Approximation by Rational Operators in $L^p$ Spaces

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The present paper introduces a kind of Kantorovich type Shepard operators. Complete results including direct and converse results, equivalence results are established. As Della Vecchia and Mastroianni ([4], [7]) did, our results involve a weighted modulus of smoothness related to step-functions $\phi(x)$ vanishing algebraically at the endpoints $\pm 1$.

1 Introduction

Let $X := \{x_{nk} = x_k \in [-1, 1], k = 0, 1, 2, \ldots, n; \ n \in \mathbb{N}\}$ be an infinite matrix where each row is a set of distinct points in $[-1, 1]$. Define for $f \in C[-1, 1]$ the Shepard operator

$$S_{n, \lambda}(f, X, x) := \sum_{k=0}^{n} \frac{|x - x_k|^{-\lambda} f(x_k)}{\sum_{k=0}^{n} |x - x_k|^{-\lambda}}, \ \lambda > 1.$$

Obviously, $S_{n, \lambda}(f, X, x)$ are positive operators preserving constants and interpolating the function at the nodes $\{x_{nk}\}_{k=0}^{n}$. Since their unique values in approximation theory and in fitting dates, curves and surfaces, Shepard operators have been studied extensively by many mathematicians, and many interesting results were achieved. Among them, Della Vecchia and Mastroianni [7] proved that $S_{n, \lambda}(f, X, x)$ give a better pointwise estimate than algebraic polynomials if only $X$ is properly chosen. Let

$$x = x(\theta) = \begin{cases} (2\theta)^{2q+1} - 1, & \theta \in [0, 1/2] \\ -(2 - 2\theta)^{2q+1} + 1, & \theta \in [1/2, 1], \end{cases} \quad q = 1, 2, \ldots,$$

and $X_1 = \{x_k = x(k/n) : k = 0, 1, \ldots, n, \ n \in \mathbb{N}\}$, Della Vecchia and Mastroianni [7] proved

Theorem 1.1 For all $f \in C[-1, 1]$, it holds that

$$|f(x) - S_{n, \lambda}(f, X_1, x)| \leq C\omega\left(f, \frac{(1 - x^2)^{2q}/(2q+1)}{n}\right),$$

where $x \in I = [-1, 1]$, $\lambda > 2$ and $C$ is a constant depending only on $q$ and $\lambda$.

Set $\psi(x) := (1 - x^2)^{2q}/(2q+1)$, define

$$K_\psi(f, t) := \inf \left\{ ||f - g|| + t\|\psi g'||, \ g \in C[-1, 1]\right\},$$
For all $a$ be different in different occurrences. Moreover, we write $C$ where $\lambda \geq 1$ and established operators can be found in [15]-[17]. Among them, Xiao and Zhou [15] introduced the following Kantorovich type Shepard approximation in an infinite interval by Shepard operators, readers could find related information in [5], [6], [8], [11], [13].

In the sequel, $S_{n, \lambda}$ as the saturation theorems for $S_{n, \lambda}(f, E, x)$ have been studied by Della Vecchia, Mastroianni, Totik, Somorjai, Szabados, and other scholars (see [9], [12], [13], [18], [19]).

Recently, Della Vecchia, Mastroianni, and Szabados considered further the weighted approximation and approximation in an infinite interval by Shepard operators, readers could find related information in [5], [6], [8], [11], [13].

Write

$$||f||_{p} = ||f||_{L_{[1,1]}^{p}} = \left\{ \int_{1}^{1} |f(x)|^{p} dx \right\}^{1/p}, 1 \leq p < \infty,$$

$$\omega(f, \delta)_{\gamma} := \sup_{0 < h \leq \delta} \|f(x + h) - f(x)\|_{L_{[-1,1]}^{p}},$$

In the sequel, $C_{t_{1}, t_{2}, \cdot \cdot \cdot, t_{l}}$ denotes a positive constant only depending on the parameter(s) $t_{1}, t_{2}, \cdot \cdot \cdot, t_{l}$ which may be different in different occurrences. Moreover, we write $a \sim b$, if there is a $C_{t_{1}, t_{2}, \cdot \cdot \cdot, t_{l}}$ such that $C_{t_{1}, t_{2}, \cdot \cdot \cdot, t_{l}} b \leq a \leq C_{t_{1}, t_{2}, \cdot \cdot \cdot, t_{l}} b$.

In $L^{p}$ spaces, there are fewer results on the approximation of rational operators. Some works on this topic can be found in [15]-[17]. Among them, Xiao and Zhou [15] introduced the following Kantorovich type Shepard operators

$$K_{n, \lambda}(f, E, x) := (n + 1) \sum_{k=0}^{n} \frac{1}{n} |x - \frac{k}{n}|^{-\lambda} \int_{k/(n+1)}^{(k+1)/(n+1)} f(t) dt,$$

and established

**Theorem 1.3** For all $f \in L^{p}_{[0,1]}$, $1 \leq p < \infty$, $||f - K_{n, \lambda}(f, E)||_{p} \leq C_{\lambda, p} \omega \left( f, \frac{1}{n} \right)_{p}, \lambda > 2$, where $C_{\lambda, p}$ is a constant only depending on $p$ and $\lambda$.

In another paper [16], Xiao and Zhou obtained the following equivalence theorem:

**Theorem 1.4** Let $f \in L^{p}_{[-1,1]}$, $1 \leq p < \infty$, $0 < \alpha < 1$ and $\lambda > 3$. Then

$$||f - K_{n, \lambda}(f, E)||_{p} \sim n^{-\alpha} \Leftrightarrow \omega(f, \delta)_{p} \sim \delta^{-\alpha}.$$
It is not difficult to find that the power \(2q + 1\) in (1) can be replaced by a more general number \(\beta \geq 1\), it implies that we can redefine the function \(x(\theta)\) in (1) by the following more general way (still denoted by \(x(\theta)\))

\[
x(\theta) := \begin{cases} 
(2\theta)^\beta - 1, & \theta \in [0, 1/2] \\
-(2 - 2\theta)^\beta + 1, & \theta \in [1/2, 1]
\end{cases}
\]

with \(\beta \geq 1\) fixed.

