Approximation by Rational Operators in L^p Spaces

Dansheng Yu[*1,2] and **Songping Zhou**[**1,2]

[1] Institute of Mathematics, Zhejiang Sci-Tech University, Xiasha Economic Development Area, Hangzhou, Zhejiang 310018 China.

[2] Department of Mathematics, Statitics & Computer Science St. Francis Xavier University Antigonish, Nova Scotia, Canada B2G 2W5

Key words Rational operators, approximation rates, L^p spaces.
MSC (2000) 41A20, 41A30

The present paper introduces a kind of Kantorovich type Shepard operators. Complete results including direct and converse results, equivalence results are established. As Della Vecchia and Mastroianni ([4], [7]) did, our results involve a weighted modulus of smoothness related to step-functions $\phi(x)$ vanishing algebraically at the endpoints ± 1.

1 Introduction

Let $X := \{x_{nk} = x_k \in [-1,1],\ k = 0,1,2,\cdots,n;\ n \in N\}$ be an infinite matrix where each row is a set of distinct points in $[-1,1]$. Define for $f \in C_{[-1,1]}$ the Shepard operator

$$S_{n,\lambda}(f,X,x) := \frac{\sum\limits_{k=0}^{n} |x-x_k|^{-\lambda} f(x_k)}{\sum\limits_{k=0}^{n} |x-x_k|^{-\lambda}},\quad \lambda > 1.$$

Obviously, $S_{n,\lambda}(f,X,x)$ are positive operators preserving constants and interpolating the function at the nodes $\{x_k\}_{k=0}^n$. Since their unique values in approximation theory and in fitting dates, curves and surfaces, Shepard operators have been studied extensively by many mathematicians, and many interesting results were achieved. Among them, Della Vecchia and Mastroianni [7] proved that $S_{n,\lambda}(f,X,x)$ give a better pointwise estimate than algebraic polynomials if only X is properly chosen. Let

$$x = x(\theta) = \begin{cases} (2\theta)^{2q+1} - 1, & \theta \in [0,1/2] \\ -(2-2\theta)^{2q+1} + 1, & \theta \in [1/2,1], \end{cases} \quad q = 1,2,\cdots, \tag{1}$$

and $X_1 = \{x_k = x(k/n) : k = 0,1,\cdots,n,\ n \in N\}$, Della Vecchia and Mastroianni [7] proved

Theorem 1.1 *For all $f \in C_{[-1,1]}$, it holds that*

$$|f(x) - S_{n,\lambda}(f,X_1,x)| \le C\omega\left(f, \frac{(1-x^2)^{2q/2q+1}}{n}\right),$$

where $x \in I = [-1,1],\ \lambda > 2$ and C is a constant depending only on q and λ.

Set $\psi(x) := (1-x^2)^{2q/(2q+1)}$, define

$$K_\psi(f,t) := \inf\left\{\|f-g\| + t\|\psi g'\|,\ g' \in C_{[-1,1]},\ g \in C_{[-1,1]}\right\},$$

* Corresponding author: e-mail: danshengyu@yahoo.com.cn, Research supported in part by NSERC RCD and AARMS of Canada.
** e-mail: szhou@zjip.com, W. F. James Chair Professor of St. Francis Xavier University. Research also supported in part by NSF of China under grant number 10471130.

and
$$\omega_\psi(f,t) := \sup_{0<h\le t} \|\Delta_{h\psi} f\|.$$
Della Vecchia [4] proved

Theorem 1.2 *For all $f \in C_{[-1,1]}$ and $\lambda > 2$, it holds that*
$$\|f - S_{n,\lambda}(f, X_1)\| \le C\omega\left(g, \frac{1}{n}\right) \sim CK_\psi\left(f, \frac{1}{n}\right) \sim C\omega_\psi\left(f, \frac{1}{n}\right),$$
$$K_\psi\left(f, \frac{1}{n}\right) \le \|f - S_{k,\lambda}(f, X_1)\| + C\frac{k}{n}K_\psi\left(f, \frac{1}{k}\right),$$
where C is a constant depending only on q and λ, $g(\theta) = f(x(\theta))$, with $x(\theta)$ given by (1)*. Moreover,*
$$\|f - S_{n,\lambda}(f, X_1)\| = O(n^{-\beta}) \Leftrightarrow \omega_\psi(f,t) = O(t^\beta),\ 0 < \beta < 1.$$

It should be noted that the results of Theorem B cannot be obtained by polynomial approximation (see [7]). For $q = 0$, that is, the nodes of matrix X are equally spaced (written as E), the direct and converse results, as well as the saturation theorems for $S_{n,\lambda}(f, E, x)$ have been studied by Della Vecchia, Mastroianni, Totik, Somorjai, Szabados, and other scholars (see [9], [12], [13], [18], [19]).

