LANDEN SURVEY

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Abstract. Landen transformations are maps on the coefficients of an integral
that preserve its value. We present a brief survey of their appearance in the
literature.

To Henry, who provides inspiration, taste and friendship

1. IN THE BEGINNING THERE WAS GAUSS

In the year 1985, one of us had the luxury of attending a graduate course on
Elliptic Functions given by Henry McKean at the Courant Institute. Among the
many beautiful results he described in his unique style, there was a calculation of
Gauss: take two positive real numbers \(a\) and \(b\), with \(a > b\), and form a new pair
by replacing \(a\) with the arithmetic mean \((a + b)/2\) and \(b\) with the geometric mean
\(\sqrt{ab}\). Then iterate:

\[
\begin{align*}
    a_{n+1} &= \frac{a_n + b_n}{2}, \\
    b_{n+1} &= \sqrt{a_n b_n}
\end{align*}
\]

starting with \(a_0 = a\) and \(b_0 = b\). Gauss [28] was interested in the initial conditions
\(a = 1\) and \(b = \sqrt{2}\). The iteration generates a sequence of algebraic numbers which
rapidly become impossible to describe explicitly, for instance,

\[
a_3 = \frac{1}{2^3} \left( (1 + \sqrt{2})^2 + 2\sqrt{2} \sqrt{2} \sqrt{1 + \sqrt{2}} \right)
\]

is a root of the polynomial

\[
G(a) = 16777216a^8 - 16777216a^7 + 5242880a^6 - 10747904a^5 \\
+ 942080a^4 - 1896448a^3 + 4436a^2 - 59840a + 1.
\]

The numerical behavior is surprising; \(a_6\) and \(b_6\) agree to 87 digits. It is simple to
check that

\[
\lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n.
\]

See (6.1) for details. This common limit is called the arithmetic-geometric mean
and is denoted by \(\text{AGM}(a, b)\). It is the explicit dependence on the initial condition
that is hard to discover.

Gauss computed some numerical values and observed that

\[
a_{11} \sim b_{11} \sim 1.198140235
\]
and then he recognized the reciprocal of this number as a numerical approximation to the elliptic integral

\[(1.5) \quad I = \frac{2}{\pi} \int_0^1 \frac{dt}{\sqrt{1 - t^4}}.\]

It is unclear to the authors how Gauss recognized this number - he simply knew it. (Stirling’s tables may have been a help; [11] contains a reproduction of the original notes and comments.) He was particularly interested in the evaluation of this definite integral as it provides the length of a lemniscate. In his diary Gauss remarked, "This will surely open up a whole new field of analysis" [26, 14].

Gauss’ procedure to find an analytic expression for \(AGM(a, b)\) began with the elementary observation

\[(1.6) \quad AGM(a, b) = AGM\left(\frac{a + b}{2}, \sqrt{ab}\right)\]

and the homogeneity condition

\[(1.7) \quad AGM(\lambda a, \lambda b) = \lambda AGM(a, b).\]

He used (1.6) with \(a = (1 + \sqrt{k})^2\) and \(b = (1 - \sqrt{k})^2\), with \(0 < k < 1\), to produce

\[
AGM(1 + k + 2\sqrt{k}, 1 + k - 2\sqrt{k}) = AGM(1 + k, 1 - k).
\]

He then used the homogeneity of \(AGM\) to write

\[
AGM(1 + k + 2\sqrt{k}, 1 + k - 2\sqrt{k}) = AGM((1 + k)(1 + k^*), (1 + k)(1 - k^*)) = (1 + k)AGM(1 + k^*, 1 - k^*),
\]

with

\[(1.8) \quad k^* = \frac{2\sqrt{k}}{1 + k}.\]

This resulted in the functional equation

\[(1.9) \quad AGM(1 + k, 1 - k) = (1 + k)AGM(1 + k^*, 1 - k^*).\]

In his analysis of (1.9), Gauss substituted the power series

\[(1.10) \quad \frac{1}{AGM(1 + k, 1 - k)} = \sum_{n=0}^{\infty} a_n k^{2n}\]

into (1.9) and, solved an infinite system of nonlinear equations, to produce

\[(1.11) \quad a_n = 2^{-2n} \binom{2n}{n}^2.\]

Then he recognized the series as that of an elliptic integral to obtain

\[(1.12) \quad \frac{1}{AGM(1 + k, 1 - k)} = \frac{2}{\pi} \int_0^{\pi/2} \frac{dx}{\sqrt{1 - k^2 \sin^2 x}}.\]

This is a remarkable tour de force.

The function

\[(1.13) \quad K(k) = \int_0^{\pi/2} \frac{dx}{\sqrt{1 - k^2 \sin^2 x}}\]
is the elliptic integral of the first kind. It can also be written in the algebraic form

\[(1.14) \quad K(k) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}.\]

In this notation, (1.9) becomes

\[(1.15) \quad K(k^*) = (1 + k)K(k).\]

This is the Landen transformation for the complete elliptic integral. John Landen [35], the namesake of the transformation, studied related integrals: for example,

\[(1.16) \quad \kappa := \int_0^1 \frac{dx}{\sqrt{x^2(1-x^2)}}.\]

He derived identities such as

\[(1.17) \quad \kappa = \varepsilon \sqrt{\varepsilon^2 - \pi}, \quad \text{where} \quad \varepsilon := \int_0^{\pi/2} \sqrt{2 - \sin^2 \theta} \ d\theta,
\]

proven mainly by suitable changes of variables in the integral for \(\varepsilon\). In [48] is a historical account of Landen’s work, including the above identities.

The reader will find in [14] and [41] proofs in a variety of styles. In trigonometric form, the Landen transformation states that

\[(1.18) \quad G(a, b) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta}}\]

is invariant under the change of parameters \((a, b) \mapsto (a + b/2, \sqrt{ab})\). D. J. Newman [44] presents a very clever proof: the change of variables \(x = b \tan \theta\) yields

\[(1.19) \quad G(a, b) = \frac{1}{2} \int_{-\infty}^{\infty} \frac{dx}{\sqrt{(a^2 + x^2)(b^2 + x^2)}}.\]

Now let \(x \mapsto x + \sqrt{x^2 + ab}\) to complete the proof. Many of the above identities can now be searched for and ‘proven’ on a computer [11].

2. An interlude: the quartic integral

The evaluation of definite integrals of rational functions is one of the standard topics in Integral Calculus. Motivated by the lack of success of symbolic languages, we began a systematic study of these integrals. A posteriori, one learns that even rational functions are easier to deal with. Thus we start with one having a power of a quartic in its denominator. The evaluation of the identity

\[(2.1) \quad \int_0^\infty \frac{dx}{(x^4 + 2ax^2 + 1)^{m+1}} = \frac{\pi}{2^{m+3/2} (a + 1)^{m+1/2}} P_m(a),\]

where

\[(2.2) \quad P_m(a) = \sum_{l=0}^{m} d_l(m)a^l\]

with

\[(2.3) \quad d_l(m) = 2^{-2m} \sum_{k=l}^m 2^k \binom{2m-2k}{m-k} \binom{m+k}{m} \binom{k}{l},\]

was first established in [6].
A standard hypergeometric argument yields
\begin{equation}
(2.4) \quad P_m(a) = P_m^{(\alpha, \beta)}(a),
\end{equation}
where
\begin{equation}
(2.5) \quad P_m^{(\alpha, \beta)}(a) = \sum_{k=0}^{m} (-1)^{m-k} \binom{m+\beta}{m-k} \binom{m+k+\alpha+\beta}{k} 2^{-k}(a+1)^k
\end{equation}

is the classical Jacobi polynomial; the parameters \( \alpha \) and \( \beta \) are given by \( \alpha = m + \frac{1}{2} \) and \( \beta = -m - \frac{1}{2} \). A general description of these functions and their properties are given in [1]. The twist here is that they depend on \( m \), which means most of the properties of \( P_m \) had to be proven from scratch. For instance, \( P_m \) satisfies the recurrence
\begin{align*}
P_m(a) &= \frac{(2m-3)(4m-3)a}{4m(m-1)(a-1)} P_{m-2}(a) - \frac{(4m-3)a(a+1)}{2m(m-1)(a-1)} P'_{m-2}(a) \\
&\quad + \frac{4m(a^2-1)+1-2a^2}{2m(a-1)} P_{m}(a).
\end{align*}
This cannot be obtained by replacing \( \alpha = m + \frac{1}{2} \) and \( \beta = -m - \frac{1}{2} \) in the standard recurrence for the Jacobi polynomials.