Set \(X = \{x_k = x(k/n) : k = 0, 1, \ldots, n\}\) and \(\phi(x) = (1 - x^2)^{\frac{2}{q+1}}\). Define

\[
N_{n,\lambda}(f, X, x) := \frac{\sum_{k=0}^{n-1} |x - x_k|^{-\lambda} \frac{1}{x_{k+1} - x_k} \int_{x_k}^{x_{k+1}} f(t) \, dt}{\sum_{k=0}^{n-1} |x - x_k|^{-\lambda}}.
\]

In the present paper, we give approximation rate by \(N_{n,\lambda}(f, X)\) for \(f \in L^p_{[-1,1]}\), and also establish some converse results and equivalent results. As Della Vecchia and Mastroianni ([4], [7]) did, our results involve a weighted modulus of smoothness and an equivalent \(K\)–functional related to step-functions \(\phi(x)\) vanishing algebraically at the endpoints \(\pm 1\). The main results are read as follows:

**Theorem 1.5** For all \(f \in L^p_{[-1,1]}\), \(1 \leq p < \infty\), \(\lambda > \beta + 1\), it holds that

\[
\|f - N_{n,\lambda}(f, X)\|_p \leq C_{\lambda,\beta, p} \omega_\phi \left( f, \frac{1}{n} \right)_p,
\]

where \(\omega_\phi(f, t)_p\) is defined by

\[
\omega_\phi(f, t)_p := \sup_{0 < h \leq t} \|\Delta_t f\|_p,
\]

and

\[
\Delta_t f(x) := \begin{cases} 
f(x + \frac{h\phi(x)}{2}) - f(x - \frac{h\phi(x)}{2}), & x + \frac{h\phi(x)}{2} \in [-1, 1], \\
0, & \text{otherwise.}
\end{cases}
\]

When \(\beta = 1\), (2) is just the result of Theorem 1.3. Theorem 1.5 can also be understood as a generalization of Theorem 1.2 to the \(L^p\) spaces.

The following result improves Theorem 1.4:

**Theorem 1.6** Let \(f \in L^p_{[-1,1]}\), \(1 \leq p < \infty\), \(0 < \alpha < 1\) and \(\lambda > \beta + 1\). Then the following two conditions are equivalent:

(A) \(\|f - N_{n,\lambda}(f, X)\|_p \sim n^{-\alpha}\).

(B) \(\omega_\phi(f, \delta)_p \sim \delta^{-\alpha}\).

When \(\beta = 1\), it becomes the result of Theorem 1.4, and we also lose the restriction \(\lambda > 3\) there to \(\lambda > 2\). Moreover, our proof has a different approach from [16].

In the following text, we always assume that \(\lambda > \beta + 1\) with \(\beta \geq 1\) fixed.
2 Auxiliary Lemmas

Lemma 2.1 Let \( x \in [x_{j-1}, x_j], \) \( j = 2, 3, \ldots, n - 1, \) then

\[
|x - x_k| \leq C_\beta \frac{(j - k + 1)^\beta}{n} \phi(u), \ u \in [x, x_k] \text{ or } [x_k, x]
\]  
(6)

for \( k = 1, 2, \ldots, n - 1. \)

For \( x \in [-1, x_1] \) and \( k = 0, 1, 2, \ldots, n, \) it holds that

\[
|x - x_k| \leq C_\beta \frac{(k + 1)^\beta}{n^\beta}
\]  
(7)

For \( x \in [x_{n-1}, 1] \) and \( k = 0, 1, 2, \ldots, n, \) it holds that

\[
|x - x_k| \leq C_\beta \frac{(n - k + 1)^\beta}{n^\beta}
\]  
(8)

For \( j, k = 0, 1, 2, \ldots, n - 1, \) it holds that

\[
\frac{x_{j+1} - x_j}{x_{k+1} - x_k} \leq C_\beta (j - k + 1)^{\beta - 1}.
\]  
(9)

Denote by \( x_j, \ 0 \leq j \leq n, \) the closest node to \( x, \) then

\[
|x - x_j| \leq C_\beta \frac{\phi(x)}{n},
\]  
(10)

\[
|x - x_k| \geq C_\beta \frac{\phi(x)}{n} |k - j|, \ k \neq j.
\]  
(11)

Proof. First, we prove (6). Let \( x \in [x_{j-1}, x_j], \) \( j = 2, 3, \ldots, n - 1, \) we verify that

\[
|x - x_k| \leq C_\beta \frac{(j - k + 1)^\beta}{n} \phi(x).
\]  
(12)

Case 1. \( 2 \leq j \leq k \leq \frac{n}{2}. \) Since \( x'(\theta) \) is nondecreasing in \([0, 1/2]\) and \( x'(\theta) \sim \phi(x), \) then

\[
|x - x_k| \leq x'(k/n) \left( \frac{k}{n} - \theta \right) \leq C_\beta \frac{|j - k| + 1}{n} \phi(x) \left( \frac{x'(k/n)}{x'(\theta)} \right) \leq C_\beta \frac{|j - k| + 1}{n} \phi(x) \left( \frac{k}{j} \right)^{\beta - 1} \leq C_\beta \frac{(j - k + 1)^\beta}{n} \phi(x).
\]

Case 2. \( 1 \leq k \leq j - 1 \leq \frac{n}{2} - 1. \) Similar to Case 1, we have

\[
|x - x_k| \leq x'(\theta) \left( \theta - \frac{k}{n} \right) \leq C_\beta \frac{|j - k| + 1}{n} \phi(x).
\]

Case 3. \( 2 \leq j \leq \frac{n}{2} \leq k \leq n - 1. \) By the definition of \( x(\theta), \) we derive that

\[
|x - x_k| \leq |x - x(1/2)| + |x(1/2) - x_k|
\]

\[
\leq x'(1/2) \left( \frac{1}{2} - \theta \right) + x'(1/2) \left( \frac{k}{n} - \frac{1}{2} \right)
\]

\[
\leq x'(1/2) \left( \frac{k}{n} - \theta \right) \leq C_\beta \frac{|j - k| + 1}{n} \phi(x) \left( \frac{x'(1/2)}{x'(\theta)} \right) \leq C_\beta \frac{|j - k| + 1}{n} \phi(x) \left( \frac{k}{j} \right)^{\beta - 1} \leq C_\beta \frac{|j - k| + 1}{n} \phi(x) \left( \frac{n/2 - j}{j} + 1 \right)^{\beta - 1} \leq C_\beta \frac{(j - k + 1)^\beta}{n} \phi(x).
\]

Note that \( x'(\theta) = 2(\theta)^{\beta - 1} = 2^{\beta - 1}(\beta - 1 + 1)^{(\beta - 1)/\beta} = 2^{\beta - 1}(x + 1)^{(\beta - 1)/\beta}. \)
**Case 4.** \(1 \leq k \leq \frac{n}{2} \leq j \leq n - 1\). Similar to Case 3, we have

\[
|x - x_k| \leq |x - x(1/2)| + |x(1/2) - x_k| \leq x'(1/2) \left( \theta - \frac{k}{n} \right).
\]

If \(j \leq \frac{3}{4}n\), then \(\phi(x) \sim 1\). Hence,

\[
|x - x_k| \leq x'(1/2) \frac{|j - k| + 1}{n} \leq C_\beta \frac{|j - k| + 1}{n} \phi(x).
\]

