z_0) = (g-h)(z) - (g-h)(z_0) > -2d > -\epsilon r_0,$$

for all $z \in B_{r_0}[z_0]$. This shows that $g \in O_{(m,n,\epsilon)}$.

(b) The set $G := \bigcap\{O_{(m,n,\epsilon)} : (n_1, n_2, n_3) \in \mathbb{N}^3\}$ is residual in $(\mathcal{X}_T, \rho)$, and for each $g \in G$ we have $\partial_ag(x) \cap (-\partial_a(-f)(x)) \neq \emptyset$ for every $x \in A$.

Indeed, if this is not the case then there exist $g \in G$ and $x_0 \in A$ such that $\partial_ag(x_0) \cap (-\partial_a(-f)(x_0)) = \emptyset$, then for some weak*-neighborhood $V$ of zero,

$$(\partial_ag(x_0) + V) \cap (-\partial_a(-f)(x_0) + V) = \emptyset. \quad (9)$$

Choose $n_3 \in \mathbb{N}$ and another weak* neighborhood $W$ of 0 such that $2W + 2/n_3 B_X \cdot \subseteq V$. If there exists a neighborhood $B_\delta(x_0)$ such that $(f - g)(B_\delta(x_0)) = \{a\}$, then $g - f \equiv a$ on $B_\delta(x_0)$; and so

$$\partial_ag(x_0) \cap (-\partial_a(-f)(x_0)) = \partial_ag(x_0) \cap (-\partial_a(-f)(x_0)) \neq \emptyset,$$

which contradicts (9). Therefore, for every $n \in \mathbb{N}$, there exists $m \in \mathbb{N}$ so that $J_m \subseteq (f - g)(B_1/n(x_0))$. We may now select $n_1, n_3 \in \mathbb{N}$ so that $B_{2/n_3}(x_0) \subseteq A_{n_1}$, and that for $x \in B_{2/n_3}(x_0)$ we have

$$-\partial_a(-f)(x) \subseteq -\partial_a(-f)(x_0) + W,$$

$$\partial_ag(x) \subseteq \partial_ag(x_0) + W.$$
Find then \( n_2 \in \mathbb{N} \) so that \( J_{n_2} \subseteq (f - g)(B_{1/n_3}(x_0)) \). Since \( g \in O(n_1, n_2, 1/n_3) \), there exist \( z_0 \in B_{1/n_3}(x_0) \) and \( 0 < r_0 < 1/n_3 \) such that

\[
\inf_{B_{r_0}[z_0]} \frac{g - f}{r_0} \geq (g - f)(z_0).
\]

By the Borwein-Preiss variational principle, there exists a Gâteaux smooth convex function \( \phi \) on \( X \) and \( \tilde{z}_0 \in X \) such that \( \|z_0 - \tilde{z}_0\| < r_0, \|\nabla \phi(\tilde{z}_0)\| \leq 2/n_3 \), and

\[
(g - f)(z) + \phi(z) \geq (g - f)(\tilde{z}_0) + \phi(\tilde{z}_0) \text{ for all } z \in B_{r_0}[z_0].
\]

As \( \nabla \phi \) is norm-to-weak* continuous and \( \partial_a \phi(\tilde{z}_0) \subset \nabla \phi(\tilde{z}_0) + W \), by the sum rule of approximate subdifferential [11] we have

\[
0 \in \partial_a g(\tilde{z}_0) + \partial_a (-f)(\tilde{z}_0) + \frac{2}{n_3}B_{X^*} + W.
\]

Therefore, there exists \( \tilde{z}_0^* \) such that \( \tilde{z}_0^* \in -\partial_a (-f)(\tilde{z}_0) \subset -\partial_a (-f)(x_0) + V \), and

\[
\tilde{z}_0^* \in \partial_a g(\tilde{z}_0) + \frac{2}{n_3}B_{X^*} + W \subset \partial_a g(x_0) + W + W + \frac{2}{n_3}B_{X^*} \subset \partial_a g(x_0) + V.
\]

which is impossible by equation (9). Therefore, for each \( g \in G \),

\[
(-\partial_a (-f)(x)) \cap \partial_a g(x) \neq \emptyset \quad \text{for all } x \in A. \]

When \( \partial_a f \) is a minimal weak* usco, \( \partial_a f \) is single-valued on a dense \( \mathcal{G} \) set \( D \subset A \) by [15]. For \( x \in D, \partial_a f(x) = -\partial_a (-f)(x) = \{ f'(x) \} \); thus \( \partial_a f(x) \subset \partial_a g(x) \) for every \( x \in A \) and \( g \in G \). Again minimality shows \( \partial_a f(x) \subset \partial_a g(x) \) for every \( x \in A \) and \( g \in G \).

In arbitrary Banach space, one can use Ekeland’s variational principle [14, page 45] and \( \partial_c f \) instead of the Borwein-Preiss variation principal and \( \partial_a f \) in the above proof to get the following analogue for the Clarke subdifferential. Note that the cusco minimality condition below is much less demanding than the usco minimality required in the prior result.

**Theorem 5.2.** Let \( A \) be a non-empty open subset of a Banach space \( X \) and \( T : A \to 2^{X^*} \) be a weak* cusco on \( A \). Then for each \( f \in \mathcal{X}_T \), the set

\[
\{ g \in \mathcal{X}_T : \partial_c g(x) \cap \partial_c f(x) \neq \emptyset \text{ for all } x \in A \}
\]

is residual in \( (\mathcal{X}_T, \rho) \).

In particular, the set \( \{ g \in \mathcal{X}_T : \partial_c g(x) \supseteq \partial_c f(x) \text{ for all } x \in A \} \) is residual in \( (\mathcal{X}_T, \rho) \) when \( \partial_c f \) is a minimal weak* cusco on \( A \).

Two applications of Theorems 5.2 and 5.1 come as follows:

**Lemma 5.3.** Let \( X \) be an arbitrary Banach space and \( k : X \to [0, \infty) \) be continuous. Suppose \( T : X \to 2^{X^*} \) is a weak* cusco such that \( k(x)B_{X^*} \subseteq T(x) \) for every \( x \in X \). Then in \( (\mathcal{X}_T, \rho) \) the set

\[
\{ f \in \mathcal{X}_T : \partial_c f(x) \supseteq k(x)B_{X^*} \text{ for every } x \in X \},
\]

is residual. In particular, the weak* cusco \( T : X \to 2^{X^*} \) given by \( T(x) := k(x)B_{X^*} \) is a Clarke subdifferential map. If \( X \) is a Banach space with a Gâteaux differentiable norm and \( B_{X^*} \) is the dual ball associated with the smooth norm, then \( T \) is also an approximate subdifferential map.
Proof. For every \( n \in \mathbb{N} \), choose a \( \frac{2}{n} \)-net in \( A := X \setminus \{ x : k(x) = 0 \} \), and denote it by \( \{ x_n^\alpha : \alpha \in \Gamma_n \} \).

Given \( x_n^\alpha \), for every \( 0 < \epsilon_n^\alpha < 1/n \) there exists \( 0 < \delta_n^\alpha < 1/(2n) \) such that \( k(y) \geq k(x_n^\alpha) - \epsilon_n^\alpha \) when \( y \in B_{\delta_n^\alpha}[x_n^\alpha] \subseteq A \). Define \( f_n^\alpha : X \to \mathbb{R} \) by

\[
\begin{align*}
    f_n^\alpha(x) := \begin{cases} 
        (k(x_n^\alpha) - \epsilon_n^\alpha)(\delta_n^\alpha - \|x - x_n^\alpha\|) & \text{if } \|x - x_n^\alpha\| \leq \delta_n^\alpha, \\
        0 & \text{otherwise.}
    \end{cases}
\end{align*}
\]

We have \( \partial_c f_n^\alpha(x) \subseteq T(x) \) for every \( x \in X \). Let \( f_n := \sup\{ f_n^\alpha : \alpha \in \Gamma_n \} \). Since the support of \( f_n^\alpha \) and the support of \( f_n^\beta \) are disjoint (at least \( 1/n \)-apart) when \( \alpha \neq \beta \), we obtain \( \partial_c f_n(x) \subseteq T(x) \) for every \( x \in X \). For each \( n \in \mathbb{N} \), the set

\[ G_n := \{ f \in \mathcal{X}_T : \partial_c f(x) \cap \partial_c f_n(x) \neq \emptyset \text{ for every } x \in X \}, \]

is residual in \( \mathcal{X}_T \) by Theorem 5.2. Because \( f_n = f_n^\alpha \) and \( f_n^\alpha \) is concave on \( B_{\delta_n^\alpha}(x_n^\alpha) \), then \( \partial_c f_n^\alpha \) is minimal [14, page 105], and consequently \( \partial_c f_n^\alpha \subseteq \partial_c f \). Then, in particular,

\[ \partial_c f_n^\alpha(x_n^\alpha) = (k(x_n^\alpha) - \epsilon_n^\alpha)B_X \subseteq \partial_c f(x_n^\alpha). \]