Recently, Della Vecchia, Mastroianni, and Szabados considered further the weighted approximation and approximation in an infinite interval by Shepard operators, readers could find related information in [5], [6], [8], [11], [13].

Write
$$\|f\|_p = \|f\|_{L^p_{[-1,1]}} = \left\{\int_{-1}^1 |f(x)|^p dx\right\}^{1/p},\ 1 \le p < \infty,$$
$$\|f\|_\infty := \|f\|_{L^\infty_{[-1,1]}} = \max_{-1\le x\le 1} |f(x)|.$$
$$\omega\left(f,\delta\right)_p = \sup_{0<h\le\delta} \|f(x+h) - f(x)\|_{L^p_{[-1,1-h]}}.$$
In the sequel, $C_{t_1,t_2,\cdots,t_l}$ denotes a positive constant only depending on the parameter(s) $t_1, t_2, \cdots, t_l$ which may be different in different occurrences. Moreover, we write $a \sim b$, if there is a $C_{t_1,t_2,\cdots,t_l}$ such that $C^{-1}_{t_1,t_2,\cdots,t_l} b \le a \le C_{t_1,t_2,\cdots,t_l} b$.

In L^p spaces, there are fewer results on the approximation of rational operators. Some works on this topic can be found in [15]-[17]. Among them, Xiao and Zhou [15] introduced the following Kantorovich type Shepard operators
$$K_{n,\lambda}(f, E, x) := (n+1)\frac{\sum_{k=0}^n \left|x - \frac{k}{n}\right|^{-\lambda} \int_{k/(n+1)}^{(k+1)/(n+1)} f(t)dt}{\sum_{k=0}^n \left|x - \frac{k}{n}\right|^{-\lambda}},$$
and established

Theorem 1.3 *For all $f \in L^p_{[0,1]},\ 1 \le p < \infty$,*
$$\|f - K_{n,\lambda}(f, E)\|_p \le C_{\lambda,p}\,\omega\left(f, \frac{1}{n}\right)_p,\ \lambda > 2,$$
where $C_{\lambda,p}$ is a constant only depending on p and λ

In another paper [16], Xiao and Zhou obtained the following equivalence theorem:

Theorem 1.4 *Let $f \in L^p_{[-1,1]},\ 1 \le p < \infty,\ 0 < \alpha < 1$ and $\lambda > 3$. Then*
$$\|f - K_{n,\lambda}(f, E)\|_p \sim n^{-\alpha} \Leftrightarrow \omega\left(f,\delta\right)_p \sim \delta^{-\alpha}.$$

It is not difficult to find that the power $2q+1$ in (1) can be replaced by a more general number $\beta \geq 1$, it implies that we can redefine the function $x(\theta)$ in (1) by the following more general way (still denoted by $x(\theta)$)

$$x(\theta) := \begin{cases} (2\theta)^\beta - 1, & \theta \in [0, 1/2] \\ -(2-2\theta)^\beta + 1, & \theta \in [1/2, 1] \end{cases}$$

with $\beta \geq 1$ fixed.

Set $X = \{x_k = x(k/n) : k = 0, 1, \cdots, n\}$ and $\phi(x) = (1-x^2)^{\frac{\beta-1}{\beta}}$. Define

$$N_{n,\lambda}(f, X, x) := \frac{\sum\limits_{k=0}^{n-1} |x - x_k|^{-\lambda} \frac{1}{x_{k+1}-x_k} \int_{x_k}^{x_{k+1}} f(t)dt}{\sum\limits_{k=0}^{n-1} |x - x_k|^{-\lambda}}.$$

In the present paper, we give approximation rate by $N_{n,\lambda}(f, X)$ for $f \in L^p_{[-1,1]}$, and also establish some converse results and equivalent results. As Della Vecchia and Mastroianni ([4], [7]) did, our results involve a weighted modulus of smoothness and an equivalent $K-$functional related to step-functions $\phi(x)$ vanishing algebraically at the endpoints ± 1. The main results are read as follows:

Theorem 1.5 *For all $f \in L^p_{[-1,1]},\ 1 \leq p < \infty,\ \lambda > \beta + 1$, it holds that*

$$\|f - N_{n,\lambda}(f, X)\|_p \leq C_{\lambda,\beta,p}\,\omega_\phi\left(f, \frac{1}{n}\right)_p, \tag{2}$$

$$\omega_\phi\left(f, \frac{1}{n}\right)_p \sim \|f - N_{n,\lambda}(f, X)\|_p + \frac{1}{n}\|\phi(x)N'_{n,\lambda}(f, X)\|_p, \tag{3}$$

$$\|f - N_{n,\lambda}(f, X)\|_p = O(n^{-\alpha}) \Leftrightarrow \omega_\phi(f, t)_p = O(t^\alpha), 0 < \alpha \leq 1, \tag{4}$$

$$\|\phi N'_{n,\lambda}(f, X)\| = O(n^{-\alpha+1}) \Leftrightarrow \omega_\phi(f, t)_p = O(t^\alpha),\ 0 < \alpha \leq 1, \tag{5}$$

where $\omega_\phi(f, t)_p$ is defined by

$$\omega_\phi(f, t)_p := \sup_{0<h\leq t} \|\Delta_{h\phi} f\|_p,$$

and

$$\Delta_{h\phi(x)} f(x) := \begin{cases} f\left(x + \frac{h\phi(x)}{2}\right) - f\left(x - \frac{h\phi(x)}{2}\right), & x \pm \frac{h\phi(x)}{2} \in [-1, 1], \\ 0, & \textit{otherwise.} \end{cases}$$

When $\beta = 1$, (2) is just the result of Theorem 1.3. Theorem 1.5 can also be understood as a generalization of Theorem 1.2 to the L^p spaces.

The following result improves Theorem 1.4:

Theorem 1.6 *Let $f \in L^p_{[-1,1]},\ 1 \leq p < \infty,\ 0 < \alpha < 1$ and $\lambda > \beta + 1$. Then the following two conditions are equivalent:*

(**A**) $\|f - N_{n,\lambda}(f, X)\|_p \sim n^{-\alpha}$.

(**B**) $\omega_\phi\left(f, \delta\right)_p \sim \delta^{-\alpha}$.

When $\beta = 1$, it becomes the result of Theorem 1.4, and we also loose the restriction $\lambda > 3$ there to $\lambda > 2$. Moreover, our proof has a different approach from [16].

In the following text, we always assume that $\lambda > \beta + 1$ with $\beta \geq 1$ fixed.

2 Auxiliary Lemmas

Lemma 2.1 *Let $x \in [x_{j-1}, x_j],\ j = 2, 3, \cdots, n-1$, then*

$$|x - x_k| \le C_\beta \frac{(|j-k|+1)^\beta}{n}\phi(u),\ u \in [x, x_k] \text{ or } [x_k, x] \tag{6}$$

for $k = 1, 2, \cdots, n-1$.

For $x \in [-1, x_1]$ and $k = 0, 1, 2, \cdots, n$, it holds that

$$|x - x_k| \le C_\beta \frac{(k+1)^\beta}{n^\beta}. \tag{7}$$

For $x \in [x_{n-1}, 1]$ and $k = 0, 1, 2, \cdots, n$, it holds that

$$|x - x_k| \le C_\beta \frac{(n-k+1)^\beta}{n^\beta}. \tag{8}$$

For $j, k = 0, 1, 2, \cdots, n-1$, it holds that

$$\frac{x_{j+1} - x_j}{x_{k+1} - x_k} \le C_\beta(|j-k|+1)^{\beta-1}. \tag{9}$$

Denote by $x_j,\ 0 \le j \le n$, the closest node to x, then

$$|x - x_j| \le C_\beta \frac{\phi(x)}{n}, \tag{10}$$

$$|x - x_k| \ge C_\beta \frac{\phi(x)}{n}|k - j|,\ k \ne j. \tag{11}$$

Proof. First, we prove (6). Let $x \in [x_{j-1}, x_j],\ j = 2, 3, \cdots, n-1$, we verify that

$$|x - x_k| \le C_\beta \frac{(|j-k|+1)^\beta}{n}\phi(x). \tag{12}$$

Case 1. $2 \le j \le k \le \frac{n}{2}$. Since $x'(\theta)$ is nondecreasing in $[0, 1/2]$ and[1] $x'(\theta) \sim \phi(x)$, then

$$\begin{aligned} |x - x_k| &\le x'\left(\frac{k}{n}\right)\left(\frac{k}{n} - \theta\right) \le C_\beta \frac{|j-k|+1}{n}\phi(x)\left(\frac{x'\left(\frac{k}{n}\right)}{x'(\theta)}\right) \\ &\le C_\beta \frac{|j-k|+1}{n}\phi(x)\left(\frac{k}{j}\right)^{\beta-1} \le C_\beta \frac{(|j-k|+1)^\beta}{n}\phi(x). \end{aligned}$$