If \(j > \frac{3}{4}n\), then

\[
|x - x_k| \leq \frac{|j - k| + 1}{n} \left( \frac{x'(1/2)}{x'(\theta)} \right) \phi(x) \leq C_\beta \frac{|j - k| + 1}{n} \phi(x) \left( \frac{n}{n - 1} \right)^{\beta - 1}.
\]

\[
\leq C_\beta \frac{|j - k| + 1}{n} \phi(x)(j + 1)^{\beta - 1} \leq C_\beta \frac{|j - k| + 1}{n} \phi(x).
\]

**Case 5.** \(\frac{n}{2} \leq k \leq n - 1\) or \(\frac{n}{2} \leq j \leq k - 1 \leq n - 2\). Similar to Case 1 and Case 2, we can easily deduce that

\[
|x - x_k| \leq C_\beta \frac{(j - k + 1)^\beta}{n} \phi(x)
\]

still holds.

Altogether, with all the above estimates, we conclude (12).

Similarly, we have

\[
|x - x_k| \leq C_\beta \frac{(j - k + 1)^\beta}{n} \phi(x_k).
\]  

(13)

Combining (12) with (13), we obtain (6).

Assume that \(x \in [-1, x_1]\), then for \(k = 1, 2, \ldots, n\), by applying (6), we get

\[
|x - x_k| \leq |x - x_1| + |x_1 - x_k| \leq \frac{2^\beta}{n^\beta} + C_\beta \frac{(k + 1)^\beta}{n} \phi(x_1)
\]

\[
\leq C_\beta \frac{(k + 1)^\beta}{n^\beta}.
\]

For \(k = 0\), the above estimate obviously holds. Thus, we have established (7). Inequality (8) can be deduced similarly.

Now, we begin to prove (9). If \(k = 0\), then (9) obviously holds for \(j = 0, n - 1\) by noting the fact

\[
x_1 - x_0 \sim x_n - x_{n-1} \sim \frac{1}{n^\beta}.
\]  

(14)

If \(1 \leq j \leq n - 2\), then

\[
x_{j+1} - x_j \sim \frac{j^{\beta - 1}}{n^\beta} \leq C_\beta \frac{(n - j)^{\beta - 1}}{n^\beta}, \ 1 \leq j \leq n/2,
\]  

(15)

and

\[
x_{j+1} - x_j \sim \frac{(n - j)^{\beta - 1}}{n^\beta} \leq C_\beta \frac{j^{\beta - 1}}{n^\beta}, \ n/2 \leq j \leq n - 2.
\]  

(16)

Thus, by (14)-(16), we finish (9) in case \(k = 0\). For \(k = n - 1\), (9) can be proved in a similar way. For \(1 \leq k \leq n - 2\), by using (14)-(16) again, an ordinary calculation also leads to (9).

Finally, (10) and (11) are the known results of [4] (see also [7]).
Lemma 2.2 Write
\[ A_k(x) := \frac{|x - x_k|^{-\lambda}}{n-1} \sum_{k=0}^{n-1} \frac{|x - x_k|^{-\lambda}}{x - x_k}. \]

Let \( x \in [x_{j-1}, x_j], \ j = 1, 2, \ldots, n, \) then
\[ A_k(x) \leq C_{\lambda, \beta}(|j-k| + 1)^{-\lambda}. \quad (17) \]

Proof. If \( k = j, j+1, \) then (17) can be directly derived by the fact \( A_k(x) \leq 1. \) Assume that \( x \in [x_{j-1}, x_j], \ j = 1, 2, \ldots, n, \) and \( x_j \) is the closest node to \( x, \) then by (10) and (11), it is clear that
\[ A_k(x) \leq \frac{|x-x_j|^\lambda}{|x-x_k|^\lambda} \leq (|j-k| + 1)^{-\lambda}. \]
We have proved Lemma 2.2.

Lemma 2.3 Let \( f \in L^p_{[-1,1]}, 1 \leq p \leq \infty, \) then
\[ \|N_{n,\lambda}(f, X)\|_p \leq C_{\lambda, \beta} \|f\|_p, \ \lambda > \beta. \quad (18) \]

Proof. When \( p = \infty, \) we see that
\[ \|N_{n,\lambda}(f, X)\|_\infty = \max_{-1 \leq x \leq 1} |N_{n,\lambda}(f, X)| \]
\[ \leq \max_{-1 \leq x \leq 1} \left| \sum_{k=0}^{n-1} A_k(x) \frac{1}{x_{k+1} - x_k} \int_{x_k}^{x_{k+1}} f(t) dt \right| \]
\[ \leq \|f\|_\infty \sum_{k=0}^{n-1} A_k(x) \|_\infty = \|f\|_\infty. \]

When \( p = 1, \) a direct calculation leads to
\[ \|N_{n,\lambda}(f, X)\|_1 \leq \sum_{k=0}^{n-1} \int_{x_k}^{x_{k+1}} |f(t)| dt \frac{1}{x_{k+1} - x_k} \int_{x_k}^{1} A_k(x) dx \]
\[ = \sum_{k=0}^{n-1} \sum_{j=0}^{n-1} \int_{x_k}^{x_{k+1}} |f(t)| dt \frac{1}{x_{k+1} - x_k} \int_{x_j}^{x_{j+1}} A_k(x) dx \]
\[ \leq C_{\beta} \sum_{k=0}^{n-1} \sum_{j=0}^{n-1} \int_{x_k}^{x_{k+1}} |f(t)| dt \frac{x_{j+1} - x_j}{x_{k+1} - x_k} ((j-k) + 1)^{-\lambda} \quad \text{(by (17))} \]
\[ \leq C_{\beta} \sum_{k=0}^{n-1} \sum_{j=0}^{n-1} \int_{x_k}^{x_{k+1}} |f(t)| dt ((j-k) + 1)^{-\lambda+\beta-1} \quad \text{(by (9))} \]
\[ \leq C_{\lambda, \beta} \sum_{k=0}^{n-1} \int_{x_k}^{x_{k+1}} |f(t)| dt = C_{\lambda, \beta} \|f\|_1. \]
Hence (18) holds for \( p = 1. \) Therefore, by Riesz-Thorin theorem, we obtain Lemma 2.3.