Since \( f_n = 0 \) on \( \text{int}\{ x \in X : k(x) = 0 \} \), we have \( \partial_c f \supseteq \{ 0 \} \). If \( f \in G := \bigcap_{n=1}^{\infty} G_n \), then \( \partial_c f(x_n^\alpha) \supseteq (k(x_n^\alpha) - \epsilon_n^\alpha)B_X \) for every \( n \in \mathbb{N} \) and \( \alpha \in \Gamma_n \), and so for every \( v \in S_X \) we have

\[ f^0(x_n^\alpha; v) \geq (k(x_n^\alpha) - \epsilon_n^\alpha) = k(x). \]

Moreover, \( \partial_c f(x) \supseteq \{ 0 \} \) for \( x \in \text{int}\{ x \in X : k(x) = 0 \} \). Since \( \bigcup_{n=1}^{\infty} \{ x_n^\alpha : \alpha \in \Gamma_n \} \) is dense in \( A \), for \( x \in A \) we have

\[ f^0(x; v) \geq \limsup_{x_n^\alpha \to x} f^0(x_n^\alpha; v) \geq \limsup_{x_n^\alpha \to x} (k(x_n^\alpha) - \epsilon_n^\alpha) = k(x). \]

Hence \( \partial_c f(x) \supseteq k(x)B_X \) for each \( x \in X \) when \( f \in G \). This proves the first part of this lemma, whereas letting \( T = kB_X \) establishes the second claim in this lemma.

When \( X \) is separable, the last assertion in this lemma follows from Theorem 4.5. So we suppose \( X \) is nonseparable and with a Gâteaux smooth norm \( \| \cdot \| \). Let \( f \) be such that \( \partial_c f(x) = k(x)B_X \) for every \( x \in X \). Because \( \partial_c f(x) = \overline{\partial f(x)} \), the Krein-Milman Theorem converse shows \( k(x)\overline{Ext B_X} \subseteq \partial_c f(x) \). Since \( X \) is infinite dimensional and \( \| \cdot \| \) is smooth, \( \overline{\text{Ext} B_X} \supseteq S_X \) and \( B_X = \overline{S_X} \). Therefore \( \partial_c f(x) = \partial_c f(x) = k(x)B_X \) for every \( x \in X \). \( \square \)

**Theorem 5.4.** Let \( X \) be an arbitrary Banach space and \( k : X \to [0, +\infty) \) be topologically robust usc. Then the \( \text{weak}^* \) cusc on map \( T : X \to 2^{X^*} \) given by \( T(x) := k(x)B_{X^*} \) is a Clarke subdifferential map.

**Proof.** By Proposition 2.1, there exists a lsc function \( g : X \to \mathbb{R} \) such that \( k = \text{usc}(g) \). By a result of R. Baire [18, page 132] there exists a sequence of real-valued continuous functions \( 0 \leq g_1 \leq g_2 \leq g_3 \leq \ldots \) on \( X \), for example

\[ g_n(x) := \inf\{ g(y) + n\|x - y\| : y \in X \}, \]

such that \( \lim_{n \to \infty} g_n(x) = g(x) \) for all \( x \in X \). Now consider the complete metric space \( (\mathcal{X}_T, \rho) \) where \( T := kB_{X^*} \). Because \( g_n \leq g \), Lemma 5.3 ensures that the set

\[ G_n := \{ f \in \mathcal{X}_T : \partial_c f(x) \supseteq g_n(x)B_X \text{ for every } x \in X \}, \]

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is residual. Let \( G := \bigcap_{n=1}^{\infty} G_n \). Then for \( f \in G \), we have
\[
\partial_{e} f(x) \supseteq g_n(x) B_{X^*}
\]
for every \( n \) and \( x \).

For \( v \in S_X \), this implies that
\[
f^0(x; v) \geq \sup_{n \geq 1} g_n(x) = g(x),
\]
so \( f^0(x; v) \geq u s e (g)(x) = k(x) \). But \( f^0(x; v) \leq k(x) \) always holds. Hence \( f^0(x; v) = k(x) \) for every \( x \in X \) and every \( v \in S_X \).

\( \square \)

**Lemma 5.5.** Let \( X \) be a Banach space admitting an equivalent Gâteaux smooth renorm, and let \( k : X \to [0, \infty) \) be continuous. Assume the original dual unit ball \( B_{X^*} \) is weak* separable. Suppose \( T : X \to 2^{X^*} \) is a weak* cusco such that \( k(x) B_{X^*} \subseteq T(x) \) for every \( x \in X \). Then in \( (X_T, \rho) \), the set
\[
\{ f \in X_T : \partial_{a} f(x) \supseteq k(x) B_{X^*} \text{ for every } x \in X \}
\]
is residual. In particular, the weak* cusco map \( T : X \to 2^{X^*} \) defined by \( T(x) := k(x) B_{X^*} \) is an approximate subdifferential.

**Proof.** We will apply Theorem 5.1. For every \( n \in \mathbb{N} \), let us fix a \( \frac{2}{n} \)-net \( \{ x_{\alpha}^n : \alpha \in \Gamma_n \} \) in \( A = X \setminus \{ x \in X : k(x) = 0 \} \) and choose \( \{ b_m^* : m \in \mathbb{N} \} \) a countable weak* dense set in \( B_{X^*} \). For each \( x_{\alpha}^n \), put \( 0 < \epsilon_{n}^{\alpha} < \min\{1/n, k(x_{\alpha}^n)\} \). Find then \( 0 < \delta_n^{\alpha} < 1/n \) such that \( k(y) \geq k(x_{\alpha}^n) - \epsilon_{n}^{\alpha} \) when \( y \in B_{\delta_n^{\alpha}}[x_{\alpha}^n] \subset A \). The function \( g \) defined by
\[
g(x) := \begin{cases} 
(k(x_{\alpha}^n) - \epsilon_{n}^{\alpha}) b_m^* (x - x_{\alpha}^n) & \text{if } \| x - x_{\alpha}^n \| \leq \delta_n^{\alpha}/4, \\
0 & \text{if } \| x - x_{\alpha}^n \| \geq \delta_n^{\alpha}/2.
\end{cases}
\]
is \( (k(x_{\alpha}^n) - \epsilon_{n}^{\alpha}) \) Lipschitz on its domain. This follows from the observation that when \( \| x - x_{\alpha}^n \| \leq \delta_n^{\alpha}/4 \) and \( \| y - x_{\alpha}^n \| \geq \delta_n^{\alpha}/2 \), we have
\[
|g(x) - g(y)| = |(k(x_{\alpha}^n) - \epsilon_{n}^{\alpha}) b_m^* (x - x_{\alpha}^n)| \leq (k(x_{\alpha}^n) - \epsilon_{n}^{\alpha}) \frac{\delta_n^{\alpha}}{4} \leq (k(x_{\alpha}^n) - \epsilon_{n}^{\alpha}) \| y - x \|.
\]
Then \( f_{n}^\alpha : B_{\delta_n^{\alpha}}[x_{\alpha}^n] \to \mathbb{R} \) defined by
\[
f_{n}^\alpha (x) := \inf \{ g(y) + (k(x_{\alpha}^n) - \epsilon_{n}^{\alpha}) \| x - y \| : y \in \text{dom} g \},
\]
is \( (k(x_{\alpha}^n) - \epsilon_{n}^{\alpha}) \) Lipschitz. In particular, \( g = f_{n}^\alpha \) on \( \text{dom} g \). Then \( \partial_{a} f_{n}^\alpha \subseteq (k(x_{\alpha}^n) - \epsilon_{n}^{\alpha}) B_{X^*} \) on \( B_{\delta_n^{\alpha}/2}[x_{\alpha}^n] \) and \( f_{n}^\alpha = 0 \) on \( B_{\delta_n^{\alpha}}[x_{\alpha}^n] \setminus B_{\delta_n^{\alpha}/2}[x_{\alpha}^n] \). We can do this at each \( \alpha \in \Gamma_n \). Then define \( f_{n,m} := f_{n}^\alpha \) on each \( B_{\delta_n^{\alpha}}[x_{\alpha}^n] \) and 0 otherwise. As \( f_{n}^\alpha \) and \( f_{n}^\beta \) have disjoint support (at least \( 1/n \)-apart) when \( \alpha \neq \beta \), we have \( f_{n,m} \in X_T \). By Theorem 5.1 the set
\[
G_{n,m} := \{ f : \partial_{a} f(x) \cap \partial_{a} f_{n,m}(x) \neq \emptyset \text{ for all } x \},
\]
is residual in \( X_T \). Let \( G_n := \bigcap_{m=1}^{\infty} G_{n,m} \) and \( G := \bigcap_{n=1}^{\infty} G_n \). Take \( f \in G \). As \( f \in G_n \), \( f \in G_{n,m} \) for every \( m \). On \( B_{\delta_{n,m}/2}(x_{\alpha}^n) \), \( f_{n,m} \) is linear by (10), then \( \partial_{a} f(x_{\alpha}^n) \supseteq \{(k(x_{\alpha}^n) - \epsilon_{n}^{\alpha}) b_m^* \} \). On \( \{ x \in X : k(x) = 0 \} \), \( f_{n,m}(x) \equiv 0 \), \( \partial_{a} f(x) \supseteq \{0\} \). Since this holds for every \( m \), we have
\[
\partial_{a} f(x_{\alpha}^n) \supseteq (k(x_{\alpha}^n) - \epsilon_{n}^{\alpha}) \{ b_m^* : m \in \mathbb{N} \} \wedge = (k(x_{\alpha}^n) - \epsilon_{n}^{\alpha}) B_{X^*},
\]
and
\[ \partial_a f(x) \supseteq \{0\} \text{ for } x \in \text{int}\{x \in X : k(x) = 0\}. \quad (11) \]