Case 2. $1 \le k \le j-1 \le \frac{n}{2} - 1$. Similar to Case 1, we have

$$|x - x_k| \le x'(\theta)\left(\theta - \frac{k}{n}\right) \le C_\beta \frac{|j-k|+1}{n}\phi(x).$$

Case 3. $2 \le j \le \frac{n}{2} \le k \le n-1$. By the definition of $x(\theta)$, we derive that

$$\begin{aligned} |x - x_k| &\le |x - x(1/2)| + |x(1/2) - x_k| \\ &\le x'(1/2)\left(\frac{1}{2} - \theta\right) + x'(1/2)\left(\frac{k}{n} - \frac{1}{2}\right) \\ &\le x'(1/2)\left(\frac{k}{n} - \theta\right) \le C_\beta \frac{|j-k|+1}{n}\phi(x)\left(\frac{x'(1/2)}{x'(\theta)}\right) \\ &\le C_\beta \frac{|j-k|+1}{n}\phi(x)\left(\frac{n/2}{j}\right)^{\beta-1} \\ &\le C_\beta \frac{|j-k|+1}{n}\phi(x)\left(\frac{n/2-j}{j} + 1\right)^{\beta-1} \\ &\le C_\beta \frac{(|j-k|+1)^\beta}{n}\phi(x). \end{aligned}$$

[1] Note that $x'(\theta) = 2\beta\theta^{\beta-1} = 2^{2-\beta}((2\theta)^\beta - 1 + 1)^{(\beta-1)/\beta} = 2^{2-\beta}(x+1)^{(\beta-1)/\beta}$.

Case 4. $1 \le k \le \frac{n}{2} \le j \le n-1$. Similar to Case 3, we have

$$|x - x_k| \le |x - x(1/2)| + |x(1/2) - x_k| \le x'(1/2)\left(\theta - \frac{k}{n}\right).$$

If $j \le \frac{3}{4}n$, then $\phi(x) \sim 1$. Hence,

$$|x - x_k| \le x'(1/2)\frac{|j-k|+1}{n} \le C_\beta \frac{|j-k|+1}{n}\phi(x).$$

If $j > \frac{3}{4}n$, then

$$|x - x_k| \le \frac{|j-k|+1}{n}\left(\frac{x'(1/2)}{x'(\theta)}\right)\phi(x) \le C_\beta \frac{|j-k|+1}{n}\phi(x)\left(\frac{n}{n-j}\right)^{\beta-1}$$

$$\le C_\beta \frac{|j-k|+1}{n}\phi(x)(j+1)^{\beta-1} \le C_\beta \frac{(|j-k|+1)^\beta}{n}\phi(x).$$

Case 5. $\frac{n}{2} \le k \le j \le n-1$ or $\frac{n}{2} \le j \le k-1 \le n-2$. Similar to Case 1 and Case 2, we can easily deduce that

$$|x - x_k| \le C_\beta \frac{(|j-k|+1)^\beta}{n}\phi(x)$$

still holds.

Altogether, with all the above estimates, we conclude (12).

Similarly, we have

$$|x - x_k| \le C_\beta \frac{(|j-k|+1)^\beta}{n}\phi(x_k). \tag{13}$$

Combining (12) with (13), we obtain (6).

Assume that $x \in [-1, x_1]$, then for $k = 1, 2, \cdots, n$, by applying (6), we get

$$\begin{aligned}|x - x_k| &\le |x - x_1| + |x_1 - x_k| \le \frac{2^\beta}{n^\beta} + C_\beta \frac{(k+1)^\beta}{n}\phi(x_1)\\ &\le C_\beta \frac{(k+1)^\beta}{n^\beta}.\end{aligned}$$

For $k = 0$, the above estimate obviously holds. Thus, we have established (7). Inequality (8) can be deduced similarly.