Lemma 2.4 Let \( f \in L^p_{[-1,1]}, \) then
\[ \|\phi(x)N^\prime_{n,\lambda}(f, X)\|_p \leq C_{\lambda, \beta, p} \|f\|_p \quad (19) \]
for \( 1 \leq p < \infty. \)
Proof. Assume that \( x \in (x_j, (x_j + x_{j+1})/2),\ 0 \leq j \leq n - 1 \), that is, \( x_j \) is the closest node to \( x \). By the definition of \( N_{n,\lambda}(f, X, x) \), we have

\[
N'_{n,\lambda}(f, X, x) = -\lambda \left( \sum_{k=0}^{n-1} |x - x_k|^{-\lambda} \right)^{-2} \left\{ \left( \sum_{k=0}^{j} |x - x_k|^{-\lambda-1} \frac{1}{x_{k+1} - x_k} \int_{x_k}^{x_{k+1}} f(t)dt \right) \left| x - x_j \right|^{-\lambda} \right. \\
- \sum_{k=j+1}^{n-1} |x - x_k|^{-\lambda-1} \frac{1}{x_{k+1} - x_k} \int_{x_k}^{x_{k+1}} f(t)dt \left| x - x_k \right|^{-\lambda} \\
- \left. \left( \sum_{k=0}^{j} |x - x_k|^{-\lambda-1} - \sum_{k=j+1}^{n-1} |x - x_k|^{-\lambda-1} \right) \right\}.
\]

Since the sum of the terms in the right hand

\[-\lambda |x - x_j|^{-\lambda-1} \frac{1}{x_{j+1} - x_j} \int_{x_j}^{x_{j+1}} f(t)dt |x - x_j|^{-\lambda} \]

and

\[\lambda |x - x_j|^{-\lambda} \frac{1}{x_{j+1} - x_j} \int_{x_j}^{x_{j+1}} f(t)dt |x - x_j|^{-\lambda-1}\]

is zero, then

\[
|\phi(x)N'_{n,\lambda}(f, X, x)|^p \leq C_{\lambda,p} \phi(x) \left| \sum_{k \neq j} |x - x_k|^{-\lambda-1} \frac{1}{x_{k+1} - x_k} \int_{x_k}^{x_{k+1}} f(t)dt \sum_{k \neq j} |x - x_k|^{-\lambda} \right|^p \left( \sum_{k=0}^{n-1} |x - x_k|^{-\lambda} \right)^2
\]
we obtain that

\[
\sum_{k \neq j} \left| x - x_k \right|^{-\lambda} \frac{1}{x_k + 1 - x_k} \int_{x_k}^{x_{k+1}} |f(t)| dt \sum_{k \neq j} \left| x - x_k \right|^{-\lambda-1} \left( \sum_{k=0}^{n-1} \left| x - x_k \right|^{-\lambda} \right)^2
\]

By (10), (11) and the obvious fact

\[
\frac{1}{\sum_{k=0}^{n-1} \left| x - x_k \right|^{-\lambda}} \leq \left| x - x_j \right|^{-\lambda},
\]

we obtain that

\[
\left| S_1 \right| \leq C_{\lambda,p} \left| x - x_j \right|^{2\lambda} \sum_{k \neq j} \left| x - x_k \right|^{-\lambda-1} \frac{1}{x_{k+1} - x_k} \int_{x_k}^{x_{k+1}} |f(t)| dt \sum_{k \neq j} \left| x - x_k \right|^{-\lambda}
\]

\[
\leq C_{\lambda,p} \sum_{k \neq j} (|j - k| + 1)^{-\lambda-1} \frac{1}{x_{k+1} - x_k} \int_{x_k}^{x_{k+1}} |f(t)| dt \sum_{k \neq j} (|j - k| + 1)^{-\lambda}
\]

\[
\leq C_{\lambda,p} \sum_{k \neq j} (|j - k| + 1)^{-\lambda-1} \frac{1}{x_{k+1} - x_k} \int_{x_k}^{x_{k+1}} |f(t)| dt
\]

\[
\leq C_{\lambda,p} n^p \sum_{k \neq j} (|j - k| + 1)^{-\lambda-1} \frac{1}{x_{k+1} - x_k} \int_{x_k}^{x_{k+1}} |f(t)|^p dt
\]

\[
\leq C_{\lambda,p} n^p \sum_{k \neq j} (|j - k| + 1)^{-\lambda} \frac{1}{x_{k+1} - x_k} \int_{x_k}^{x_{k+1}} |f(t)|^p dt,
\]
where in the fourth inequality, we used Hölder’s inequality for \( p > 1 \). Therefore, by (9), we get
\[
\int_{x_j}^{(x_j+x_{j+1})/2} |S_1| \, dx \leq \sum_{k \neq j} C_{\lambda,p} n^p \int_{x_j}^{x_{j+1}} \left( \sum_{k \neq j} |x_j - x_k| \right)^p \, dx \leq \sum_{k \neq j} C_{\lambda,p} n^p \int_{x_j}^{x_{j+1}} |x_j - x_k|^{-\lambda} \, dx.
\]
A similar discussion leads to
\[
\int_{x_j}^{(x_j+x_{j+1})/2} |S_i| \, dx \leq C_{\lambda,p} n^p \sum_{k \neq j} (|j-k| + 1)^{-\lambda + \beta - 1} \int_{x_k}^{x_{k+1}} |f(t)|^p \, dt, \quad i = 2, 4, 6.
\]
Now, we can deduce (19) from (26) and (27) by estimating
\[
\|\phi(x)N_{n,\lambda}^p(f, X, x)\|^p \leq \sum_{j=0}^{n-1} \int_{x_j}^{x_{j+1}} |\phi(x)N_{n,\lambda}^p(f, x)|^p dx
\]
\[
\leq C_{\lambda, p} n^p \sum_{j=0}^{n-1} \sum_{k \neq j} ((j-k)+1)^{-\lambda+\beta-1} \int_{x_k}^{x_{k+1}} |f(t)|^p dt
\]
\[
+ C_{\lambda, p} n^p \sum_{j=0}^{n-1} \int_{x_j}^{x_{j+1}} |f(t)|^p dt
\]
\[
\leq C_{\lambda, \beta, p} n^p \left( \sum_{k \neq j} \int_{x_k}^{x_{k+1}} |f(t)|^p dt + \|f\|^p \right)
\]
\[
\leq C_{\lambda, \beta, p} n^p \|f\|^p.
\]

**Lemma 2.5** Let \( f' \in L^p_{[-1,1]} \) then
\[
\|\phi(x)N_{n,\lambda}^p(f, X, x)\|_p \leq C_{\lambda, \beta, p} \left( \|\phi f\|_p + \frac{1}{n^{\beta-1}} \|f'\|_p \right)
\]
for \( 1 \leq p < \infty \).