As \( f \in G \), we have
\[ \partial_a f(x_n^\alpha) \supseteq (k(x_n^\alpha) - \epsilon_n^\alpha)B_{X^*} \text{ for every } n \text{ and } \alpha \in \Gamma_n. \]

Since \( C = \{x_n^\alpha : n \in \mathbb{N}, \alpha \in \Gamma_n\} \) is dense in \( A \). For every \( x \in A \), there exists a sequence \( \{x_n\} \) from \( C \) such that \( x_n \) converges to \( x \) as \( n \to \infty \). By the upper semicontinuity of \( \partial_a f \), we obtain \( \partial_a f(x) \supseteq k(x)B_{X^*} \) for every \( x \in A \). Together with (11), we have \( \partial_a f(x) \supseteq k(x)B_{X^*} \) for every \( x \in X \).

We close this section with our main result concerning approximate subdifferentials.

**Theorem 5.6.** Let \( X \) be a Banach space admitting an equivalent Gâteaux smooth renorm, and let \( k : X \to \mathbb{R} \) be topologically robust usc. Assume that the original dual unit ball \( B_{X^*} \) is weak* separable. Then the weak* cusco map \( T : X \to 2^{X^*} \) given by \( T(x) := k(x)B_{X^*} \) is an approximate subdifferential map.

**Proof.** Invoking Lemma 5.5, we may use similar arguments as in Theorem 5.4 to get a residual set \( G \subset X_T \) such that \( \partial_a f(x) \supseteq g_n(x)B_{X^*} \) for all \( n, f \in G \) and \( x \in X \), where \( \{g_n\} \) is an increasing sequence of Lipschitz functions as given in the proof of Theorem 5.4. Then \( \partial_a f(x) \supseteq \sup_n g_n(x)B_{X^*} \) for every \( x \in X \). By the norm to weak* upper semicontinuity of \( \partial_a f \), we have \( \partial_a f(x) \supseteq k(x)B_{X^*} \). The opposite inclusion holds because \( f \in X_T \).

\[ \] 6 \quad \sigma\text{-minimal Clarke subdifferentials plus } k(x)B_{X^*}

We say that a Clarke subdifferential map \( T \) is \( \sigma\text{-minimal} \) if there exists a countable family of minimal Clarke subdifferential maps \( \{\partial_z f_i : i \in \mathbb{N}\} \) such that \( T = \text{CSC}(\bigcup_{i \in \mathbb{N}} \partial_z f_i) \). The main result of this section says that such subdifferentials plus \( k(x)B_{X^*} \) are again Clarke subdifferentials if \( k \) is a continuous function on \( X \). To prove this, we will use the following three preparatory lemmas.

**Lemma 6.1 (minimal subdifferential invariance).** [6] Let \( f \) be a real-valued locally Lipschitz function defined on a non-empty open subset \( A \) of a Banach space \( X \). If \( \partial_z f \) is a minimal weak* cusco on \( A \) and \( g \) is a convex function, then \( \partial_z (f + g) \) is a minimal weak* cusco on \( A \).

**Lemma 6.2.** Let \( f \) be a locally Lipschitz function defined on a non-empty open subset \( A \) of a Banach space \( X \). Assume \( \partial_z f \) is a minimal weak* cusco. Then for each \( y \in X \) and each dense set \( D \subset A \) we have \( f^0(x; y) = \limsup_{z \in D, z \to x} f^-(z; y) \).

**Proof.** Given a dense set \( D \), suppose for some \( y, f^0(x; y) > \alpha > \limsup_{z \in D, z \to x} f^-(z; y) \). Then there exists a open neighborhood \( U \) of \( x \) such that \( f^-(z; y) < \alpha \) for all \( z \in D \cap U \). On the other hand, \( f^0(x; y) > \alpha \) shows \( \partial_z f(U) \not\subseteq \{x^* \in X^* : \langle x^*, y \rangle \leq \alpha \} \). Since \( \partial_z f \) is minimal, there exists a non-empty open \( V \subset U \) such that \( \partial_z f(V) \cap \{x^* \in X^* : \langle x^*, y \rangle \leq \alpha \} = \emptyset \). Then for all \( z \in V \cap D \), \( f^-(z; y) \geq -f^0(z; -y) \geq \alpha \), a contradiction.

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Lemma 6.3. Let $X$ be a Banach space. Suppose that $T : X \to 2^{X^*}$ is a weak* cusco and $l : X \to [0, \infty)$ be continuous. If $\partial_c f$ is a minimal weak* cusco and $\partial_c f(x) + l(x)B_{X^*} \subseteq T(x)$ for all $x \in X$, then

$$\{g \in \mathcal{X}_T : \partial_c g(x) \supseteq \partial_c f(x) + l(x)B_{X^*} \text{ for all } x \in X\}$$

is residual in $(\mathcal{X}_T, \rho)$.

Proof. For each $n \in \mathbb{N}$ we choose a $2/n$-net $\{x^n_\alpha : \alpha \in \Gamma_n\}$ in $A := \{x \in X : l(x) \neq 0\}$. Given $x^n_\alpha$, let $0 < \epsilon_n^\alpha < \min\{l(x^n_\alpha), 1/n\}$. There exists $0 < \delta_n^\alpha < 1/(2n)$ such that $B_{\delta_n^\alpha}[x^n_\alpha] \subset A$ and $l(x) \geq l(x^n_\alpha) - \epsilon_n^\alpha$ for $x \in B_{\delta_n^\alpha}[x^n_\alpha]$. Define

$$f_n^\alpha(x) := \begin{cases} (l(x^n_\alpha) - \epsilon_n^\alpha)(\|x - x^n_\alpha\| - \delta_n^\alpha) & \text{if } \|x - x^n_\alpha\| \leq \delta_n^\alpha, \\ 0 & \text{otherwise}. \end{cases}$$

Then let $f_n := \inf_{\alpha \in \Gamma_n} f_n^\alpha$. When $\|x - x^n_\alpha\| \leq \delta_n^\alpha$ we have $f_n = f_n^\alpha$, so

$$\partial_c(f + f_n)(x) \subseteq \partial_c f(x) + (l(x^n_\alpha) - \epsilon_n^\alpha)B_{X^*} \subseteq \partial_c f(x) + l(x)B_{X^*},$$

for $x \in B_{\delta_n^\alpha}[x^n_\alpha]$. Because the distance between $B_{\delta_n^\alpha}[x^n_\alpha]$ and $B_{\delta_n^\beta}[x^n_\beta]$ are at least $1/n$ for $\alpha \neq \beta$, the set $X \setminus \bigcup_{\alpha \in \Gamma_n} B_{\delta_n^\alpha}[x^n_\alpha]$ is open and $f_n \equiv 0$, so

$$\partial_c(f + f_n) = \partial_c f.$$ 

This shows $g_n := f + f_n \in \mathcal{X}_T$. By Theorem 5.2, there exists a residual set $G_n$ in $(\mathcal{X}_T, \rho)$ such that for each $g \in G_n$, $\partial_c g(x) \cap \partial_c g_n(x) \neq \emptyset$ for all $x \in X$. Set $G := \bigcap_{n=1}^\infty G_n$. We will show that for each $g \in G$, $\partial_c g(x) \supseteq \partial_c f(x) + l(x)B_{X^*}$ for all $x \in X$. Take $g \in G$. On $B_{\delta_n^\alpha}[x^n_\alpha]$, $g_n(x) = f(x) + f_n^\alpha(x)$, thus $\partial_c g_n$ is a minimal weak* cusco on $B_{\delta_n^\alpha}[x^n_\alpha]$ by Lemma 6.1, and so $\partial_c g_n \subseteq \partial_c g$ on $B_{\delta_n^\alpha}[x^n_\alpha]$. In particular, for each $v \in X$ we have

$$f^-(x^n_\alpha;v) + (l(x^n_\alpha) - \epsilon_n^\alpha)\|v\| = g_n^-(x^n_\alpha;v) \leq g_n^0(x^n_\alpha;v) \leq g^0(x^n_\alpha;v).$$