Now, we begin to prove (9). If $k = 0$, then (9) obviously holds for $j = 0, n-1$ by noting the fact

$$x_1 - x_0 \sim x_n - x_{n-1} \sim \frac{1}{n^\beta}. \tag{14}$$

If $1 \le j \le n-2$, then

$$x_{j+1} - x_j \sim \frac{j^{\beta-1}}{n^\beta} \le C_\beta \frac{(n-j)^{\beta-1}}{n^\beta},\ 1 \le j \le n/2, \tag{15}$$

and

$$x_{j+1} - x_j \sim \frac{(n-j)^{\beta-1}}{n^\beta} \le C_\beta \frac{j^{\beta-1}}{n^\beta},\ n/2 \le j \le n-2. \tag{16}$$

Thus, by (14)-(16), we finish (9) in case $k = 0$. For $k = n-1$, (9) can be proved in a similar way. For $1 \le k \le n-2$, by using (14)-(16) again, an ordinary calculation also leads to (9).

Finally, (10) and (11) are the known results of [4] (see also [7]).

Lemma 2.2 *Write*

$$A_k(x) := \frac{|x-x_k|^{-\lambda}}{\sum\limits_{k=0}^{n-1}|x-x_k|^{-\lambda}}.$$

Let $x \in [x_{j-1}, x_j],\ j=1,2,\cdots,n$, *then*

$$A_k(x) \le C_{\lambda,\beta}(|j-k|+1)^{-\lambda}. \tag{17}$$

Proof. If $k=j-1,j,j+1$, then (17) can be directly derived by the fact $A_k(x)\le 1$. Assume that $x\in[x_{j-1},x_j],\ j=1,2,\cdots,n$, and x_j is the closest node to x, then by (10) and (11), it is clear that

$$A_k(x) \le \frac{|x-x_j|^{\lambda}}{|x-x_k|^{\lambda}} \le (|j-k|+1)^{-\lambda}.$$

We have proved Lemma 2.2.

Lemma 2.3 *Let* $f\in L^p_{[-1,1]},\ 1\le p\le\infty$, *then*

$$\|N_{n,\lambda}(f,X)\|_p \le C_{\lambda,p}\|f\|_p,\ \lambda>\beta. \tag{18}$$

Proof. When $p=\infty$, we see that

$$\begin{aligned}
\|N_{n,\lambda}(f,X)\|_\infty &= \max_{-1\le x\le 1}|N_{n,\lambda}(f,X,x)| \\
&\le \max_{-1\le x\le 1}\left|\sum_{k=0}^{n-1}A_k(x)\frac{1}{x_{k+1}-x_k}\int_{x_k}^{x_{k+1}}f(t)dt\right| \\
&\le \|f\|_\infty\left\|\sum_{k=0}^{n-1}A_k(x)\right\|_\infty = \|f\|_\infty.
\end{aligned}$$

When $p=1$, a direct calculation leads to

$$\begin{aligned}
\|N_{n,\lambda}(f,X)\|_1 &\le \sum_{k=0}^{n-1}\int_{x_k}^{x_{k+1}}|f(t)|dt\frac{1}{x_{k+1}-x_k}\int_{-1}^{1}A_k(x)dx \\
&= \sum_{k=0}^{n-1}\sum_{j=0}^{n-1}\int_{x_k}^{x_{k+1}}|f(t)|dt\frac{1}{x_{k+1}-x_k}\int_{x_j}^{x_{j+1}}A_k(x)dx \\
&\le C_\beta\sum_{k=0}^{n-1}\sum_{j=0}^{n-1}\int_{x_k}^{x_{k+1}}|f(t)|dt\frac{x_{j+1}-x_j}{x_{k+1}-x_k}(|j-k|+1)^{-\lambda}\quad \text{(by (17))} \\
&\le C_\beta\sum_{k=0}^{n-1}\sum_{j=0}^{n-1}\int_{x_k}^{x_{k+1}}|f(t)|dt(|j-k|+1)^{-\lambda+\beta-1}\quad \text{(by (9))} \\
&\le C_{\lambda,\beta}\sum_{k=0}^{n-1}\int_{x_k}^{x_{k+1}}|f(t)|dt = C_{\lambda,\beta}\|f\|_1.
\end{aligned}$$

Hence (18) holds for $p=1$. Therefore, by Riesz-Thorin theorem, we obtain Lemma 2.3.

Lemma 2.4 *Let* $f\in L^p_{[-1,1]}$, *then*

$$\|\phi(x)N'_{n,\lambda}(f,X)\|_p \le C_{\lambda,\beta,p}n\|f\|_p \tag{19}$$

for $1\le p<\infty$.