The polynomials \( P_m(a) \) makes a surprising appearance in the expansion
\begin{equation}
(2.6) \quad \sqrt{a + \sqrt{1 + c}} = \sqrt{a + 1} \left[ 1 - \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \frac{P_{k-1}(a) c^k}{2^{k+1}(a+1)^k} \right]
\end{equation}
as described in [8]. The special case \( a = 1 \) appears in [21], page 191, exercise 21. Ramanujan [3] had a more general expression, but only for the case \( c = a^2 \):
\begin{equation}
(2.7) \quad (a + \sqrt{1 + a^2})^n = 1 + na + \sum_{k=2}^{\infty} \frac{b_k(n) a^k}{k!},
\end{equation}
where, for \( k \geq 2 \),
\begin{equation}
(2.8) \quad b_k(n) = \begin{cases} 
n^2(n^2 - 2^2)(n^2 - 4^2) \cdots (n^2 - (k-2)^2) & \text{if } k \text{ is even}, \\
n(n^2 - 1^2)(n^2 - 3^2) \cdots (n^2 - (k-2)^2) & \text{if } k \text{ is odd},
\end{cases}
\end{equation}
This result appears in Berndt [3] as Corollary 2 to Entry 14 and is machine-checkable, as are many of the identities in this section.

The coefficients \( d_l(m) \) in (2.3) have many interesting properties:

- They form a unimodal sequence: there exists an index \( 0 \leq m^* \leq m \) such that \( d_j(m) \) increases up to \( j = m^* \) and decreases from then on. See [5] for a proof of the more general statement: If \( P(x) \) is a polynomial with nondecreasing, nonnegative coefficients, then the coefficient sequence of \( P(x+1) \) is unimodal.

- They form a log-concave sequence: define the operator \( \mathcal{L}(\{a_k\}) := \{a_k^2 - a_{k-1}a_{k+1}\} \) acting on sequences of positive real numbers. A sequence \( \{a_k\} \) is called log-concave if its image under \( \mathcal{L} \) is again a sequence of positive numbers; i.e. \( a_k^2 - a_{k-1}a_{k+1} \geq 0 \). Note that this condition is satisfied if and only if the sequence \( \{b_k := \log(a_k)\} \) is concave, hence the name. We refer the reader to [49] for a detailed introduction.
The log-concavity of $d_l(m)$ was established in [33] using Computer Algebra techniques: in particular, cylindrical algebraic decompositions as developed in [23] and [25].

- They produce interesting polynomials: in [10] one finds the representation

$$d_l(m) = \frac{A_{l,m}}{l!m!2^{m+l}},$$

with

$$A_{l,m} = \alpha_l(m) \prod_{k=1}^{m} (4k - 1) - \beta_l(m) \prod_{k=1}^{m} (4k + 1).$$

Here $\alpha_l$ and $\beta_l$ are polynomials in $m$ of degrees $l$ and $l-1$, respectively. For example, $\alpha_1(m) = 2m + 1$ and $\beta_1(m) = 1$, so that the coefficient of the linear term of $P_m(a)$ is

$$d_1(m) = \frac{1}{m!2^{m+1}} \left( (2m + 1) \prod_{k=1}^{m} (4k - 1) - \prod_{k=1}^{m} (4k + 1) \right).$$

J. Little established in [38] the remarkable fact that the polynomials $\alpha_l(m)$ and $\beta_l(m)$ have all their roots on the vertical line $\text{Re } m = -\frac{1}{2}$.

When we showed this to Henry, he simply remarked: *the only thing you have to do now is to let $l \to \infty$ and get the Riemann hypothesis.* The proof in [38] consists in a study of the recurrence

$$y_{l+1}(s) = 2sy_l(s) - \left( s^2 - (2l - 1)^2 \right) y_{l-1}(s),$$

satisfied by $\alpha_l((s - 1)/2)$ and $\beta_l((s - 1)/2)$. There is no Number Theory in the proof, so it is not likely to connect to the Riemann zeta function $\zeta(s)$, but one never knows.

The arithmetical properties of $A_{l,m}$ are beginning to be elucidated. We have shown that their 2-adic valuation satisfies

$$\nu_2(A_{l,m}) = \nu_2((m + 1 - l)/2) + l,$$

where $(a)_k = a(a+1)(a+2)\cdots(a+k-1)$ is the Pochhammer symbol. This expression allows for a combinatorial interpretation of the block structure of these valuations. See [2] for details.

3. The incipient rational Landen transformation

The clean analytic expression in (2.1) is not expected to extend to rational functions of higher order. In our analysis we distinguish according to the domain of integration: the finite interval case, mapped by a bilinear transformation to $[0, \infty)$, and the whole line. In this section we consider the definite integral,

$$U_0(a, b; c, d, e) = \int_0^\infty \frac{cx^4 + dx^2 + e}{x^6 + ax^4 + bx^2 + 1} \, dx,$$

as the simplest case on $[0, \infty)$. The case of the real line is considered below. The integrand is chosen to be even by necessity: *none of the techniques in this section work for the odd case.* We normalize two of the coefficients in the denominator in order to reduce the number of parameters. The standard approach for the evaluation of (3.1) is to introduce the change of variables $x = \tan \theta$. This leads to an intractable trigonometric integral.
A different result is obtained if one first symmetrizes the denominator: we say that a polynomial of degree $d$ is reciprocal if $Q_d(1/x) = x^{-d}Q_d(x)$, that is, the sequence of its coefficients is a palindrome. Observe that if $Q_d$ is any polynomial of degree $d$, then

$$T_{2d}(x) = x^dQ_d(x)Q_d(1/x)$$

is a reciprocal polynomial of degree $2d$. For example, if

$$Q_6(x) = x^6 + ax^4 + bx^2 + 1,$$

then

$$T_{12}(x) = x^{12} + (a + b)x^{10} + (a + b + ab)x^8 + (2 + a^2 + b^2)x^6 + (a + b + ab)x^4 + (a + b)x^2 + 1.$$

The numerator and denominator in the integrand of (3.1) are now scaled by $x^6Q_6(1/x)$ to produce a new integrand with reciprocal denominator:

$$U_6 = \int_0^\infty \frac{S_{10}(x)}{T_{12}(x)} \, dx,$$

where we write

$$S_{10}(x) = \sum_{j=0}^5 s_j x^{2j} \text{ and } T_{12}(x) = \sum_{j=0}^6 t_j x^{2j}.$$

The change of variables $x = \tan \theta$ now yields

$$U_6 = \int_0^{\pi/2} \frac{S_{10}(\tan \theta) \cos^{10}(\theta)}{T_{12}(\tan \theta) \cos^{12}(\theta)} \, d\theta.$$