**Proof.** Since \( \sum_{k=0}^{n-1} A_k(x) \equiv 1 \), then \( \sum_{k=0}^{n-1} A'_k(x) \equiv 0 \). For any \( x \in (x_j, (x_j + x_{j+1})/2) \), \( 0 \leq j \leq n-1 \), in a similar way to Lemma 2.4, it follows that
\[
N_{n,\lambda}^p(f, x, x) = \sum_{k=0}^{n-1} A'_k(x) \frac{1}{x_{k+1} - x_k} \int_{x_k}^{x_{k+1}} (f(t) - f(x)) dt
\]
\[
= - \sum_{k=0}^{n-1} A'_k(x) \frac{1}{x_{k+1} - x_k} \int_{x_k}^{x_{k+1}} \int_{x_k}^{x} f'(u) du dt
\]
\[
= \lambda \left( \sum_{k=0}^{n-1} |x - x_k|^{-\lambda} \right)^{-2} \left\{ \left( \sum_{k=0}^{n-1} |x - x_k|^{-\lambda-1} \frac{1}{x_{k+1} - x_k} \int_{x_k}^{x_{k+1}} \int_{x_k}^{x} f'(u) du dt - \sum_{k=0}^{n-1} |x - x_k|^{-\lambda-1} \int_{x_k}^{x_{k+1}} \int_{x_k}^{x} f'(u) du dt \right) \sum_{k=0}^{n-1} |x - x_k|^{-\lambda}
\]
\[
- \sum_{k=0}^{n-1} \left[ |x - x_k|^{-\lambda-1} - \sum_{k=0}^{n-1} |x - x_k|^{-\lambda-1} \right] \right\}.
\]
Hence
\[
|\phi(x)N_{n,\lambda}^p(f, X, x)|^p \leq C_{\lambda, p} \left| \phi(x) \frac{\sum_{k \neq j} |x - x_k|^{-\lambda-1} \frac{1}{x_{k+1} - x_k} \int_{x_k}^{x_{k+1}} |f'(u)| du dt \sum_{k \neq j} |x - x_k|^{-\lambda}}{\left( \sum_{k=0}^{n-1} |x - x_k|^{-\lambda} \right)^2} \right|^p.
\]
Define the Hardy-Littlewood maximum function $M(f, x)$ by

$$M(f, x) := \sup_{-1 \leq t \leq 1} \frac{1}{x-t} \int_{t}^{x} |f(u)| du.$$  

(29)

It is well known that

$$\|M(f)\|_{p} \leq C_{p}\|f\|_{p}, \quad p > 1.$$  

If $j = 0$, then by (7), (10), (11), (21), we get

$$|T_{1}| \leq C_{\lambda,p} \left| \phi(x)|x-x_{j}|^{2\lambda} \sum_{k \neq j} |x-x_{k}|^{-\lambda-1} \frac{1}{x_{k+1}-x_{k}} \int_{x_{k}}^{x_{k+1}} |f'(u)| du \right| dt \sum_{k \neq j} |x-x_{k}|^{-\lambda-1-1}^{p} \left| \sum_{k \neq j} \frac{1}{x_{k+1}-x_{k}} \int_{x_{k}}^{x_{k+1}} |f'(u)| du \right| dt$$

$$\leq C_{\lambda,p} \left[ \sum_{k \neq j} (k+1)^{-\lambda-1} \frac{1}{x_{k+1}-x_{k}} \int_{x_{k}}^{x_{k+1}} |x-t| \right] \left| \frac{1}{x-t} \int_{t}^{x} |f'(u)| du \right| dt$$

$$\leq C_{\lambda,\beta,n^{\beta}} \left[ \sum_{k \neq j} (k+1)^{-\lambda-1} \frac{1}{x_{k+1}-x_{k}} \int_{x_{k}}^{x_{k+1}} |x-t| dt \right] \left| \frac{1}{x-t} \int_{t}^{x} |f'(u)| du \right| dt$$

$$\leq C_{\lambda,\beta,n^{\beta}} \left[ \sum_{k \neq j} (k+1)^{-\lambda-1} |x_{k+1}-x| \right] \left| \frac{1}{x-t} \int_{t}^{x} |f'(u)| du \right| dt$$

$$\leq C_{\lambda,\beta,n^{\beta}} \left[ \sum_{k \neq j} (k+1)^{-\lambda-1+\beta} |x_{k+1}-x| \right] \left| \frac{1}{x-t} \int_{t}^{x} |f'(u)| du \right| dt$$

$$\leq C_{\lambda,\beta,n^{\beta}} \left[ \sum_{k \neq j} (k+1)^{-\lambda-1+\beta} \right] \left| \frac{1}{x-t} \int_{t}^{x} |f'(u)| du \right| dt$$

$$\leq C_{\lambda,\beta,n^{\beta}} \left[ \sum_{k \neq j} (k+1)^{-\lambda-1+\beta} \right] \left| \frac{1}{x-t} \int_{t}^{x} |f'(u)| du \right| dt.$$
which implies for \( p > 1 \) that (by (29))
\[
\int_{-1}^{(1+x_1)/2} |T_1| \, dx \leq C_{\lambda, \beta, p} n^{-\beta p + p} \int_{-1}^{(1+x_1)/2} |M(f', x)|^p \, dx \\
\leq C_{\lambda, \beta, p} n^{-\beta p + p} \|f'\|_p^p.
\] (30)

Set
\[
t(j, k) := \begin{cases} 
  x_{k+1}, & j \leq k, \\
  x_k, & j > k, 
\end{cases}
\]
and
\[
x(j, k) := \begin{cases} 
  x_j, & j \leq k, \\
  x_{j+1}, & j > k 
\end{cases}
\]
for \( j, k = 0, 1, \ldots, n-1 \). For \( p = 1 \), we have
\[
|T_1| \leq C_{\lambda, n} \sum_{k \neq 0} (k + 1)^{-\lambda - 1} \left| \int_{x(0,k)}^{t(0,k)} |f'(u)| \, du \right|,
\]
which indicates that
\[
\int_{-1}^{(1+x_1)/2} |T_1| \, dx \leq C_{\lambda, \beta, n} n^{-\beta + 1} \|f'\|_1 \\
\leq C_{\lambda, \beta, p} n^{-\beta + 1} \|f'\|_1.
\] (31)

Combining (30) and (31), we obtain that
\[
\int_{-1}^{(1+x_1)/2} |T_1| \, dx \leq C_{\lambda, \beta, p} n^{-\beta p + p} \|f'\|_p^p, \quad 1 \leq p < \infty.
\]

Similar arguments lead to that²
\[
\int_{-1}^{(1+x_1)/2} |T_1| \, dx \leq C_{\lambda, \beta, p} n^{-\beta p + p} \|f'\|_p^p, \quad i = 2, 3, 4, 5, 6,
\]
and
\[
\int_{1+x_{n-1}/2}^1 |T_i| \, dx \leq C_{\lambda, \beta, p} n^{-\beta p + p} \|f'\|_p^p, \quad i = 1, 2, 3, 4, 5, 6.
\]