When $x \in \text{int}\{x : l(x) = 0\}$, $g_n = f$, again by the minimality of $\partial_c f$ we have $\partial_c f(x) \subseteq \partial_c g(x)$. Now $C := \{x^n_\alpha : n \in \mathbb{N}, \alpha \in \Gamma_n\}$ is dense in $A$. For every $x \in A$, there exists a sequence $\{x^n_\alpha\}$ from $C$ such that $x^n_\alpha \to x$ as $n \to \infty$. By Lemma 6.2 we obtain

$$f^0(x;v) + l(x)\|v\| = \limsup_{n \to \infty} f^-(x^n_\alpha;v) + (l(x^n_\alpha) - \epsilon_n^\alpha)\|v\| \leq \limsup_{n \to \infty} g_n^0(x^n_\alpha;v) \leq g^0(x;v).$$

Hence, $\partial_c g(x) \supseteq \partial_c f(x) + l(x)B_{X^*}$ for all $x \in X$. \qed

With these results in hand, we are now ready for

Theorem 6.4. Assume $\{f_j : j \in \mathbb{N}\}$ are equi-locally Lipschitz on a Banach space $X$ with each $\partial_c f_j$ being a minimal weak* cusco. Define $\Omega : X \to 2^{X^*}$ by $\Omega(x) := \bigcup \{\partial_c f_j(x) : j \in \mathbb{N}\}$ for $x \in X$. Suppose $l : X \to [0, \infty)$ is lsc. Then the weak* cusco $T : X \to 2^{X^*}$ defined

$$T(x) := \text{CSC}(\Omega + lB_{X^*})(x),$$

is a Clarke subdifferential map. In particular, when $l$ is continuous on $X$, we have $T(x) = \text{CSC}(\Omega)(x) + l(x)B_{X^*}$ for each $x \in X$. \hfill \square
Proof. Define a weak* cuso $T : X \to 2^{X^*}$ by $T(x) := \text{CSC} (\Omega + lB_{X^*} ) (x)$ for $x \in X$. As in the proof of Theorem 5.4, we may find an increasing sequence of Lipschitz functions $g_n : X \to [0, \infty)$ such that $l = \sup_{n \in \mathbb{N}} g_n$. For each $f_j$ and $g_n$, as $g_n \leq l$ we have

$$\partial_c f_j (x) + g_n (x) B_{X^*} \subseteq T(x) \quad \text{for every } x \in X.$$  

By Lemma 6.3 there exists a residual set $G_{j,n}$ in $(\mathcal{X}_T, \rho)$ such that $\partial_c f_j (x) + g_n (x) B_{X^*} \subseteq \partial_c g(x)$ for all $x \in X$ when $g \in G_{j,n}$. Then $G := \bigcap_{j \in \mathbb{N}, n \in \mathbb{N}} G_{j,n}$ is residual in $(\mathcal{X}_T, \rho)$. If $g \in G$ we have $\partial_c f_j (x) + g_n (x) B_{X^*} \subseteq \partial_c g(x)$ for all $x \in X$ and $j, n \in \mathbb{N}$. Then

$$\bigcup_{j \in \mathbb{N}} \partial_c f_j (x) + \sup_{n \in \mathbb{N}} g_n (x) B_{X^*} \subseteq \partial_c g(x), \text{ and so}$$

$$\text{CSC} (\Omega + lB_{X^*} ) (x) \subseteq \partial_c g(x).$$

Because $g \in \mathcal{X}_T$, we have $\text{CSC} (\Omega + lB_{X^*} ) (x) = \partial_c g(x)$ for $g \in G$ and $x \in X$. When $l : X \to \mathbb{R}$ is continuous on $X$, we have

$$\text{USC} (\Omega + lB_{X^*} ) = \text{USC} (\Omega) + lB_{X^*}.$$  

Therefore, by (1), for every $x \in X$ we have

$$\text{CSC} (\Omega + lB_{X^*} ) (x) = \sigma^w (\text{USC} (\Omega + lB_{X^*}) ) (x) = \text{CSC} (\Omega) (x) + l(x) B_{X^*}.$$  

\[ \square \]

7 The local Lipschitz-constant function and its calculus

Let $A$ be a nonempty open subset of a Banach space $X$. For a locally Lipschitz function $f : A \to \mathbb{R}$ we define the local Lipschitz-constant function by

$$\text{Lip}_f (x) := \lim_{\delta \to 0} \omega (x; \delta) = \inf_{\delta > 0} \omega (x; \delta) \text{ where}$$

$$\omega (x; \delta) := \sup \left\{ \frac{|f(y) - f(z)|}{\| y - z \|} : y, z \in B_{\delta} (x), y \neq z \right\}.$$  

Directly from the definition, we see that $\text{Lip}_f$ is upper semicontinuous on $A$. Another reason for studying $kB_{X^*}$ is that for each locally Lipschitz function $f : X \to \mathbb{R}$ one might define the largest ‘reasonable’ subdifferential map $Df : X \to 2^{X^*}$ by $Df(x) := \text{Lip}_f (x) B_{X^*}$ for $x \in X$.

**Proposition 7.1.** Assume that $X$ is a Banach space and $A \subset X$ is nonempty and open. If $f : A \to \mathbb{R}$ is locally Lipschitz, then $\text{Lip}_f$ is robust usc, that is, for every Haar null set $N \subset X$ we have

$$\text{Lip}_f (x) = \limsup_{y \to x, y \not\in N} \text{Lip}_f (y).$$

**Proof.** Let $N$ be a fixed Haar null set in $X$. For $\epsilon > 0$, we have $\text{Lip}_f (x) - \epsilon < \text{Lip}_f (x) \leq \omega (x; \delta)$ holds for every $\delta > 0$. Then for every $\delta > 0$ there exist distinct $y, z \in B_{\delta} (x)$ such that

$$\text{Lip}_f (x) - \epsilon < \frac{f(y) - f(z)}{\| y - z \|}.$$
As $f$ is locally Lipschitz and $N$ is a Haar null set, by a version of Fubini’s theorem [5] we may find distinct $y', z' \in B_\delta(x)$ near $y, z$ such that $[y', z'] \cap N$ has Lebesgue measure 0 and
\[
\text{Lip}_f(x) - \epsilon < \frac{f(y') - f(z')}{\|y' - z\|}.
\]
Applying Lebesgue’s mean value theorem, we see that the set
\[
\{ u \in [y', z'] : \text{Lip}_f(x) - \epsilon < f'(u; \frac{y' - z'}{\|y' - z\|}) \leq \text{Lip}_f(u) \}
\]
has positive one dimensional Lebesgue measure in $[y', z']$. Consequently, we may choose $u \notin N$ such that
\[
\text{Lip}_f(x) - \epsilon < f'(u; \frac{y' - z'}{\|y' - z\|}) \leq \text{Lip}_f(u).
\]
This shows $\text{Lip}_f(x) - \epsilon < \sup \{ \text{Lip}_f(y) : y \in B_\delta(x) \setminus N \}$. Letting $\delta \to 0$, we have
\[
\text{Lip}_f(x) - \epsilon \leq \limsup_{y \to x, y \notin N} \text{Lip}_f(y) \leq \limsup_{y \to x} \text{Lip}_f(y) = \text{Lip}_f(x).
\]
As $\epsilon$ is arbitrary, we have $\text{Lip}_f(x) = \limsup_{y \to x, y \notin N} \text{Lip}_f(y)$, as required. \qed

We next present a few characterizations of $\text{Lip}_f$ via gradients or subgradients.

**Proposition 7.2.** Let $A$ be a non-empty open subset of a Banach space $X$ and let $f : A \to \mathbb{R}$ be locally Lipschitz. Then

(i) In every Banach space
\[
\text{Lip}_f(x) = \limsup_{y \to x} \left( \sup_{\|v\|=1} f^0(y; v) \right).
\]  

If $X$ is finite dimensional, then (12) reduces to $\text{Lip}_f(x) = \max_{\|v\|=1} f^0(x; v)$. Correspondingly, if $X$ is a Gâteaux smoothable Banach space, then $\text{Lip}_f(x) = \limsup_{y \to x} \|f'(y)\|$.

(ii) The set
\[
\left\{ x \in A : \text{Lip}_f(x) = \limsup_{t \downarrow 0} \sup_{\|v\|=1} f(x + tv) - f(x) \right\},
\]

is residual in $A$. When $X$ is finite dimensional, (13) implies that
\[
\{ x \in A : f^+(x; v) = \text{Lip}_f(x) \text{ for some } v \in X \text{ with } \|v\| = 1 \},
\]
is residual in $A$.  