Now let $w = \cos 2\theta$ and use $\sin^2 \theta = \frac{1}{2}(1 - w)$ and $\cos^2 \theta = \frac{1}{2}(1 + w)$ to check that the numerator and denominator of the new integrand,

$$S_{10}(\tan \theta) \cos^{10}(\theta) = \sum_{j=0}^5 s_j \sin^{2j} \theta \cos^{10-2j} \theta$$

and

$$T_{12}(\tan \theta) \cos^{12}(\theta) = \sum_{j=0}^6 t_j \sin^{2j} \theta \cos^{12-2j} \theta$$

are both polynomials in $w$. The mirror symmetry of $T_{12}$, reflected in $t_j = t_{6-j}$, shows that the new denominator is an even polynomial in $w$. The symmetry of cosine about $\pi/2$ shows that the terms with odd power of $w$ have a vanishing integral. Thus, with $\psi = 2\theta$, and using the symmetry of the integrand to reduce the integral from $[0, \pi]$ to $[0, \pi/2]$, we obtain

$$U_6 = \int_0^{\pi/2} \frac{r_4 \cos^4 \psi + r_2 \cos^2 \psi + r_0}{q_6 \cos^6 \psi + q_4 \cos^4 \psi + q_2 \cos^2 \psi + q_0} \, d\psi.$$
The parameters \( r_j, q_j \) have explicit formulas in terms of the original parameters \( U_6 \). This even rational function of \( \cos \psi \) can now be expressed in terms of \( \cos 2 \psi \) to produce (letting \( \theta \leftrightarrow 2 \psi \))

\[
U_6 = \int_0^\infty \frac{\alpha_2 \cos^2 \theta + \alpha_1 \cos \theta + \alpha_0}{\beta_3 \cos^3 \theta + \beta_2 \cos^2 \theta + \beta_1 \cos \theta + \beta_0} \, d\theta.
\]

The final change of variables \( y = \tan \frac{\theta}{2} \) yields a new rational form of the integrand:

\[
(3.7) \quad U_6 = \int_0^\infty \frac{c_1 y^4 + d_1 y^2 + e_1}{y^6 + a_1 y^4 + b_1 y^2 + 1} \, dy.
\]

Keeping track of the parameters, we have established:

**Theorem 3.1.** The integral

\[
(3.8) \quad U_6 = \int_0^\infty \frac{c x^4 + d x^2 + e}{x^6 + a x^4 + b x^2 + 1} \, dx
\]

is invariant under the change of parameters

\[
(3.9) \quad a_1 \leftarrow \frac{ab + 5a + 5b + 9}{(a + b + 2)^{4/3}},
\]

\[
b_1 \leftarrow \frac{a + b + 6}{(a + b + 2)^{2/3}},
\]

for the denominator parameters and

\[
c_1 \leftarrow \frac{c + d + e}{(a + b + 2)^{2/3}},
\]

\[
d_1 \leftarrow \frac{(b + 3)c + 2d + (a + 3)e}{a + b + 2},
\]

\[
e_1 \leftarrow \frac{c + e}{(a + b + 2)^{1/3}}
\]

for those of the numerator.

Theorem 3.1 is the precise analogue of the elliptic Landen transformation (1.1) for the case of a rational integrand. We call (3.9) a rational Landen transformation. This construction was first presented in [7].

3.1. **Even rational Landen transformations.** More generally, there is a similar transformation of coefficients for any even rational integrand; details appear in [9]. We call these even rational Landen Transformations. The obstruction in the general case comes from (3.6); one does not get a polynomial in \( w = \cos 2\theta \).

The method of proof for even rational integrals can be summarized as follows.

1) Start with an even rational integral:

\[
U_{2p} = \int_0^\infty \frac{\text{even polynomial in } x}{\text{even polynomial in } x} \, dx.
\]
2) Symmetrize the denominator to produce

\[ U_{2p} = \int_{0}^{\infty} \frac{\text{even polynomial in } x}{\text{even reciprocal polynomial in } x} \, dx. \]

The degree of the denominator is doubled.

3) Let \( x = \tan \theta \). Then

\[ U_{2p} = \int_{0}^{\pi/2} \frac{\text{polynomial in } \cos 2\theta}{\text{even polynomial in } \cos 2\theta} \, d\theta. \]

4) Symmetry produced the vanishing of the integrands with an odd power of \( \cos \theta \) in the numerator. We obtain

\[ U_{2p} = \int_{0}^{\pi/2} \frac{\text{even polynomial in } \cos 2\theta}{\text{even polynomial in } \cos 2\theta} \, d\theta. \]

5) Let \( \psi = 2\theta \) to produce

\[ U_{2p} = \int_{0}^{\pi} \frac{\text{even polynomial in } \cos \psi}{\text{even polynomial in } \cos \psi} \, d\psi. \]

Using symmetry this becomes an integral over \([0, \pi/2]\).

6) Let \( y = \tan \psi \) and use \( \cos \psi = 1/\sqrt{1 + y^2} \) to obtain

\[ U_{2p} = \int_{0}^{\infty} \frac{\text{even polynomial in } y}{\text{even polynomial in } y} \, dy. \]

The degree of the denominator is half of what it was in Step 5.

Keeping track of the degrees one checks that the degree of the new rational function is the same as the original one, with new coefficients that appear as functions of the old ones.

4. A GEOMETRIC INTERPRETATION

We now present a geometric foundation of the general even rational Landen transformation (3.9) using the theory of Riemann surfaces. [47] provides an introduction to this theory, including definitions of objects we will refer to here. The sequence of transformations in section 3 can be achieved in one step by relating \( \tan 2\theta \) to \( \tan \theta \). For historical reasons (this is what we did first) we present the details with \( \cotangent \) instead of tangent.

Consider the even rational integral

\[ I = \int_{0}^{\infty} R(x) \, dx = \frac{1}{2} \int_{-\infty}^{\infty} R(x) \, dx. \]

Introduce the new variable

\[ y = R_2(x) = \frac{x^2 - 1}{2x}, \]

motivated by the identity \( \cot 2\theta = R_2(\cot \theta) \). The function \( R_2 : \mathbb{R} \to \mathbb{R} \) is a two-to-one map. The sections of the inverse are

\[ x = \sigma_{\pm}(y) = y \pm \sqrt{y^2 + 1}. \]
Splitting the original integral as

\[ I = \int_{-\infty}^{0} R(x) \, dx + \int_{0}^{\infty} R(x) \, dx \]

and introducing \( x = \sigma_+(y) \) in the first and \( x = \sigma_-(y) \) in the second integral, yields

\[ I = \int_{-\infty}^{\infty} (R_+(y) + R_-(y)) \, dy \]

where

\[ R_+(y) = R(\sigma_+(y)) + R(\sigma_-(y)) \quad \text{and} \quad R_-(y) = \frac{y}{\sqrt{y^2 + 1}} \left( R(\sigma_+(y)) - R(\sigma_-(y)) \right). \]

A direct calculation shows that \( R_+ \) and \( R_- \) are rational functions of degree at most that of \( R \).

The change of variables \( y = R_2(x) \) converts the meromorphic differential \( \varphi = R(x) \, dx \) into

\[ \frac{d\sigma_+}{dy} + R(\sigma_-(y)) \frac{d\sigma_-}{dy} = \left( (R_+(y) + R_-(y)) + \frac{y(R(\sigma_+) - R(\sigma_-))}{\sqrt{y^2 + 1}} \right) \, dy = (R_+(y) + R_-(y)) \, dy. \]

The general situation is this: start with a finite ramified cover \( \pi : X \to Y \) of Riemann surfaces and a meromorphic differential \( \varphi \) on \( X \). Let \( U \subset Y \) be a simply connected domain that contains no critical values of \( \pi \), and let \( \sigma_1, \ldots, \sigma_k : U \to X \) be the distinct sections of \( \pi \). Define

\[ \pi_* \varphi \big|_U = \sum_{j=1}^{k} \sigma_j^* \varphi. \]

In [31] we show that this construction preserves analytic 1-forms, that is, if \( \varphi \) is an analytic 1-form in \( X \) then \( \pi_* \varphi \) is an analytic 1-form in \( Y \). Furthermore, for any rectifiable curve \( \gamma \) on \( Y \), we have

\[ \int_{\gamma} \pi_* \varphi = \int_{\pi^{-1}\gamma} \varphi. \]

In the case of projective space, this leads to the following:

**Lemma 4.1.** If \( \pi : \mathbb{P}^1 \to \mathbb{P}^1 \) is analytic, and \( \varphi = R(z) \, dz \) with \( R \) a rational function, then \( \pi_* \varphi \) can be written as \( R_1(z) \, dz \) with \( R_1 \) a rational function of degree at most the degree of \( R \).