Thus,
\[
\int_{-1}^{(1+x_1)/2} |\phi(x)N_{n, \lambda}^p(f, X, x)|^p \, dx \leq C_{\lambda, \beta, p} n^{-\beta p + p} \|f'\|_p^p, \quad i = 1, 2, 3, 4, 5, 6,
\] (32)
and
\[
\int_{1+x_{n-1}/2}^1 |\phi(x)N_{n, \lambda}^p(f, X, x)|^p \, dx \leq C_{\lambda, \beta, p} n^{-\beta p + p} \|f'\|_p^p, \quad i = 1, 2, 3, 4, 5, 6.
\] (33)

² Note that \( \lambda > \beta + 1 \).
Assume that \( x \in [x_j, (x_j + x_{j+1})/2] \), \( 1 \leq j \leq n - 2 \). By (6), (10), (11) and (21), we get for \( p > 1 \) that
\[
|T_1| \leq C_{\lambda,\beta} n \sum_{k \neq j} |(j-k)|^{-p} \frac{1}{|x-t(j,k)|} \int_{t(j,k)}^{x} |x-t(j,k)||f'(u)|\,du \leq C_{\lambda,\beta} n \sum_{k \neq j} |(j-k)|^{-p} \frac{1}{|x-t(j,k)|} \int_{t(j,k)}^{x} |\phi(u)f'(u)|\,du
\]
for \( j < k \), with another case being treated exactly in the same way. It follows from (37) that another case can be treated exactly in the same way. Altogether, we get for \( p > 1 \) that
\[
\int_{x_j}^{(x_j+x_{j+1})/2} |\phi(x) N_{n,\lambda}(f, X, x)|^p \,dx \leq C_{\lambda,\beta} n \int_{x_j}^{(x_j+x_{j+1})/2} |M(f', x)|^p \,dx + C_{\lambda,\beta} n^{-\beta p + \beta p} \int_{x_j}^{(x_j+x_{j+1})/2} |M(f', x)|^p \,dx.
\]
For \( p = 1 \), we have
\[
|T_1| \leq C_{\lambda,\beta} n \sum_{k \neq j} |(j-k)|^{-1} \frac{1}{|x-t(j,k)|} \int_{t(j,k)}^{x} |f'(u)||x-t(j,k)|\,du \leq C_{\lambda,\beta} n \sum_{k \neq j} |(j-k)|^{-1} \frac{1}{|x-t(j,k)|} \int_{t(j,k)}^{x} |\phi(u)f'(u)|\,du
\]

We need the following inequality to continue our work:
\[
\frac{x_{j+1} - x_j}{|x(j,k) - t(j,k)|} \leq C_\beta |(j-k)|^{-1}, \quad 1 \leq j \leq n - 2.
\]
We only prove (36) for the case \( j < k \), while another case can be treated exactly in the same way. It follows from (16) and (11) for \( j \geq 1 \) that
\[
|f(x_j) - f(x_{j+1})| \leq C_\beta \frac{\phi(x_j)}{n},
\]
and

\[ |x_{k+1} - x_j| \geq C_\beta \frac{\phi(x_j)}{n} (|j - k| + 1). \]  

(38)

Combining (37)-(38), we obtain (36).

Now, by (35) and (36), we can drive that

\[
\int_{x_j}^{(x_j+x_{j+1})/2} |T_1|dx \leq C_{\lambda,\beta} \sum_{k \neq j} ((j - k) + 1)^{-\lambda - 1 + \beta} \frac{x_{j+1} - x_j}{|x(j,k) - t(j,k)|} \int_{x(j,k)}^{t(j,k)} \phi(u) f'(u)du \]

\[
\leq C_{\lambda,\beta} \sum_{k \neq j} ((j - k) + 1)^{-\lambda + \beta - 1} \int_{x(j,k)}^{t(j,k)} \phi(u) f'(u)du \]

Similarly, we can prove the above estimate to hold when \( T_1 \) is replaced by \( T_i \) for \( 1 \leq i \leq 6 \). Therefore,

\[
\int_{x_j}^{(x_j+x_{j+1})/2} |\phi(x) N'_{n,\lambda}(f, X, x)|dx \leq C_{\lambda,\beta,p} \sum_{k = 0}^{n-1} ((j - k) + 1)^{-\lambda + \beta - 1} \int_{x(j,k)}^{t(j,k)} \phi(u) f'(u)du . \]

(39)

The same arguments also lead to that

\[
\int_{(-1+x_{1})/2}^{x_{1}} |\phi(x) N'_{n,\lambda}(f, X, x)|^pdx \leq C_{\lambda,\beta,p} n^{-\beta p + p} \|f'\|_p^p, \; i = 1, 2, 3, 4, 5, 6, \]

(40)

\[
\int_{x_{n-1}}^{(1+x_{n-1})/2} |\phi(x) N'_{n,\lambda}(f, X, x)|^pdx \leq C_{\lambda,\beta,p} n^{-\beta p + p} \|f'\|_p^p, \; i = 1, 2, 3, 4, 5, 6, \]

(41)

\[
\int_{(x_{j}+x_{j+1})/2}^{x_{j+1}} |\phi(x) N'_{n,\lambda}(f, X, x)|^pdx \leq C_{\lambda,\beta,p} \int_{(x_{j}+x_{j+1})/2}^{x_{j+1}} |M(\phi f', x)|^pdx + C_{\lambda,\beta,p} n^{-\beta p + p} \int_{(x_{j}+x_{j+1})/2}^{x_{j+1}} |M(f', x)|^pdx, \; p > 1, \]

(42)

and

\[
\int_{(x_{j}+x_{j+1})/2}^{x_{j+1}} |\phi(x) N'_{n,\lambda}(f, X, x)|dx \leq C_{\lambda,\beta,p} \sum_{k = 0}^{n-1} ((j - k) + 1)^{-\lambda + \beta - 1} \int_{x(j,k)}^{t(j,k)} \phi(u) f'(u)du . \]

(43)
Suppose that $(1)$ inequality $M$ where

Then there exists a constant $t$ and

We complete Lemma 2.5 by (44) and (45).

and for $p > 1$ that

and for $p = 1$ that

We complete Lemma 2.5 by (44) and (45).

By [10: Theorem 2.1.1] and [10: Theorem 3.1.2], we have

Lemma 2.6 Suppose that $f \in L^p_{[-1, 1]}$, $1 \leq p < \infty$. Define

where $M_2$ is a positive constant only depending on $M, \alpha, r$ and $a$. 