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Proof. (i) For \( \epsilon > 0 \), as \( \text{Lip}_f(x) = \inf_{\delta > 0} \omega(x; \delta) \) we have
\[
\text{Lip}_f(x) - \epsilon < \text{Lip}_f(x) \leq \omega(x; \delta) \quad \text{for every } \delta > 0.
\]
For each \( \delta > 0 \) there exist distinct \( y, z \in B_\delta(x) \) such that
\[
\text{Lip}_f(x) - \epsilon < \frac{f(y) - f(z)}{\|y - z\|} = \left\langle \xi, \frac{y - z}{\|y - z\|} \right\rangle \quad \text{where } \xi \in \partial f(u) \text{ and } u \in [y, z],
\]
\[
\leq f^0(u; \frac{y - z}{\|y - z\|}) \leq \sup_{\|v\| = 1} f^0(u; v) \leq \sup_{\|v\| = 1} \sup_{w \in B_\delta(x)} f^0(w; v).
\]
As \( \delta \) is arbitrary, we have \( \text{Lip}_f(x) - \epsilon \leq \lim \sup_{\|v\| = 1} \sup_{w \rightarrow x} f^0(w; v) \). Letting \( \epsilon \to 0 \), we obtain
\[
\text{Lip}_f(x) \leq \lim \sup_{\|v\| = 1} \sup_{w \rightarrow x} f^0(w; v).
\]
Now \( f^0(w; v) \leq \text{Lip}_f(w) \) for every \( \|v\| = 1 \), and \( \text{Lip}_f \) isusc. We, thus, have
\[
\lim \sup_{\|v\| = 1} \sup_{w \rightarrow x} f^0(w; v) \leq \lim \sup_{w \rightarrow x} \text{Lip}_f(w) \leq \text{Lip}_f(x).
\]
Hence \( \text{Lip}_f(x) = \lim \sup_{w \rightarrow x} \sup_{\|v\| = 1} f^0(w; v) \), which is (12).

Let us now assume \( X \) finite dimensional. By (12), we may find \( y_n \to x \) and \( v_n \in S_X \) such that \( \text{Lip}_f(x) = \lim_{n \to \infty} f^0(y_n; v_n) \). Since \( S_X \) is compact, \( v_n \) has a convergent subsequence. For simplicity, we may assume that \( v_n \to \hat{v} \) and \( \|\hat{v}\| = 1 \). By the upper semicontinuity of \( f^0(\cdot; \cdot) \), we have
\[
f^0(x; \hat{v}) \geq \lim \sup_{n \to \infty} f^0(x_n; v_n) = \text{Lip}_f(x) \geq f^0(x; \hat{v}).
\]
Consequently, \( \text{Lip}_f(x) = \max_{\|v\| = 1} f^0(x; v) \).

When \( X \) admits a smooth norm, Preiss' Theorem [16] ensures that for every \( \delta > 0 \), when \( y, z \in B_\delta(x) \) we have
\[
f(y) - f(z) \leq \sup\{\langle y - z, f'(u) \rangle : u \in B_\delta(x), f'(u) \text{ exists}\}.
\]
Then
\[
\omega(x; \delta) \leq \sup\{\|f'(u)\| : u \in B_\delta(x) \text{ and } f'(u) \text{ exists}\}.
\]
Letting \( \delta \downarrow 0 \) yields \( \text{Lip}_f(x) \leq \lim \sup_{u \rightarrow x} \|f'(u)\| \). As \( f'(u) \leq \text{Lip}_f(u) \) and \( \text{Lip}_f \) isusc, we have
\[
\text{Lip}_f(x) = \lim \sup_{u \rightarrow x} \|f'(u)\|.
\]

(ii) For \( k \in \mathbb{N} \), let \( O_k \) be defined by
\[
O_k := \left\{ x \in A : \frac{f(x + tv) - f(x)}{t} - \text{Lip}_f(x) > -\frac{1}{k} \text{ for some } 0 < t < \frac{1}{k} \text{ and some } \|v\| = 1 \right\}.
\]
The set \( O_k \) is open because \(-\text{Lip}_f\) is lsc. We next show that \( O_k \) is dense. Because \( \text{Lip}_f \) is usc, there is a dense \( G_\delta \) subset \( G \) of \( A \) on which \( \text{Lip}_f \) is continuous. Let \( x \in X \) and \( \epsilon > 0 \) be arbitrary. Choose
\(y \in G\) such that \(\|y - x\| < \epsilon/2\). For this \(y\), there exist distinct \(y_1, y_2 \in X\) with \(\|y_i - y\| < \epsilon/2\) such that \(\|y_1 - y_2\| < 1/k\),

\[
\frac{f(y_2) - f(y_1)}{\|y_2 - y_1\|} > \text{Lip}_f(y) - \frac{1}{2k} \quad \text{and} \quad |\text{Lip}_f(y_1) - \text{Lip}_f(y)| < \frac{1}{2k}.
\]

Then

\[
\frac{f(y_1 + \|y_2 - y_1\| \cdot (y_2 - y_1)/\|y_2 - y_1\|) - f(y_1)}{\|y_2 - y_1\|} > \text{Lip}_f(y) - \frac{1}{2k} > \text{Lip}_f(y) - \frac{1}{k}.
\]

Thus \(y_1 \in O_k, \|y_1 - x\| < \epsilon\), and so \(O_k\) is shown to be dense in \(A\). Now \(O := \bigcap_{k=1}^{\infty} O_k\) is a dense \(G_\delta\) subset of \(A\). If \(x \in O\), then for every \(k\) there exists \(0 < t_k < 1/k\) and \(v_k \in S_X\) such that

\[
\sup_{\|v\| = 1} \frac{f(x + t_kv) - f(x)}{t_k} \geq \frac{f(x + t_kv_k) - f(x)}{t_k} > \text{Lip}_f(x) - \frac{1}{k}.
\]

Letting \(k \to \infty\), one obtains

\[
\limsup_{t \to 0} \sup_{\|v\| = 1} \frac{f(x + tv) - f(x)}{t} \geq \text{Lip}_f(x).
\]

Since the opposite inequality always holds, \(O\) is a subset of that given by (13).

Assume now that \(X\) is finite dimensional. When \(x \in O\), we may find \(t_n \downarrow 0\) and \(v_n \in S_X\) such that

\[
\text{Lip}_f(x) = \lim_{n \to \infty} \frac{f(x + t_nv_n) - f(x)}{t_n}.
\]

Again, for simplicity (as \(S_X\) is compact) we may assume \(v_n \to \tilde{v}\) for some \(\tilde{v} \in S_X\). Then

\[
\text{Lip}_f(x) = \lim_{n \to \infty} \frac{f(x + t_nv) - f(x)}{t_n} \leq f^+(x; \tilde{v}) \leq \text{Lip}_f(x).
\]

It follows that \(O\) is a subset of that given in (14). \(\square\)

Note that the expression for \(\text{Lip}_f\) in the finite dimensional case is also given in [17, page 359]. The next lemma is a convenient building block for our next corollary.

**Lemma 7.3.** Let \(\{f_i\}\) be a sequence of equi-locally Lipschitz functions on a Banach space \(X\). Define \(\Omega : X \to 2^X\) by \(\Omega(x) := \bigcap_i \{\partial f_i(x) : i \in \mathbb{N}\}\) for every \(x \in X\). Suppose that \(f : X \to \mathbb{R}\) is locally Lipschitz and \(\partial_x f = \text{CSC}(\Omega)\). Then \(\text{Lip}_f = \text{usc}(\sup_i \text{Lip}_{f_i})\).

**Proof.** Because \(f^0(y; v) \geq f^0_i(y; v),\) Proposition 7.2(i) (equation (12)) ensures that for every \(i\) we have \(\text{Lip}_f \geq \text{Lip}_{f_i}\); thus \(\text{Lip}_f \geq \sup_i (\text{Lip}_{f_i})\). Since \(\text{Lip}_f\) is \(\text{usc}\), we have \(\text{Lip}_f \geq \text{usc}(\sup_i \text{Lip}_{f_i})\). Fix \(\epsilon > 0\). Since \(\partial_x f(x) = \text{CSC}(\Omega)(x),\) by (1) we have \(\partial_x f(x) \subset \overline{\partial}^{\text{w}*}\Omega(B_x(x))\). Then for \(v \in X\),

\[
f^0(x; v) \leq \sup\{\langle y^*, v \rangle : y^* \in \Omega(y), y \in B_x(x)\} = \sup\{\sup_i f^0_i(y; v) : y \in B_x(x)\} \quad \text{and so,}
\]

\[
\sup_{\|v\| = 1} f^0(x; v) \leq \sup_{\|v\| = 1} \sup_i \text{Lip}_{f_i}(y) : y \in B_x(x)\}.
\]

When \(\epsilon \downarrow 0\) we obtain

\[
\sup_{\|v\| = 1} f^0(x; v) \leq \text{usc}(\sup_i \text{Lip}_{f_i})(x).
\]

By Proposition 7.2(i), we have \(\text{Lip}_f(x) \leq \text{usc}(\sup_i \text{Lip}_{f_i})(x). \square\)
Corollary 7.4. Let \( \{f_i\} \) be a sequence of locally equi-Lipschitz functions on a Banach space \( X \).