This is the generalization of the fact that the integrals in (4.1) and (4.5) are the same.

### 5. A FURTHER GENERALIZATION

The procedure described in Section 3 can be extended with the rational map \( R_m \), defined by the identity

\[ \cot m\theta = R_m(\cot \theta). \]

Here \( m \in \mathbb{N} \) is arbitrary greater or equal than 2. We present some elementary properties of the rational function \( R_m \).
Proposition 5.1. The rational function $R_m$ satisfies:

1) For $m \in \mathbb{N}$ define

$$P_m(x) := \sum_{j=0}^{\lfloor m/2 \rfloor} (-1)^j \binom{m}{2j} x^{m-2j}$$
and

$$Q_m(x) := \sum_{j=0}^{\lfloor (m-1)/2 \rfloor} (-1)^j \binom{m}{2j+1} x^{m-(2j+1)}.$$

Then $R_m := P_m/Q_m$.

2) The function $R_m$ is conjugate to $f_m(x) := x^m$ via $M(x) := \frac{x+i}{x-i}$; that is, $R_m = M^{-1} \circ f_m \circ M$.

3) The polynomials $P_m$ and $Q_m$ have simple real zeros given by

$$p_k := \cot \left( \frac{(2k+1)\pi}{2m} \right) \text{ for } 0 \leq k \leq m-1,$$
and

$$q_k := \cot \left( \frac{k\pi}{m} \right) \text{ for } 1 \leq k \leq m-1.$$

If we change the domain to the entire real line, we can, using the rational substitutions $R_m(x) \to x$, produce a rational Landen transformation for an arbitrary integrable rational function $R(x) = B(x)/A(x)$ for each integer value of $m$. The result is a new list of coefficients, from which one produces a second rational function $R^{(1)}(x) = J(x)/H(x)$ with

$$\int_{-\infty}^{\infty} \frac{B(x)}{A(x)} \, dx = \int_{-\infty}^{\infty} \frac{J(x)}{H(x)} \, dx.$$

Iteration of this procedure yields a sequence $x_n$, that has a limit $x_\infty$ with convergence of order $m$, that is,

$$\|x_{n+1} - x_\infty\| \leq C\|x_n - x_\infty\|^m.$$

We describe this procedure here in the form of an algorithm; proofs appear in [39].

Lemma 4.1 applied to the map $\pi(x) = R_m(x)$, viewed as ramified cover of $\mathbb{P}^1$, guarantees the existence of a such new rational function $R^{(1)}$. The question of effective computation of the coefficients of $J$ and $H$ is discussed below. In particular, we show that all these calculations can be done symbolically.

• Algorithm for Deriving Rational Landen Transformations

**Step 1.** The initial data is a rational function $R(x) := B(x)/A(x)$. We assume that $A$ and $B$ are polynomials with real coefficients and $A$ has no real zeros and write

$$A(x) := \sum_{k=0}^{p} a_k x^{p-k} \text{ and } B(x) := \sum_{k=0}^{p-2} b_k x^{p-2-k}.$$

**Step 2.** Choose a positive integer $m \geq 2$.

**Step 3.** Introduce the polynomial

$$H(x) := \text{Res}_z (A(z), P_m(z) - xQ_m(z))$$
and write it as

\[(5.6) \quad H(x) := \sum_{l=0}^{P} e_l x^{p-l}.\]

The polynomial \(H\) is thus defined as the determinant of the Sylvester matrix which is formed of the polynomial coefficients. As such, the coefficients \(e_l\) of \(H(x)\) themselves are integer polynomials in the \(a_i\). Explicitly,

\[(5.7) \quad e_l = (-1)^l a_0^m \prod_{j=1}^{P} Q_m(x_j) \times \sigma_l^{(p)}(R_m(x_1, R_m(x_2), \ldots, R_m(x_p))),\]

where \(\{x_1, x_2, \ldots, x_p\}\) are the roots of \(A\), each each written according to multiplicity. The functions \(\sigma_l^{(p)}\) are the elementary symmetric functions in \(p\) variables defined by

\[(5.8) \quad \prod_{l=1}^{p} (y - y_l) = \sum_{l=0}^{p} (-1)^l \sigma_l^{(p)}(y_1, \ldots, y_p)y^{p-l}.\]

It is possible to compute the coefficients \(e_l\) symbolically from the coefficients of \(A\), without the knowledge of the roots of \(A\).

Also define

\[(5.9) \quad E(x) := H(R_m(x)) \times Q_m(x)^p.\]

**Step 4.** The polynomial \(A\) divides \(E\) and we denote the quotient by \(Z\). The coefficients of \(Z\) are integer polynomials in the \(a_i\).

**Step 5.** Define the polynomial \(C(x) := B(x)Z(x)\).

**Step 6.** There exists a polynomial \(J(x)\), whose coefficients have an explicit formula in terms of the coefficients \(c_j\) of \(C(x)\), such that

\[(5.10) \quad \int_{-\infty}^{\infty} \frac{B(x)}{A(x)} \, dx = \int_{-\infty}^{\infty} \frac{J(x)}{H(x)} \, dx.\]

This new integrand is the rational function whose existence is guaranteed by Lemma 4.1. The explicit computation of the coefficients of \(J\) can be found in [39]. This is the rational Landen transformation of order \(m\).

**Example 5.1.** Completing the algorithm with \(m = 3\) and the rational function

\[(5.11) \quad R(x) = \frac{1}{ax^2 + bx + c},\]

produces the result stated below. Notice that the values of the iterates are ratios of integer polynomials of degree 3, as was stated above. The details of this example appear in [40].

**Theorem 5.2.** The integral

\[(5.12) \quad I = \int_{-\infty}^{\infty} \frac{dx}{ax^2 + bx + c}.\]
is invariant under the transformation
\begin{align*}
a & \mapsto \frac{a}{\Delta} (a + 3c)^2 - 3b^2), \\
b & \mapsto \frac{b}{\Delta} (3(a - c)^2 - b^2), \\
c & \mapsto \frac{c}{\Delta} ((3a + c)^2 - 3b^2),
\end{align*}
where $\Delta := (3a + c)(a + 3c) - b^2$. The condition $b^2 - 4ac < 0$, imposed to ensure convergence of the integral, is preserved by the iteration.

The convergence of the iterations of rational Landen transformations is discussed in the next section.

6. The issue of convergence

The convergence of the double sequence $(a_n, b_n)$ appearing in the elliptic Landen transformation (1.1) is easily established. Assume $0 < b_0 \leq a_0$, then the arithmetic-geometric inequality yields $b_n \leq b_{n+1} \leq a_{n+1} \leq a_n$. Also
\begin{equation}
0 \leq a_{n+1} - b_{n+1} = \frac{1}{2} \frac{(a_n - b_n)^2}{(\sqrt{a_n} + \sqrt{b_n})^2}.
\end{equation}
This shows that $a_n$ and $b_n$ have a common limit: $M = \text{AGM}(a, b)$, the arithmetic-geometric of $a$ and $b$. The convergence is quadratic:
\begin{equation}
|a_{n+1} - M| \leq C|a_n - M|^2,
\end{equation}
for some constant $C > 0$ independent of $n$. Details can be found in [14].