Lemma 2.7 ([11]) Suppose that $\rho(t)$ increases on $[0, a]$, $0 < \alpha < 1$, $r > \alpha$. If for any $h, t \in [0, a]$, the inequality

holds, then

\[ \rho(t) \leq M_2 t^\alpha, \quad t \in [0, a], \]
3 Proof of Theorems.

Proof of Theorem 1.5 By Lemma 2.6, for any \( f \in L^p_{[-1, 1]} \), \( 1 \leq p < \infty \), we can choose a \( g \) such that
\[
\| f - g \|_p \leq C\omega_f \left( \frac{f}{\lambda} \right)_p, \tag{46}
\]
\[
\| g' \|_p \leq Cn^\beta \omega_f \left( \frac{f}{\lambda} \right)_p, \tag{47}
\]
\[
\| \phi g' \|_p \leq nC\omega_f \left( \frac{f}{\lambda} \right)_p. \tag{48}
\]
By the definition of \( N_{n, \lambda}(f, X) \), we have
\[
\| g - N_{n, \lambda}(g, X) \|_p^p \leq \int_{-1}^{1} \left| \sum_{k=0}^{n-1} A_k(x) \frac{1}{x_{k+1} - x_k} \int_{t}^{x} |g'(u)|du \right|^p dx
+ \int_{-1}^{1} \sum_{k=0}^{n-1} A_k(x) \frac{1}{x_{k+1} - x_k} \int_{t}^{x} |g'(u)|du \left| dx \right|^p
+ \int_{-1}^{1} \sum_{k=0}^{n-1} A_k(x) \frac{1}{x_{k+1} - x_k} \int_{t}^{x} |g'(u)|du \left| dx \right|^p
=: I_1 + I_2 + I_3. \tag{49}
\]
First assume that \( p > 1 \). By (17) and (7), we deduce that
\[
I_1 \leq C_\beta \int_{-1}^{1} \sum_{k=0}^{n-1} (k + 1)^{-\lambda} \left| \int_{t}^{(1,k)} g'(u)du \right|^p dx + C_\beta \int_{-1}^{1} \left| \int_{t}^{(1,k)} g'(u)du \right|^p dx
+ C_\beta (1 + x_1)^p \int_{-1}^{x_1} |g'(x)|^p dx
\leq C_\beta n^{-\beta p} \int_{-1}^{x_1} |M(g', x)|^p \left| \sum_{k=0}^{n-1} (k + 1)^{-\lambda + \beta} \right| dx + C_\beta n^{-\beta p} \int_{-1}^{x_1} |g'(x)|^p dx
\leq C_{\lambda, \beta} n^{-\beta p} \left( \int_{-1}^{1} |M(g', x)|^p dx + \int_{-1}^{x_1} |g'(x)|^p dx \right) \leq C_{\lambda, \beta} n^{-\beta p} \| g' \|_p^p. \tag{50}
\]
Similarly, we produce that
\[
I_2 \leq C_{\lambda, \beta} n^{-\beta p} \| g' \|_p^p. \tag{51}
\]
By (6) and (17), we have
\[
I_3 \leq C_\beta \sum_{j=1}^{n-2} \int_{x_j}^{x_{j+1}} \sum_{k=0}^{n-1} (j - k + 1)^{-\lambda} \left| \frac{1}{x - t(j,k)} \int_{t}^{(j,k)} g'(u)|x - t(j,k)|du \right|^p dx
\leq C_\beta n^{-p} \sum_{j=1}^{n-2} \int_{x_j}^{x_{j+1}} \sum_{k=0}^{n-1} (j - k + 1)^{-\lambda + \beta} \left| \frac{1}{x - t(j,k)} \int_{t}^{(j,k)} \phi(u)g'(u)du \right|^p dx
\leq C_{\lambda, \beta} n^{-p} \sum_{j=1}^{n-2} \int_{x_j}^{x_{j+1}} |M(\phi g', x)|^p dx
\leq C_{\lambda, \beta} n^{-p} \| M(\phi g') \|_p^p \leq C_{\lambda, \beta} n^{-p} \| \phi g' \|_p^p. \tag{52}
\]
Thus we have proved (2) for $p > 1$.

Next, we consider the case $p = 1$. We surely have

$$I_1 \leq C_\beta \int_{-1}^{1} \sum_{k=0}^{n-1} (k + 1)^{-\lambda} \left| \int_{0}^{1} |g'(u)|du \right| dx$$

$$\leq C_\beta n^{-\lambda} \sum_{k=0}^{n-1} (k + 1)^{-\lambda} \int_{0}^{1} |g'(u)|du$$

$$\leq C_\lambda, \beta n^{-\lambda} \|g'\|_1.$$  

Similarly,

$$I_2 \leq C_\lambda, \beta n^{-\lambda} \|g'\|_1.$$  

For $I_3$, we see that

$$I_3 \leq C_\beta n^{-\lambda} \sum_{j=1}^{n-2} \sum_{k=0}^{n-1} (|j - k| + 1)^{-\lambda+\beta} \left| \frac{1}{x - t(j,k)} \int_{x}^{t(j,k)} \phi(u)g'(u)du \right| dx$$

$$\leq C_\beta n^{-\lambda} \sum_{j=1}^{n-2} \sum_{k=0}^{n-1} (|j - k| + 1)^{-\lambda+\beta} \left| \frac{x_j + x_j - x_j}{x(j,k) - t(j,k)} \int_{x(j,k)}^{t(j,k)} \phi(u)g'(u)du \right|$$

$$\leq C_\beta n^{-\lambda} \sum_{j=1}^{n-2} \sum_{k=0}^{n-1} (|j - k| + 1)^{-\lambda+\beta-1} \int_{x(j,k)}^{t(j,k)} \phi(u)g'(u)du$$

(by (36))

$$\leq C_\beta n^{-\lambda} \sum_{j=1}^{n-2} \sum_{k=0}^{n-1} (|j - k| + 1)^{-\lambda+\beta-1} \int_{x(j,k)}^{t(j,k)} \phi(u)g'(u)du$$

$$\leq C_\beta n^{-\lambda} \sum_{j=1}^{n-2} \sum_{m=1}^{n-1} (m + 1)^{-\lambda+\beta-1} \int_{x(j,k)}^{t(j,k)} \phi(u)g'(u)du$$

$$\leq C_\beta n^{-\lambda} \sum_{j=1}^{n-2} \sum_{m=1}^{n-1} (m + 1)^{-\lambda+\beta} \|\phi(u)g'(u)\|_1 \leq C_\lambda, \beta n^{-\lambda} \|\phi(u)g'(u)\|_1.$$  

 Altogether, the estimates (54)-(56), with the same argument as that of $p > 1$, yield (2) for $p = 1$.