(i) If \( X \) be a separable Banach space, then \( \text{usc}(\sup_i \text{Lip}_{f_i}) \) is a locally Lipschitz-constant function. In particular, the maximum of a finite number of local Lipschitz-constant functions is a local Lipschitz-constant function.

(ii) If \( X \) is an arbitrary Banach space, each \( \partial_c f_i \) is a minimal weak* usco, and that \( l : X \to [0, \infty) \) is continuous, then \( \text{usc}(\sup_i \text{Lip}_{f_i}) + l \) is a local Lipschitz-constant function.

Proof. (i) Let \( X \) be separable. By Proposition 4.3, there exists a locally Lipschitz function \( f \) such that

\[
\partial_c f(x) = \text{CSC}(\bigcup \{ \partial_c f_i : i \in \mathbb{N} \})(x) \text{ for every } x \in X.
\]

Thus (i) follows from Lemma 7.3.

(ii) Let \( X \) be an arbitrary Banach space. By Theorem 6.4, there exists a locally Lipschitz function \( f \) such that

\[
\partial_c f(x) = \text{CSC}(\bigcup \{ \partial_c f_i : i \in \mathbb{N} \})(x) + l(x)B_X.
\]

In particular, there exists a locally Lipschitz function \( g : X \to \mathbb{R} \) such that

\[
\partial_c g(x) = \text{CSC}(\bigcup \{ \partial_c f_i : i \in \mathbb{N} \})(x) \text{ for every } x \in X. \tag{15}
\]

Then for \( x \in X \), \( \sup_{\|v\|=1} f^0(x; v) = \sup_{\|v\|=1} g^0(x; v) + l(x) \). This together with Proposition 7.2 imply \( \text{Lip}_f(x) = \text{Lip}_g(x) + l(x) \). Now (15) and Lemma 7.3 imply \( \text{Lip}_g(x) = \text{usc}(\sup_i \text{Lip}_{f_i})(x) \). Putting this together we conclude that

\[
\text{Lip}_f(x) = \text{usc}(\sup_i \text{Lip}_{f_i})(x) + l(x).
\]

When \( X \) admits an equivalent smooth renorm, (ii) also follows from Propositions 7.10, 2.2, and Theorem 5.4. \( \square \)

In contrast to the previous corollary, we present the following example.

Example 7.5. The class of local Lipschitz-constant functions is not closed under either addition or taking minima. Consider the local Lipschitz-constant functions:

\[
f_1(x) := \begin{cases} 
1 & \text{if } x \geq 0 \\
0 & \text{otherwise}
\end{cases} \quad f_2(x) := \begin{cases} 
0 & \text{if } x > 0 \\
1 & \text{otherwise}
\end{cases}.
\]

Because \( f_1 + f_2 \) and \( \min\{f_1, f_2\} \) are not robust usc, they are not local Lipschitz-constant functions.

The following theorem summarizes some of the main results concerning local Lipschitz-constants and Clarke subdifferentials.

Theorem 7.6. (i) On \( \mathbb{R} \), a non-negative function is a local Lipschitz-constant function if and only if it is robust usc.
(ii) On a general Banach space \( X \), every nonnegative topologically robust usc function is a local Lipschitz-constant function.

**Proof.** (i) This follows from [3] which shows that on the real line \( T := [-\lambda, \lambda] \) is a Clarke subdifferential if and only if \( \lambda \) is a robustly usc function. According to Proposition 7.2 (i), (ii) is a reformulation of Theorem 5.4. \( \square \)

**Question.** Does the equivalence of Theorem 7.6(i) remain true in arbitrary Banach spaces? Or in other words, is it enough to assume the function in Theorem 7.6(ii) is only robust usc?

Local Lipschitz-constant functions enjoy a nice calculus which we shall exhibit in the next few results. If \( f_i : X \rightarrow \mathbb{R} \) is locally Lipschitz at \( \tilde{x} \), then the definition directly implies:

\[
\text{Lip}_{k^i}(\tilde{x}) = |k|\text{Lip}_{f_1}(\tilde{x}) \quad \text{for every } k \in \mathbb{R},
\]

\[
\text{Lip}_{f_1 + f_2}(\tilde{x}) \leq \text{Lip}_{f_1}(\tilde{x}) + \text{Lip}_{f_2}(\tilde{x}),
\]

\[
\text{Lip}_{\text{max}\{f_1, f_2\}}(\tilde{x}) \leq \max\{\text{Lip}_{f_1}(\tilde{x}), \text{Lip}_{f_2}(\tilde{x})\},
\]

\[
\text{Lip}_{\text{min}\{f_1, f_2\}}(\tilde{x}) \leq \min\{\text{Lip}_{f_1}(\tilde{x}), \text{Lip}_{f_2}(\tilde{x})\}.
\]

We now establish a chain rule in equality form, which is similar to the chain rule for the sequential subdifferential given by Mordukhovich and Shao in [13].

**Theorem 7.7.** Let \( X \) and \( Y \) be Banach spaces. Assume that \( \Phi : X \rightarrow Y \) is Lipschitz continuous around \( \tilde{x} \) and \( \varphi : X \times Y \rightarrow \mathbb{R} \) is strictly differentiable at \( (\tilde{x}, \Phi(\tilde{x})) \). Then for \( m(x) := \varphi(x, \Phi(x)) \) one has

\[
\text{Lip}_m(\tilde{x}) = \text{Lip}_{\langle \varphi'(\tilde{x}, \tilde{y}), \Phi \rangle}(\tilde{x}).
\]

For a function \( f : X \rightarrow \mathbb{R} \) being strictly differentiable at \( \tilde{x} \), we have \( \text{Lip}_f(\tilde{x}) = \text{Lip}_{f'(\tilde{x})}(\tilde{x}) = \| f'(\tilde{x}) \| \).

**Proof.** As \( \varphi \) is strictly differentiable at \( (\tilde{x}, \tilde{y}) \), for every \( \epsilon > 0 \) there exists \( \delta > 0 \) such that when \( \| x - \tilde{x} \| < \delta, \| y - \tilde{y} \| < \delta \) with \( x \neq y \) we have \( \| \Phi(y) - \Phi(x) \| \leq l \| x - y \| \) and

\[
\frac{|\varphi(y, \Phi(y)) - \varphi(x, \Phi(x)) - \langle \varphi'_x(\tilde{x}, \tilde{y}), y - x \rangle - \langle \varphi'_y(\tilde{x}, \tilde{y}), \Phi(y) - \Phi(x) \rangle|}{\| y - x \|} < \epsilon.
\]

Then

\[
\frac{|\varphi(y, \Phi(y)) - \varphi(x, \Phi(x)) - \langle \varphi'_x(\tilde{x}, \tilde{y}), y - x \rangle - \langle \varphi'_y(\tilde{x}, \tilde{y}), \Phi(y) - \Phi(x) \rangle|}{\| y - x \|} < (1 + l)\epsilon.
\]

This implies

\[
\frac{|\varphi(y, \Phi(y)) - \varphi(x, \Phi(x))|}{\| y - x \|} \leq \frac{|\langle \varphi'_x(\tilde{x}, \tilde{y}), y - x \rangle + \langle \varphi'_y(\tilde{x}, \tilde{y}), \Phi(y) - \Phi(x) \rangle|}{\| y - x \|} + (1 + l)\epsilon, \tag{17}
\]

\[
\frac{|\varphi(y, \Phi(y)) - \varphi(x, \Phi(x))|}{\| y - x \|} \geq \frac{|\langle \varphi'_x(\tilde{x}, \tilde{y}), y - x \rangle + \langle \varphi'_y(\tilde{x}, \tilde{y}), \Phi(y) - \Phi(x) \rangle|}{\| x - y \|} - (1 + l)\epsilon. \tag{18}
\]

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When $y, x \to \bar{x}$, taking the limsup for (17) and (18), we have
\[
\text{Lip}(\varphi'_x(x, \bar{y}) + (\varphi'_x(x, \bar{y}), \Phi))(\bar{x}) - (1 + l)\epsilon \leq \text{Lip}_m(\bar{x}) \leq \text{Lip}(\varphi'_x(x, \bar{y}) + (\varphi'_x(x, \bar{y}), \Phi))(\bar{x}) + (1 + l)\epsilon.
\]
When $\epsilon \downarrow 0$, we obtain (16). When $f : X \to \mathbb{R}$ is strictly differentiable at $\bar{x}$, with $\varphi(x, \Phi(x)) = f(x)$ we may apply (16).

Letting $\Phi : X \to \mathbb{R}^2$ be defined by $\Phi(x) := (f_1(x), f_2(x))$ and $\varphi : \mathbb{R}^2 \to \mathbb{R}$ defined by $\varphi(y_1, y_2) := y_1 y_2$ or $y_1/y_2$ we obtain the following versions of the product and quotient rules as direct consequences of the previous result.