The Landen transformations produce maps on the space of coefficients of the integrand. In this section, we discuss the convergence of the rational Landen transformations. This discussion is divided in two cases:

**Case 1: the half-line.** Let $R(x)$ be an even rational function, written as $R(x) = P(x)/Q(x)$, with
\begin{equation}
P(x) = \sum_{k=0}^{p-1} b_k x^{2(p-1-k)}, \quad Q(x) = \sum_{k=0}^{p} a_k x^{2(p-k)},
\end{equation}
and $a_0 = a_p = 1$. The parameter space is
\begin{equation}
\mathcal{P}^+_{2p} = \{(a_1, \cdots, a_{p-1}; b_0, \cdots, b_{p-1}) \in \mathbb{R}^{p-1} \times \mathbb{R}^p.
\end{equation}
We write
\begin{equation}
a := (a_1, \cdots, a_{p-1}) \quad \text{and} \quad b := (b_0, \cdots, b_p).
\end{equation}
Define
\begin{equation}
\Lambda_{2p} = \{(a_1, \cdots, a_{p-1}) \in \mathbb{R}^{p-1} : \int_0^\infty R(x) \, dx \text{ is finite} \}.
\end{equation}
Observe that the convergence of the integral depends only on the parameters in the denominator.

The Landen transformations provide a map
\begin{equation}
\Phi_{2p} : \mathcal{P}^+_{2p} \rightarrow \mathcal{P}^+_{2p}
\end{equation}
that preserves the integral. Introduce the notation
\begin{equation}
\mathbf{a}_n = (a_1^{(n)}, \ldots, a_{p-1}^{(n)}) \text{ and } \mathbf{b}_n = (b_0^{(n)}, \ldots, b_{p}^{(n)}),
\end{equation}
where
\begin{equation}
(\mathbf{a}_n, \mathbf{b}_n) = \Phi_{2p}(\mathbf{a}_{n-1}, \mathbf{b}_{n-1})
\end{equation}
are the iterates of the map $\Phi_{2p}$.

The result that one expects is this:

**Theorem 6.1.** The region $\Lambda_{2p}$ is invariant under the map $\Phi_{2p}$. Moreover
\begin{equation}
\mathbf{a}_n \to \left( \left( \begin{array}{c} p \vspace{1pt} \\ 1 \end{array} \right), \left( \begin{array}{c} p \\
2 \end{array} \right), \ldots, \left( \begin{array}{c} p \\
p-1 \end{array} \right) \right),
\end{equation}
and there exists a number $L$, that depends on the initial conditions, such that
\begin{equation}
\mathbf{b}_n \to \left( \left( \begin{array}{c} p-1 \\
0 \end{array} \right)L, \left( \begin{array}{c} p-1 \\
1 \end{array} \right)L, \ldots, \left( \begin{array}{c} p-1 \\
p-1 \end{array} \right)L \right).
\end{equation}
This is equivalent to say that the sequence of rational functions formed by the Landen transformations, converge to $L/(x^2 + 1)$.

This was established in [31] using the geometric language of Landen transformations which, while unexpected, is satisfactory.

**Theorem 6.2.** Let $\varphi$ be a 1-form, holomorphic in a neighborhood of $\mathbb{R} \subset \mathbb{P}^1$. Then
\begin{equation}
\lim_{n \to \infty} (\pi_*)^n \varphi = \frac{1}{\pi} \left( \int_{-\infty}^{\infty} \varphi \right) \frac{dz}{1 + z^2},
\end{equation}
where the convergence is uniform on compact subsets of $U$, the neighborhood in the definition of $\pi_*$. The proof is detailed for the map $\pi(z) = \frac{z^2 - 1}{2z} = R_3(z)$, but it extends without difficulty to the generalization $R_m$.

Theorem 6.2 can be equivalently reformulated as:

**Theorem 6.3.** The iterates of the Landen transformation starting at $(\mathbf{a}_0, \mathbf{b}_0) \in \mathbb{P}_{2p}^+$ converge (to the limit stated in Theorem 6.1) if and only if the integral formed by the initial is finite.

It would be desirable to establish this result by purely dynamical techniques. This has been established only for the case $p = 3$. In that case the Landen transformation for
\begin{equation}
U_6 := \int_0^\infty \frac{cx^4 + dx^2 + e}{x^6 + ax^4 + bx^2 + 1} \, dx
\end{equation}
is
\begin{align}
\mathbf{a}_1 & \leftarrow \frac{ab + 5a + 5b + 9}{(a + b + 2)^{4/3}}, \\
\mathbf{b}_1 & \leftarrow \frac{a + b + 6}{(a + b + 2)^{2/3}},
\end{align}
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Figure 1. Zero locus of \( R(a, b) \) and Symmetry

coupled with

\[
\begin{align*}
c_1 & \leftarrow \frac{c + d + e}{(a + b + 2)^{2/3}}, \\
d_1 & \leftarrow \frac{(b + 3)c + 2d + (a + 3)e}{a + b + 2}, \\
e_1 & \leftarrow \frac{c + e}{(a + b + 2)^{1/3}}.
\end{align*}
\]

The region

\[
\Lambda_6 = \{(a, b) \in \mathbb{R}^2 : U_6 < \infty\}
\]

is described by the discriminant curve \( \mathcal{R} \), the zero set of the polynomial

\[
R(a, b) = 4a^3 + 4b^3 - 18ab - a^2b^2 + 27.
\]

This zero set (Fig. 1) has two connected components: the first one \( \mathcal{R}_+ \) contains \((3, 3)\) as a cusp and the second one \( \mathcal{R}_- \), given by \( R_-(a, b) = 0 \), is disjoint from the first quadrant. The branch \( \mathcal{R}_- \) is the boundary of the set \( \Lambda_6 \).

The identity

\[
R(a_1, b_1) = \frac{(a - b)^2 R(a, b)}{(a + b + 2)^4},
\]

shows that \( \partial \mathcal{R} \) is invariant under \( \Phi_6 \). By examining the effect of this map along lines of slope \(-1\), we obtain a direct parametrization of the flow on the discriminant curve. Indeed, this curve is parametrized by

\[
a(s) = \frac{s^3 + 4}{s^2} \quad \text{and} \quad b(s) = \frac{s^3 + 16}{4s}.
\]
\[
\varphi(s) = \left( \frac{4(s^2 + 4)^2}{s(s + 2)^2} \right)^{1/3}
\]
gives the image of the Landen transformation \( \Phi_6 \); that is,
\[
\Phi_6(a(s), b(s)) = (a(\varphi(s)), b(\varphi(s))).
\]

The map \( \Phi_6 \) has three fixed points: \((3, 3)\), that is super-attracting, a saddle point \( P_2 \) on the lower branch \( \mathcal{R}_- \) of the discriminant curve, and a third unstable spiral below this lower branch. In [24] we prove:

**Theorem 6.4.** The lower branch of the discriminant curve is the curve \( \Lambda_6 \). This curve is also the global unstable manifold of the saddle point \( P_2 \). Therefore the iterations of \( \Phi_6 \) starting at \((a, b)\) converge if and only if the integral \( U_6 \), formed with the parameters \((a, b)\), is finite. Moreover, \((a_n, b_n) \to (3, 3)\) quadratically and there exists a number \( L \) such that \((c_n, d_n, e_n) \to (1, 2, 1)\).

The next result provides an analogue of the AGM (1.12) for the rational case. The main differences here are that our iterates converge to an algebraic number and we achieve order-\( m \) convergence.

**Case 2: The whole-line:** This works for any choice of positive integer \( m \). Let \( R(x) \) be a rational function, written as \( R(x) = B(x)/A(x) \). Assume that the coefficients of \( A \) and \( B \) are real, that \( A \) has no real zeros and that \( \deg(B) \leq \deg(A) - 2 \). These conditions are imposed to guarantee the existence of
\[
I = \int_{-\infty}^{\infty} R(x) \, dx.
\]
In particular \( A \) must be of even degree, and we write
\[
A(x) = \sum_{k=0}^{p} a_k x^{p-k} \quad \text{and} \quad B(x) = \sum_{k=0}^{p-2} b_k x^{p-2-k}.
\]
We can also require that \( \deg(\gcd(A, B)) = 0 \).