Now we turn to prove (3). By Lemmas 2.4-2.6, we can choose a proper $g \in A.C., \ g' \in L^{p}_{[-1,1]}'$, such that

$$\|\phi N_{n,\lambda}'(f, X)\|_p \leq \|\phi N_{n,\lambda}'(f - g, X)\|_p + \|\phi N_{n,\lambda}'(g, X)\|_p$$

$$\leq C_\lambda, \beta, p (n\|f - g\|_p + \|\phi g'\|_p + n^{-\lambda+\beta} \|\phi g'\|_p)$$

$$\leq C_\lambda, \beta, p nK_{\phi} \left( f, \frac{1}{n} \right)_p \leq C_\lambda, \beta, p n\omega_{\phi} \left( f, \frac{1}{n} \right)_p.$$  

(57)
By Lemma 2.6, (2) and (57), we obtain that
\[
\omega_{\phi} \left( f, \frac{1}{n} \right)_p \leq CK_{\phi} \left( f, \frac{1}{n} \right)_p \\
\leq C \left( \|f - N_{n,\lambda}(f, X)\|_p + n^{-1} \|\phi N'_{n,\lambda}(f, X)\|_p \right) \\
\leq C_{\lambda,\beta,p} \omega_{\phi} \left( f, \frac{1}{n} \right)_p.
\]
Hence, we finish (3).

By using (2) again, if \( \omega_{\phi} (f, t)_p = O(t^\alpha) \), it is easy to obtain that
\[
\|f - N_{n,\lambda}(f, X)\|_p = O(n^{-\alpha}).
\]
We verify the converse inequality. By the definition of \( K_{\phi}(f, t)_p \), Lemma 2.4 and Lemma 2.5 yield that\(^3\)
\[
K_{\phi} \left( f, \frac{1}{n} \right)_p \leq \|f - N_{k,\lambda}(f, X)\|_p + \frac{1}{n} \|\phi N'_{k,\lambda}(f, X)\|_p \\
\leq \|f - N_{k,\lambda}(f, X)\|_p + \frac{1}{n} \left( \|\phi N'_{k,\lambda}(f - g, X)\|_p + \|\phi N'_{k,\lambda}(g, X)\|_p \right) \\
\leq \|f - N_{k,\lambda}(f, X)\|_p + \frac{k}{n} \left( \|f - g\|_p + \frac{1}{K} \|\phi g^\prime\|_p + \frac{1}{K^\beta} \|g^\prime\|_p \right) \\
\leq \|f - N_{k,\lambda}(f, X)\|_p + C_{\lambda,\beta,p} \frac{k}{n} K_{\phi} \left( f, \frac{1}{K} \right)_p \\
\leq \|f - N_{k,\lambda}(f, X)\|_p + C_{\lambda,\beta,p} \frac{k}{n} K_{\phi} \left( f, \frac{1}{K} \right)_p.
\]
Hence, if
\[
\|f - N_{n,\lambda}(f, X)\|_p = O(n^{-\alpha}),
\]
then
\[
K_{\phi} \left( f, \frac{1}{n} \right)_p \leq C_{\lambda,\beta,p} k^{-\beta} + C_{\lambda,\beta,p} \frac{k}{n} K_{\phi} \left( f, \frac{1}{K} \right)_p.
\]
By the well known result of Berens and Lorentz [2], we have finished (4). Finally, (5) can be deduced by the same way as [5] or [10].

**Proof of Theorem 1.6** First, we verify that \( (A) \Rightarrow (B) \). From (2) and the Condition (A), it follows that
\[
\omega_{\phi} \left( f, \frac{1}{n} \right)_p \geq C_{\lambda,\beta,p} n^{-\alpha}.
\]
For any \( \delta \in (0, 1) \), take \( n \) such that \( 1/n \leq \delta < 1/(n - 1) \). The properties of modulus of continuity and (58) lead to that
\[
\omega_{\phi}(f, \delta)_p \geq \omega_{\phi} \left( f, \frac{1}{n} \right)_p \geq C_{\lambda,\beta,p} n^{-\alpha} \geq C_{\lambda,\beta,p} \delta^\alpha.
\]
We verify the converse inequality. For \( 0 < h \leq \delta \), working as [10: (2.4.3), (2.4.4)], we obtain that
\[
\|\Delta_{h,\phi} f\|_p \leq \|\Delta_{h,\phi}(f - N_{n,\lambda}(f, X))\|_p + \|\Delta_{h,\phi} N_{n,\lambda}(f, X)\|_p \\
\leq C \left( \|f - N_{n,\lambda}(f, X)\|_p + h \|\phi N'_{n,\lambda}(f, X)\|_p \right).
\]
\(^3\) Choose \( g \in A.C. \) such that \( \|f - g\|_p + k^{-1} \|\phi g^\prime\|_p + k^{-\beta} \|g^\prime\|_p \leq C K_{\phi} (f, k^{-1})_p. \)
Hence, it follows from Condition (A) and (57) that
\[ \|\Delta_{n,\delta}(x)f\|_p \leq C_{\lambda,\beta,\xi}(n^{-\alpha} + \delta_n \omega(f, \frac{1}{n})_p), \]
which implies
\[ \omega(f, \delta)_p \leq C_{\lambda,\beta,\xi}(n^{-\alpha} + \delta_n \omega(f, \frac{1}{n})_p). \]
The above inequality with Lemma 2.7 thus yields that
\[ \omega(f, \delta)_p \leq C_{\lambda,\beta,\xi} \delta^\alpha. \]
Combining (59) and (60), we finally achieve that
\[ \omega(f, \delta)_p \sim \delta^\alpha. \]

The proof of (B) \( \Rightarrow \) (A) can be proceeded exactly in the same way as that of [16], we omit the details here.

**Remark.** There is a lack of symmetry in the definition of \( N_{n,\lambda}(f, x) \) in contrast to the operators defined by Della Vecchia [7], for \( f(x) = f(-x) \) does not imply \( N_{n,\lambda}(f, -x) = N_{n,\lambda}(f, x) \). This lack could be repaired in obvious way, for example by \( \frac{1}{2} \left( N_{n,\lambda}(f, x) + \tilde{N}_{n,\lambda}(f, x) \right) \), where
\[ \tilde{N}_{n,\lambda}(f, x) = \left( \frac{1}{n} \sum_{k=1}^{n} |x - x_k|^{-\lambda} \right)^{-1} \left( \frac{1}{n} \sum_{k=1}^{n} |x - x_k|^{-\lambda} \frac{1}{x_k - x_{k-1}} \int_{x_{k-1}}^{x_k} f(t) dt \right), \]
has analogous properties as \( N_{n,\lambda}(f, x) \).

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**References**