**Corollary 7.8 (product rule).** Let $f_i : X \to \mathbb{R}$, $i = 1, 2$ be Lipschitz continuous around $\bar{x}$. Then:
\[
\text{Lip}_{f_1, f_2}(\bar{x}) = \text{Lip}_{f_1(x)f_2 + f_1 f_2}(\bar{x}) \leq \text{Lip}_{f_1(f_2)}(\bar{x}) + \text{Lip}_{f_1, f_2(\bar{x})}(\bar{x}).
\]

**Corollary 7.9 (quotient rule).** Let $f_i : X \to \mathbb{R}$, $i = 1, 2$ be Lipschitz continuous around $\bar{x}$. Let $f_2(\bar{x}) \neq 0$. Then:
\[
\text{Lip}_{f_1/f_2}(\bar{x}) = \frac{\text{Lip}_{f_2(\bar{x})f_1 - f_1(\bar{x})f_2(\bar{x})}}{f_2(\bar{x})} \leq \frac{\text{Lip}_{f_2(\bar{x})f_1(\bar{x})} + \text{Lip}_{f_1(\bar{x})f_2(\bar{x})}}{f_2(\bar{x})}.
\]

The following result provides conditions ensuring that $\text{Lip}_f$ is topologically robust usc, and so many of our prior results apply.

**Proposition 7.10.** Let $X$ be a Banach space with an equivalent Gâteaux smooth renorm and $f : X \to \mathbb{R}$ be locally Lipschitz with minimal Clarke subdifferential. Then $\text{Lip}_f$ is topologically robust usc.

**Proof.** As $\partial_c f$ is a minimal norm-to-weak* cusco, $\partial_c f$ is single-valued and norm-to-weak* continuous at the points of a residual set $G$ [15]. Then
\[
\partial_c f = \text{CSC}\{f'(x) : x \in G\} \text{ and so,}
\]
\[
\text{Lip}_f(x) = \limsup_{y \in G, y \to x} \|f'(y)\| \text{ for every } x \in X. \tag{19}
\]
It suffices to show that $\text{Lip}_f$ is quasi lsc on $X$ by Proposition 2.1(i). Let $\epsilon > 0$ and $x \in X$. For every neighborhood $U$ of $x$, there exists $y \in G$ such that $\|f'(y)\| > \text{Lip}_f(x) - \epsilon$ by (19). Since $\partial_c f$ is norm-to-weak* continuous, $f'(z) \rightharpoonup f'(y)$ if $z \in G$ and $z \to y$. As the dual norm on $X^*$ is weak* lsc, we have
\[
\liminf_{z \to y, z \in G} \|f'(z)\| \geq \|f'(y)\|.
\]
Thus, there exists $\delta > 0$ such that $B_\delta(y) \subset U$ and for $z \in B_\delta(y) \cap G$ we have $\|f'(z)\| \geq \|f'(y)\| - \epsilon$, so $\|f'(z)\| \geq \text{Lip}_f(x) - 2\epsilon$. By (19) we have
\[
\text{Lip}_f(z) \geq \text{Lip}_f(x) - 2\epsilon \text{ when } z \in B_\delta(y) \subset U.
\]
Hence $\text{Lip}_f$ is quasi lsc at $x$. 

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**A minimality criterion:** Let $X$ be a Banach space with an equivalent Gâteaux smooth renorm. Suppose $f$ is a locally Lipschitz function on an open subset $A$ with the property that wherever $f$ is Gâteaux differentiable it is strictly Gâteaux differentiable. Then $f$ has a minimal Clarke subdifferential. This is the case in particular when $f$ is convex or $C^1$.

**Proof.** By [16] and [2], for every $x \in A$ we have
\[
\partial_c f(x) = \bigcap_{\delta > 0} \overline{co}^\gamma \{ f'(y) : y \in B_{\delta}(x) \} = \overline{co}^\gamma \{ x^* : f'(x_n) \xrightarrow{w} x^* \text{ with } x_n \to x \}.
\]
Let $T : A \to 2^{X^*}$ be a weak* cusco with $T \subset \partial_c f$. Whenever $f'(x)$ exists, $\partial_c f(x) = \{ f'(x) \}$, thus $T(x) = \{ f'(x) \}$. As $T$ is a weak* cusco and $f'$ exists densely on $A$, we have $T \supset \text{CSC}(f') = \partial_c f$. Hence $T = \partial_c f$ and $\partial_c f$ is minimal on $A$. \hfill \Box

**Example 7.11.** The local Lipschitz-constant function of a nowhere monotone function with bounded derivative is not topologically robust usc. Indeed, if $f : \mathbb{R} \to \mathbb{R}$ is nowhere monotone and has bounded derivative [18, page 337], then $f'$ is continuous generically with value 0. If $\text{Lip}_f$ is lower semicontinuous at $x$, then for every $\epsilon > 0$ we have
\[ 0 = |f'(y)| = \text{Lip}_f(y) > \text{Lip}_f(x) - \epsilon, \]
when $f'$ is continuous at $y$ being sufficiently close to $x$. Hence $\text{Lip}_f(x) = 0$ whenever $\text{Lip}_f$ is continuous at $x$. As $\text{Lip}_f \neq 0$, $\text{Lip}_f$ is not topologically robust usc. By Proposition 2.2, $\text{Lip}_f$ is not the upper envelope of the supremum of a countable topologically robust usc functions either. However, $\text{Lip}_f$ is robust usc by Proposition 7.1.

While most naturally occurring Lipschitz functions have minimal Clarke subdifferentials [2], the following says that these and other minimal Clarke subdifferentials are naturally paired with maximal Clarke subdifferentials! Recall that a real-valued locally Lipschitz function $f$ defined on a nonempty open subset $A$ of a Banach space $X$ is called essentially strictly differentiable on $A$ if for every $y \in X$ the set
\[ \{ x \in A : f^0(x; y) \neq -f^0(x; -y) \}, \]
is a Haar null subset of $X$.

**Corollary 7.12.** Let $X$ be a Banach space with an equivalent Gâteaux smooth renorm and $f : X \to \mathbb{R}$ be locally Lipschitz with $\partial_c f$ being minimal. Then $Df := \text{Lip}_f B_{X^*}$ is a Clarke subdifferential map. In particular, this holds when $f$ is essentially strictly differentiable.

**Proof.** This follows from Theorem 5.4 and Proposition 7.10. Note that each essentially strictly differentiable function on an arbitrary Banach space has a minimal Clarke subdifferential [2, 5]. \hfill \Box

We conclude this section by establishing a norm convergence result for derivatives.

**Theorem 7.13.** Let $X$ be a reflexive Banach space with a Fréchet differentiable norm. Let $f : X \to \mathbb{R}$ be a locally Lipschitz function with $\partial_c f = kB_{X^*}$ for a topologically robust usc function $k : X \to [0, +\infty)$. Then for every $x \in X$ we have
\[ k(x)S_{X^*} \subseteq \{ y^* : \exists f(y_n) \xrightarrow{w} y^* \text{ with } y_n \to x \}. \]
Proof. Fix \( x^* \in S_X \). Since \( X \) is reflexive, \( x^* \) is norm attaining. Select a vector \( v \in S_X \) with \( \langle x^*, v \rangle = 1 \). We have

\[
\text{Lip}_f(x) = k(x) = \langle k(x)x^*, v \rangle \leq f^0(x; v).
\]

Because \( X \) is reflexive, it follows from Preiss' Theorem [16] that \( f \) is densely Fréchet differentiable and

\[
f^0(x; v) = \limsup \{ f'(y), v : y \to x \} \leq \limsup \{ \| f'(y) \| : y \to x \} \leq \text{Lip}_f(x).
\]

Hence for some \( y_n \to x \), \( \langle f'(y_n), v \rangle \to \text{Lip}_f(x) \), and so

\[
\| f'(y_n) \| \to \text{Lip}_f(x) = k(x).
\]

If \( k(x) = 0 \), then \( f'(y_n) \rightharpoonup 0 \). If \( k(x) \neq 0 \), for \( n \) large we have \( \| f'(y_n) \| \neq 0 \) and

\[
\lim_{n \to \infty} \left\langle \frac{f'(y_n)}{\| f'(y_n) \|}, v \right\rangle = 1 = \langle x^*, v \rangle.
\]

As \( \| \cdot \| \) is Fréchet differentiable, \( \| f'(y_n) / \| f'(y_n) \| - x^* \| \to 0 \), thus \( f'(y_n) \to k(x)x^* \) in norm. \qed

When \( X \) is finite dimensional, in Theorem 7.13 one can choose the \( y_n \)'s to lie outside any given Lebesgue null set.