The class of such rational functions will be denoted by \( \mathcal{R}_p \).

The algorithm presented in Section 5 provides a transformation on the parameters
\[
\mathcal{P}_p := \{a_0, a_1, \ldots, a_p; b_0, b_1, \ldots, b_{p-2}\} = \mathbb{R}^{p+1} \times \mathbb{R}^{p-1}
\]
of \( R \in \mathcal{R}_p \) that preserves the integral \( I \). In fact, we produce a family of maps, indexed by \( m \in \mathbb{N} \),
\[
\mathcal{L}_{m,p} : \mathcal{R}_p \to \mathcal{R}_p,
\]
such that
\[
\int_{-\infty}^{\infty} R(x) \, dx = \int_{-\infty}^{\infty} \mathcal{L}_{m,p}(R(x)) \, dx.
\]
The maps \( \mathcal{L}_{m,p} \) induce a rational Landen transformation on the parameter space:
\[
\Phi_{m,p} : \mathcal{P}_p \to \mathcal{P}_p
\]
by simply listing the coefficients of the function \( \mathcal{L}_{m,p}(R(x)) \).
The original integral is written in the form
\[
I = \frac{b_0}{a_0} \int_{-\infty}^{\infty} \frac{x^{p-2} + b_0^{-1}b_1x^{p-3} + b_0^{-1}b_2x^{p-4} + \cdots + b_0^{-1}b_{p-2}}{x^p + a_0^{-1}a_1x^{p-1} + a_0^{-1}a_2x^{p-2} + \cdots + a_0^{-1}a_p} \, dx.
\]
(6.26)

The Landen transformation generates a sequence of coefficients,
\[
P_{p,n} := \left\{ a_0^{(n)}, a_1^{(n)}, \cdots, a_p^{(n)}; b_0^{(n)}, b_1^{(n)}, \cdots, b_{p-2}^{(n)} \right\},
\]
with \( P_{p,0} = P_p \) as in (6.23). We expect that, as \( n \to \infty \),
\[
x_n := \left( \frac{a_1^{(n)}}{a_0^{(n)}}, \frac{a_2^{(n)}}{a_0^{(n)}}, \cdots, \frac{a_p^{(n)}}{a_0^{(n)}}, \frac{b_1^{(n)}}{b_0^{(n)}}, \frac{b_2^{(n)}}{b_0^{(n)}}, \cdots, \frac{b_{p-2}^{(n)}}{b_0^{(n)}} \right)
\]
(6.27)

converges to
\[
x_\infty := \left( 0, \frac{q}{1}, 0, \frac{q}{2}, \cdots, \frac{q}{q}, 0, \frac{q-1}{1}, 0, \frac{q-1}{2}, \cdots, \frac{q-1}{q-1} \right),
\]
(6.28)

where \( q = p/2 \). Moreover, we should have
\[
\|x_{n+1} - x_\infty\| \leq C\|x_n - x_\infty\|^m.
\]
(6.30)

The invariance of the integral then shows that
\[
\frac{b_0^{(n)}}{a_0^{(n)}} \to \frac{1}{\pi} I.
\]
(6.31)

This produces an iterative method to evaluate the integral of a rational function. The method’s convergence is of order-\( m \).

The convergence of these iterations, and in particular the bound (6.30), can be established by the argument presented in section 4. Thus, the transformation \( L_{m,p} \) leads to a sequence that has order-\( m \) convergence. We expect to develop these ideas into an efficient numerical method for integration.

We choose to measure the convergence of a sequence of vectors to 0 is in the \( L_2 \)-norm,
\[
\|v\|_2 = \frac{1}{\sqrt{2p-2}} \left( \sum_{k=1}^{2p-2} \|v_k\|^2 \right)^{1/2},
\]
(6.32)

and also the \( L_\infty \)-norm,
\[
\|v\|_\infty = \text{Max} \{ \|v_k\| : 1 \leq k \leq 2p-2 \}.
\]
(6.33)

The rational functions appearing as integrands have rational coefficients, so, as a measure of their complexity, we take the largest number of digits of these coefficients. This appears in the column marked size.

The following tables illustrate the iterates of rational Landen transformations of order 2, 3 and 4, applied to the example
\[
I = \int_{-\infty}^{\infty} \frac{3x + 5}{x^4 + 14x^3 + 74x^2 + 184x + 208} \, dx = -\frac{7\pi}{12}.
\]
The first column gives the $L_2$-norm of $u_n - u_\infty$, the second its $L_\infty$-norm, the third presents the relative error in (6.31), and in the last column we give the size of the rational integrand. At each step, we verify that the new rational function integrates to $-7\pi/12$.

### Method of order 2

<table>
<thead>
<tr>
<th>$n$</th>
<th>$L_2$-norm</th>
<th>$L_\infty$-norm</th>
<th>Error</th>
<th>Size</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>58.7171</td>
<td>69.1000</td>
<td>1.02060</td>
<td>5</td>
</tr>
<tr>
<td>2</td>
<td>7.444927</td>
<td>9.64324</td>
<td>1.04473</td>
<td>10</td>
</tr>
<tr>
<td>3</td>
<td>4.04691</td>
<td>5.36256</td>
<td>0.945481</td>
<td>18</td>
</tr>
<tr>
<td>4</td>
<td>1.81592</td>
<td>2.41658</td>
<td>1.15092</td>
<td>41</td>
</tr>
<tr>
<td>5</td>
<td>0.360422</td>
<td>0.411437</td>
<td>0.262511</td>
<td>82</td>
</tr>
<tr>
<td>6</td>
<td>0.02988922</td>
<td>0.0249128</td>
<td>0.0189903</td>
<td>164</td>
</tr>
<tr>
<td>7</td>
<td>0.000256824</td>
<td>0.000299728</td>
<td>0.0000362352</td>
<td>327</td>
</tr>
<tr>
<td>8</td>
<td>1.92454 $\times 10^{-8}$</td>
<td>2.24568 $\times 10^{-8}$</td>
<td>1.47053 $\times 10^{-8}$</td>
<td>659</td>
</tr>
<tr>
<td>9</td>
<td>1.0823 $\times 10^{-16}$</td>
<td>1.2609 $\times 10^{-16}$</td>
<td>8.2207 $\times 10^{-17}$</td>
<td>1318</td>
</tr>
</tbody>
</table>

As expected, for the method of order 2, we observe quadratic convergence in the $L_2$-norm and also in the $L_\infty$-norm. The size of the coefficients of the integrand is approximately doubled at each iteration.

### Method of order 3

<table>
<thead>
<tr>
<th>$n$</th>
<th>$L_2$-norm</th>
<th>$L_\infty$-norm</th>
<th>Error</th>
<th>Size</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>15.2207</td>
<td>20.2945</td>
<td>1.03511</td>
<td>8</td>
</tr>
<tr>
<td>2</td>
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<td>1.83067</td>
<td>0.85941</td>
<td>23</td>
</tr>
<tr>
<td>3</td>
<td>0.41100</td>
<td>0.338358</td>
<td>0.197044</td>
<td>69</td>
</tr>
<tr>
<td>4</td>
<td>0.00842346</td>
<td>0.00815475</td>
<td>0.00597363</td>
<td>208</td>
</tr>
<tr>
<td>5</td>
<td>5.05016 $\times 10^{-8}$</td>
<td>5.75969 $\times 10^{-8}$</td>
<td>1.64059 $\times 10^{-9}$</td>
<td>626</td>
</tr>
<tr>
<td>6</td>
<td>1.09651 $\times 10^{-23}$</td>
<td>1.02510 $\times 10^{-23}$</td>
<td>3.86286 $\times 10^{-24}$</td>
<td>1878</td>
</tr>
<tr>
<td>7</td>
<td>1.12238 $\times 10^{-70}$</td>
<td>1.22843 $\times 10^{-70}$</td>
<td>8.59237 $\times 10^{-71}$</td>
<td>5634</td>
</tr>
</tbody>
</table>

### Method of order 4

<table>
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<tr>
<th>$n$</th>
<th>$L_2$-norm</th>
<th>$L_\infty$-norm</th>
<th>Error</th>
<th>Size</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>7.44927</td>
<td>9.64324</td>
<td>1.04473</td>
<td>10</td>
</tr>
<tr>
<td>2</td>
<td>1.81592</td>
<td>2.41858</td>
<td>1.15092</td>
<td>41</td>
</tr>
<tr>
<td>3</td>
<td>0.02988922</td>
<td>0.0249128</td>
<td>0.0189903</td>
<td>164</td>
</tr>
<tr>
<td>4</td>
<td>1.92454 $\times 10^{-8}$</td>
<td>2.249128 $\times 10^{-8}$</td>
<td>1.47053 $\times 10^{-8}$</td>
<td>659</td>
</tr>
<tr>
<td>5</td>
<td>3.40769 $\times 10^{-33}$</td>
<td>3.96407 $\times 10^{-33}$</td>
<td>2.56817 $\times 10^{-33}$</td>
<td>2637</td>
</tr>
</tbody>
</table>

**Example 6.1.** A method of order 3 for the evaluation of the quadratic integral

$$I = \int_{-\infty}^{\infty} \frac{dx}{ax^2 + bx + c},$$

has been analyzed in [40]. We refer to Example 5.1 for the explicit formulas of this Landen transformation, and define the iterates accordingly. From there, we prove that the error term,

$$e_n := \left( a_n - \frac{1}{2} \sqrt{4ac - b^2}, b_n, c_n - \frac{1}{2} \sqrt{4ac - b^2} \right)$$
satisfies $e_n \to 0$ as $n \to \infty$, with cubic rate:

$$\|e_{n+1}\| \leq C\|e_n\|^3.$$  

The proof of convergence is elementary. Therefore we have

$$\lim_{n \to \infty} (a_n, b_n, c_n) = \left(\sqrt{ac - b^2/4}, 0, \sqrt{4ac - b^2/4}\right),$$

which, in conjunction with (6.34), implies that

$$I = 2\sqrt{4} \int_{-\infty}^{\infty} \frac{dx}{x^2 + 1},$$

exactly as one would have concluded by completing the square. Unlike completing the square, our method extends to a general rational integral over the real line.

7. The appearance of the AGM in diverse contexts

The (elliptic) Landen transformation

$$a_1 \leftarrow \frac{1}{2}(a + b) \text{ and } b_1 \leftarrow \sqrt{ab}$$

leaving invariant the elliptic integral

$$G(a, b) = \int_0^{\pi/2} \frac{d\varphi}{\sqrt{a^2 \cos^2 \varphi + b^2 \sin^2 \varphi}}$$

appears in many different forms. In this last section we present a partial list of them.

7.1. The Elliptic Landen Transformation. For the lattice $\mathbb{L} = \mathbb{Z} \oplus \omega \mathbb{Z}$, introduce the theta-functions

$$\vartheta_3(x, \omega) = \sum_{n=-\infty}^{\infty} z^{2n} q^{n^2} \text{ and } \vartheta_4(x, \omega) = \sum_{n=-\infty}^{\infty} (-1)^n z^{2n} q^{n^2},$$

where $z = e^{\pi i x}$ and $q = e^{\pi i \omega}$. The condition $\Im \omega > 0$ is imposed to ensure convergence of the series. These functions admit a variety of remarkable identities. In particular, the null-values (those with $x = 0$) satisfy

$$\vartheta_3^2(0, 2\omega) = \vartheta_3(0, \omega) \vartheta_4(0, \omega) \text{ and } \vartheta_3^2(0, 2\omega) = \frac{1}{2} \left( \vartheta_3^2(0, \omega) + \vartheta_4^2(0, \omega) \right),$$

and completely characterize values of the AGM, leading to the earlier result [14]. Grayson [29] has used the doubling of the period $\omega$ to derive the arithmetic-geometric mean from the cubic equations describing the corresponding elliptic curves. See Chapter 3 in [41] for more information.
7.2. A time-one map. We now present a deeper and more modern version of a result known to Gauss: given a sequence of points \( \{x_n\} \) on a manifold \( X \), decide whether there is a differential equation

\[
\frac{dx}{dt} = V(x),
\]

starting at \( x_0 \) such that \( x_n = x(n, x_0) \). Here \( x(t, x_0) \) is the unique solution to (7.4) satisfying \( x(0, x_0) = x_0 \). Denote by

\[
(7.5) \quad \phi_{\text{ellip}}(a, b) = \left( \frac{1}{2}(a + b), \sqrt{ab} \right)
\]

the familiar elliptic Landen transformation. Now take \( a, b \in \mathbb{R} \) with \( a > b > 0 \). Use the null-values of the theta functions to find unique values \( (\tau, \rho) \) such that

\[
(7.6) \quad a = \rho \vartheta^2_3(0, \tau) \quad \text{and} \quad b = \rho \vartheta^2_4(0, \tau).
\]

Finally define

\[
(7.7) \quad x_{\text{ellip}}(t) = (a(t), b(t)) = \rho \left( \vartheta^2_3(0, 2t\tau), \vartheta^2_4(0, 2t\tau) \right),
\]

with \( x_{\text{ellip}}(0) = (a, b) \). The remarkable result is [27]:

**Theorem 7.1.** (Deift, Li, Previato, Tomei). The map \( t \to x_{\text{ellip}}(t) \) is an integrable Hamiltonian flow on \( X \) equipped with an appropriate symplectic structure. The Hamiltonian is the complete elliptic integral \( G(a, b) \) and the angle is (essentially the logarithm of) the second period of the elliptic curve associated with \( \tau \). Moreover

\[
(7.8) \quad x_{\text{ellip}}(k) = \phi^k_{\text{ellip}}(a, b).
\]

Thus the arithmetic-geometric algorithm is the time-one map of a completely integrable Hamiltonian flow.

Notice that this theorem shows that the result in question respects some additional structures whose invention postdates Gauss.

A natural question is whether the map (3.9) appears as a time-one map of an interesting flow.


\[
(7.9) \quad a_{n+1} = \frac{a_n + b_n + c_n + d_n}{4},
\]

\[
b_{n+1} = \frac{\sqrt{a_nb_n} + \sqrt{c_nd_n}}{2},
\]

\[
c_{n+1} = \frac{\sqrt{a_n c_n} + \sqrt{b_n d_n}}{2},
\]

\[
d_{n+1} = \frac{\sqrt{a_n d_n} + \sqrt{b_n c_n}}{2},
\]

starting with \( a_0 = a, b_0 = b, c_0 = c \) and \( d_0 = d \). The common limit, denoted by \( G(a, b, c, d) \), is given by

\[
(7.10) \quad \frac{1}{G(a, b, c, d)} = \frac{1}{\pi^2} \int_0^{\alpha_3} \int_0^{\alpha_2} \frac{(x - y) \, dx \, dy}{\sqrt{R(x)R(y)}},
\]
where \( R(x) = x(x - \alpha_0)(x - \alpha_1)(x - \alpha_2)(x - \alpha_3) \) and the numbers \( \alpha_j \) are given by explicit formulas in terms of the parameters \( a, b, c, d \). Details are given by Mestre in [42].

The initial conditions \((a, b, c, d) \in \mathbb{R}^4\) for which the iteration converges has some interesting invariant subsets. When \( a = b \) and \( c = d \), we recover the AGM iteration (1.1). In the case that \( b = c = d \), we have another invariant subset, linking to an iterative mean described below.