## 8 \( \beta \)-subdifferentiability via local Lipschitz-constant functions

Assume that \( f \) is a locally Lipschitz function defined on a nonempty open subset \( A \) of a finite dimensional space \( X \), by Proposition 7.2 (i), for every \( x \in A \) both \( f^0(x; \cdot) \) and \( (-f)^0(x; \cdot) \) attain \( \text{Lip}_f(x) \) in some direction. In this section, we will see that if \( f \) is not differentiable at \( x \), then either \( f^-(x; \cdot) \) or \( (-f)^-(x; \cdot) \) must be less than \( \text{Lip}_f(x) \) for every direction (Theorem 8.1(ii)). A bornology on \( X \), denoted by \( \beta \), is any family of bounded sets \( S \) whose union is all of \( X \), which is closed under reflection through the origin, under multiplication by positive scalars and is directed upwards. For a locally Lipschitz \( f : X \to \mathbb{R} \), its \( \beta \)-subdifferential [14] and \( W\beta \)-subdifferential at \( x \) are defined respectively as

\[
\partial f(x) := \{ x^* \in X^* : \liminf_{t \downarrow 0} \inf_{s \in S} t^{-1} (f(x + ts) - f(x) - \langle x^*, ts \rangle) \geq 0 \text{ for all } S \in \beta \},
\]

\[
\partial W\beta f(x) := \{ x^* \in X^* : \limsup_{t \downarrow 0} \inf_{s \in S} t^{-1} (f(x + ts) - f(x) - \langle x^*, ts \rangle) \geq 0 \text{ for all } S \in \beta \}.
\]

We now show that one can always find a maximal \( \beta \)-subderivative whenever the one-sided directional derivatives equal the local Lipschitz-constant. Our proof follows [9].

**Theorem 8.1.** Let \( X \) be a Banach space and let \( f : X \to \mathbb{R} \) be Lipschitz near \( x \in X \). Fix \( u \in S_X \), and suppose the norm of \( X \) is \( \beta \)-differentiable at \( u \) with \( \beta \)-derivative \( u^* \).

(i) If \( f^+(x; u) = \text{Lip}_f(x) \), then \( \text{Lip}_f(x) u^* \in \partial W\beta f(x) \).

(ii) If \( f^-(x; u) = \text{Lip}_f(x) \), then \( \text{Lip}_f(x) u^* \in \partial \beta f(x) \).
Proof. (i) Let \( 1 > \epsilon > 0 \). Fix \( S \in \beta \) with \( \|s\| \leq 1 \) for \( s \in S \). Choose \( 0 < \nu \leq \epsilon \) such that

\[
\|u + ts\| - \|u\| \leq \langle u^*, ts \rangle + \epsilon|t| \quad \text{for all } |t| \leq \nu, \text{ and } s \in S.
\]

By the definitions of \( f^+(x; u) \) and \( \text{Lip}_f(x) \), for this \( \nu \) we may choose \( \delta > 0 \) and \( t_n \downarrow 0 \) such that \( t_n < \delta \);

\[
\frac{f(x + t_n u) - f(x)}{t_n} > f^+(x; u) - \nu^2, \quad \text{and}
\]

\[
|f(x + y) - f(x + z)| \leq (\nu^2 + \text{Lip}_f(x))\|y - z\|,
\]

whenever \( \|y - x\| \leq \delta, \|z - x\| \leq \delta \). Let \( t'_n = \nu t_n \). We have, for all \( s \in S \),

\[
f(x + t'_n s) - f(x) = f(x + t'_n s) - f(x + t_n u) + f(x + t_n u) - f(x) \\
\geq -(\nu^2 + \text{Lip}_f(x))\|t'_n s - t_n u\| + (f^+(x; u) - \nu^2) t_n \quad \text{(by } (21)\text{)} \\
= -\text{Lip}_f(x)(\|t'_n s - t_n u\| - \|t_n u\|) - (\nu^2 t_n + \nu^2 \|t'_n s - t_n u\|) \quad \text{(since } f^+(x; u) = \text{Lip}_f(x)\text{)} \\
= -\text{Lip}_f(x) t_n(\|u - \nu s\| - \|u\|) - \nu^2 t_n(1 + \|\nu s - u\|) \\
\geq -\text{Lip}_f(x) t_n(\|u^* - \nu s\| + \nu \|u^* - s\|) - 3t'_n \epsilon \quad \text{(by } (20)\text{)} \\
= \langle \text{Lip}_f(x) u^*, t'_n s \rangle - t'_n(\text{Lip}_f(x) + 3) \epsilon.
\]

That is, for \( t'_n \) sufficiently small we have

\[
\inf_{s \in S} \frac{f(x + t'_n s) - f(x) - \langle \text{Lip}_f(x) u^*, t'_n s \rangle}{t'_n} \geq -(\text{Lip}_f(x) + 3) \epsilon, \quad \text{and so}
\]

\[
\limsup_{t \downarrow 0} \inf_{s \in S} \frac{f(x + ts) - f(x) - \langle \text{Lip}_f(x) u^*, ts \rangle}{t} \geq -(\text{Lip}_f(x) + 3) \epsilon.
\]

Letting \( \epsilon \downarrow 0 \), we obtain \( \text{Lip}_f(x) u^* \in \partial_{W\beta} f(x) \).

(ii). We will proceed as in (i), with the exception that (21) is replaced by:

\[
\frac{f(x + tu) - f(x)}{t} > f^-(x; u) - \nu^2 \quad \text{for all } 0 < t < \delta.
\]

When \( t' = \nu t \) where \( 0 < t < \delta \), we have

\[
f(x + t' s) - f(x) \geq \langle \text{Lip}_f(x) u^*, t' s \rangle - t'(\text{Lip}_f(x) + 3) \epsilon, \quad \text{uniformly for } s \in S.
\]

Then for all \( t' \) with \( 0 < t' < \nu \delta \), we have

\[
\inf_{s \in S} \frac{f(x + t' s) - f(x) - \langle \text{Lip}_f(x) u^*, t' s \rangle}{t'} \geq -(\text{Lip}_f(x) + 3) \epsilon.
\]

Hence \( \text{Lip}_f(x) u^* \in \partial_{\beta} f(x) \). \qed

Corollary 8.2 (Fitzpatrick [9]). Let \( X \) be a Banach space and \( f : X \to \mathbb{R} \) be locally Lipschitz. Suppose for some direction \( u \in S_X \), the two-sided directional derivative \( f'(x; u) = \text{Lip}_f(x) \) at \( x \). If the norm of \( X \) is \( \beta \)-differentiable at \( u \) with derivative \( u^* \), then \( f \) is \( \beta \)-differentiable at \( x \) and \( f'(x) = \text{Lip}_f(x) u^* \).
Proof. The assumption gives
\[
\lim_{t \to 0} \frac{f(x + tu) - f(x)}{t} = \operatorname{Lip}_f(x), \quad \text{and} \quad \lim_{t \to 0} \frac{f(x + tu) - f(x)}{t} = \operatorname{Lip}_f(x).
\]
Since \( \operatorname{Lip}_f(x) = \operatorname{Lip}_{-f}(x) \), we have
\[
\lim_{t \to 0} \frac{(-f)(x + (-t)(-u)) - (-f)(x)}{-t} = \operatorname{Lip}_{-f}(x).
\]
By Theorem 8.1(ii), we have \( \operatorname{Lip}_f(x)u^* \in \partial_\beta f(x) \) and \( \operatorname{Lip}_f(x)(-u^*) \in \partial_\beta (-f)(x) \), hence \( f \) is \( \beta \)-differentiable at \( x \).

Corollary 8.3. Let \( C \) be a non-empty closed subset of a Banach space \( X \). Assume \( x \not\in C \) and the metric distance \( d_C(x) = \|x - \bar{x}\| \) for some \( \bar{x} \in C \). If the norm of \( X \) is \( \beta \)-differentiable, then \( \partial_\beta (-d_C)(x) \neq \emptyset \). In particular, when \( X \) is finite dimensional with a smooth norm, \( -d_C \) is Fréchet subdifferentiable at every \( x \in X \setminus C \).

Proof. For \( 0 < \lambda < 1 \), we have \( d_C(x + \lambda(\bar{x} - x)) = (1 - \lambda)d_C(x) \). Then
\[
\frac{-d_C(x + \lambda(\bar{x} - x)) + d_C(x)}{\lambda} = d_C(x).
\]
As \( \lambda \to 0 \), we have \( (-d_C)^-(x; (\bar{x} - x)/d_C(x)) = 1 \). Theorem 8.1 shows \( \partial_\beta (-d_C)(x) \neq \emptyset \). For every \( y \) in the open segment \( (x, \bar{x}) \), the two-sided derivative \( (-d_C)'(y; (\bar{x} - x)/d_C(x)) = 1 \), thus \( -d_C \) is \( \beta \)-differentiable at \( y \).

Combining Theorem 8.1 (i) and Proposition 7.2 (ii), we obtain:

Corollary 8.4. Let \( X \) be a finite-dimensional Banach space with a smooth norm. If \( f : X \to \mathbb{R} \) is locally Lipschitz, then the set \( \{x \in X : \partial_{W_\beta} f(x) \cap (\operatorname{Lip}_f(x) S_X) \neq \emptyset \} \) is residual in \( X \).

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References


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