7.4. Variations of AGM with hypergeometric limit. Let \( N \in \mathbb{N} \). The analysis of

\[
\begin{align*}
& a_{n+1} = \frac{a_n + (N-1)b_n}{N} \quad \text{and} \quad c_{n+1} = \frac{a_n - b_n}{N}, \\
& b_n = (a_n^N - c_n^N)^{1/N},
\end{align*}
\]

with \( b_n = (a_n^N - c_n^N)^{1/N}, \) is presented in [17]. All the common ingredients appear there: a common limit, fast convergence, theta functions and sophisticated iterations for the evaluation of \( \pi \). The common limit is denoted by \( AG_N(a,b) \). The convergence is of order \( N \) and the limit is identified for small \( N \): for \( 0 < k < 1 \),

\[
\begin{align*}
1 \over AG_2(1,k) &= 2F_1(1/2, 1/2; 1; 1 - k^2) \quad \text{and} \\
1 \over AG_3(1,k) &= 2F_1(1/3, 2/3; 1; 1 - k^2).
\end{align*}
\]

where

\[
\begin{align*}
2F_1(a, b; c; x) &= \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} x^k 
\end{align*}
\]

is the classical hypergeometric function. There are integral representations of these as well which parallel (1.12), see [12], Section 6.1 for details.

Other hypergeometric values appear from similar iterations. For example,

\[
\begin{align*}
a_{n+1} &= \frac{a_n + 3b_n}{4} \quad \text{and} \quad b_{n+1} = \sqrt{b_n(a_n + b_n)/2}, \\
1 \over A_4(1,k) &= 2F_1^2(1/4, 3/4; 1; 1 - k^2) .
\end{align*}
\]

To compute \( \pi \) quartically, start at \( a_0 = 1, b_0 = (12\sqrt{2} - 16)^{1/4} \). Now compute \( a_n \) from two steps of \( AG_2 \):

\[
\begin{align*}
a_{n+1} &= \frac{a_n + b_n}{2}, \quad \text{and} \quad b_{n+1} = \left(\frac{a_n b_n^3 + b_n a_n^3}{2}\right)^{1/4}.
\end{align*}
\]

Then

\[
\begin{align*}
1 \over a_{n+1}^4 &= \left(1 - \sum_{j=0}^{n} 2^{j+1}(a_j^4 - a_{j+1}^4)\right)^{-1}.
\end{align*}
\]
with $|a_{n+1} - \pi| \leq C|a_n - \pi|^4$, for some constant $C > 0$. This is much better than the partial sums of

$$\pi = 4 \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}. \tag{7.17}$$

The sequence $(a_n, b_n)$ defined by the iteration

$$a_{n+1} = \frac{a_n + 2b_n}{3}, \quad b_{n+1} = \left(\frac{b_n(a_n + a_n b_n + b_n^2)}{3}\right)^{\frac{1}{3}}, \tag{7.18}$$

starting at $a_0 = 1$, $b_0 = x$ are analyzed in [16]. They have a common limit $F(x)$ given by

$$\frac{1}{F(x)} = \text{}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; 1-x^3\right). \tag{7.19}$$

### 7.5. Iterations where the limit is not explicit. J. Borwein and P. Borwein [15] studied the iteration of

$$(a, b) \to \left(\frac{a + 3b}{4}, \frac{\sqrt{ab} + b}{2}\right), \tag{7.20}$$

and showed the existence of a common limit $B(a_0, b_0)$. Define $B(x) = B(1, x)$. The study of the iteration (7.20) is based on the functional equation

$$B(x) = \frac{1 + 3x}{4} B\left(\frac{2(\sqrt{2} + x)}{1 + 3x}\right). \tag{7.21}$$

and a parametrization of the iterates by theta functions [15]. The complete analysis of (7.20) starts with the purely computational observation that

$$B(x) \sim \frac{\pi^2}{3} \log^{-2}(x/4) \text{ as } x \to 0. \tag{7.22}$$

### 7.6. Fast computation of elementary functions. The fast convergence of the elliptic Landen recurrence (1.1) to the arithmetic-geometric mean provides a method for numerical evaluation of the elliptic integral $G(a, b)$. The same idea provides for the fast computation of elementary functions. For example, in [13] we find the estimate

$$|\log x - (G(1, 10^{-n}) - G(1, 10^{-n} x))| < n10^{-2(n-1)}, \tag{7.23}$$

for $0 < x < 1$ and $n \geq 3$. 

7.7. A continued fraction. The continued fraction

\begin{equation}
R_\eta(a, b) = \frac{a}{\eta + \frac{b^2}{\eta + \frac{4a^2}{\eta + \frac{9b^2}{\eta + \cdots}}}},
\end{equation}

has an interesting connection to the AGM. In their study of the convergence of \( R_\eta(a, b) \), J. Borwein, R. Crandall and G. Fee [18] established the identity

\begin{equation}
R_\eta \left( \frac{a + b^2}{2}, \sqrt{ab} \right) = \frac{1}{2} \left( R_\eta(a, b) + R_\eta(b, a) \right).
\end{equation}

This identity originates with Ramanujan; the similarity with AGM is now direct.

The continued fraction converges for positive real parameters, but for \( a, b \in \mathbb{C} \) the convergence question is quite delicate. For example, the even/odd parts of \( R_1(1, i) \) converge to distinct limits. See [18] and [19] for more details.

7.8. Elliptic Landen with complex initial conditions. The iteration of (1.1) with \( a_0, b_0 \in \mathbb{C} \) requires a choice of square root at each step. Let \( a, b \in \mathbb{C} \) be non-zero and assume \( a \neq \pm b \). A square root \( c \) of \( ab \) is called the right choice if

\begin{equation}
\left| \frac{a + b^2}{2} - c \right| = \left| \frac{a + b^2}{2} + c \right|.
\end{equation}

It turns out that in order to have a limit for (1.1) one has to make the right choice for all but finitely many indices \( n \geq 1 \). This is described in detail by Cox [26].

7.9. Elliptic Landen with \( p \)-adic initial conditions. Let \( p \) be a prime and \( a, b \) be non-zero \( p \)-adic numbers. In order to guarantee that the \( p \)-adic series

\begin{equation}
c = a \sum_{i=0}^{\infty} \left( \frac{1}{2} \right) \left( \frac{b}{a} - 1 \right)^i
\end{equation}

converges, and thus defines a \( p \)-adic square root of \( ab \), one must assume

\begin{equation}
b/a \equiv 1 \mod p^\alpha,
\end{equation}

where \( \alpha = 3 \) for \( p = 2 \) and 1 otherwise. The corresponding sequence defined by (1.1) converges for \( p \neq 2 \) to a common limit: the \( p \)-adic AGM. In the case \( p = 2 \) one must assume that the initial conditions satisfy \( b/a \equiv 1 \mod 16 \). In the case \( b/a \equiv 1 \mod 8 \) but not 1 modulo 16, the corresponding sequence \((a_n, b_n)\) does not converge, but the sequence of so-called absolute invariants

\begin{equation}
j_n = \frac{2^8(a_n^4 - a_n^2b_n^2 + b_n^4)^3}{a_n^4b_n^2(a_n^2 - b_n^2)^2}
\end{equation}

converges to a 2-adic integer. Information about these issues can be found in [30].

D. Kohel [34] has proposed a generalization of the AGM for elliptic curves over a field of characteristic \( p \in \{2, 3, 5, 7, 13\} \). Mestre [43] has developed an AGM theory for ordinary hyperelliptic curves over a field of characteristic 2. This has been extended to non-hyperelliptic curves of genus 3 curves by C. Ritzenthaler [36]. An algorithm for counting points for ordinary elliptic curves over finite fields of characteristic \( p > 2 \) based on the AGM is presented in R. Carls [22].
7.10. Higher genus AGM. An algorithm analog to the AGM for abelian integrals of genus 2 was discussed by Richelot [45], [46] and Hummel [32]. Some details are discussed by J. Bost and J. F. Mestre in [20]. The case of abelian integrals of genus 3 is due to D. Lehavi [36] and D. Lehavi and C. Ritzenthaler [37].

Gauss was correct: his numerical calculation (1.4) has grown in many unexpected directions.

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References

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