Proof. Assume that $x \in \big(x_j, (x_j + x_{j+1})/2\big),\ 0 \le j \le n-1$, that is, x_j is the closest node to x. By the definition of $N_{n,\lambda}(f, X, x)$, we have

$$
\begin{aligned}
N'_{n,\lambda}(f,X,x) &= -\lambda\left(\sum_{k=0}^{n-1}|x-x_k|^{-\lambda}\right)^{-2}\Bigg\{\Bigg(\sum_{k=0}^{j}|x-x_k|^{-\lambda-1}\frac{1}{x_{k+1}-x_k}\int_{x_k}^{x_{k+1}}f(t)dt\\
&\quad -\sum_{k=j+1}^{n-1}|x-x_k|^{-\lambda-1}\frac{1}{x_{k+1}-x_k}\int_{x_k}^{x_{k+1}}f(t)dt\Bigg)\sum_{k=0}^{n-1}|x-x_k|^{-\lambda}\\
&\quad -\sum_{k=0}^{n-1}|x-x_k|^{-\lambda}\frac{1}{x_{k+1}-x_k}\int_{x_k}^{x_{k+1}}f(t)dt\ \times\\
&\quad \times\left(\sum_{k=0}^{j}|x-x_k|^{-\lambda-1}-\sum_{k=j+1}^{n-1}|x-x_k|^{-\lambda-1}\right)\Bigg\}.
\end{aligned}
$$

Since the sum of the terms in the right hand

$$-\lambda|x-x_j|^{-\lambda-1}\frac{1}{x_{j+1}-x_j}\int_{x_j}^{x_{j+1}}f(t)dt|x-x_j|^{-\lambda}$$

and

$$\lambda|x-x_j|^{-\lambda}\frac{1}{x_{j+1}-x_j}\int_{x_j}^{x_{j+1}}f(t)dt|x-x_j|^{-\lambda-1}$$

is zero, then

$$|\phi(x)N'_{n,\lambda}(f,X,x)|^p \le C_{\lambda,p}\left|\phi(x)\frac{\sum\limits_{k\neq j}|x-x_k|^{-\lambda-1}\frac{1}{x_{k+1}-x_k}\int_{x_k}^{x_{k+1}}|f(t)|dt\sum\limits_{k\neq j}|x-x_k|^{-\lambda}}{\left(\sum\limits_{k=0}^{n-1}|x-x_k|^{-\lambda}\right)^2}\right|^p$$

$$\begin{aligned}
&+C_{\lambda,p}\left|\phi(x)\frac{\sum\limits_{k\neq j}|x-x_k|^{-\lambda}\frac{1}{x_{k+1}-x_k}\int_{x_k}^{x_{k+1}}|f(t)|dt\sum\limits_{k\neq j}|x-x_k|^{-\lambda-1}}{\left(\sum\limits_{k=0}^{n-1}|x-x_k|^{-\lambda}\right)^2}\right|^p\\
&+C_{\lambda,p}\left|\phi(x)\frac{\frac{1}{x_{j+1}-x_j}\int_{x_j}^{x_{j+1}}|f(t)|dt|x-x_j|^{-\lambda-1}\sum\limits_{k\neq j}|x-x_k|^{-\lambda}}{\left(\sum\limits_{k=0}^{n-1}|x-x_k|^{-\lambda}\right)^2}\right|^p\\
&+C_{\lambda,p}\left|\phi(x)\frac{\sum\limits_{k\neq j}|x-x_k|^{-\lambda-1}\frac{1}{x_{k+1}-x_k}\int_{x_k}^{x_{k+1}}|f(t)|dt|x-x_j|^{-\lambda}}{\left(\sum\limits_{k=0}^{n-1}|x-x_k|^{-\lambda}\right)^2}\right|^p\\
&+C_{\lambda,p}\left|\phi(x)\frac{\frac{1}{x_{j+1}-x_j}\int_{x_j}^{x_{j+1}}|f(t)|dt|x-x_j|^{-\lambda}\sum\limits_{k\neq j}|x-x_k|^{-\lambda-1}}{\left(\sum\limits_{k=0}^{n-1}|x-x_k|^{-\lambda}\right)^2}\right|^p\\
&+C_{\lambda,p}\left|\phi(x)\frac{\sum\limits_{k\neq j}|x-x_k|^{-\lambda}\frac{1}{x_{k+1}-x_k}\int_{x_k}^{x_{k+1}}|f(t)|dt|x-x_j|^{-\lambda-1}}{\left(\sum\limits_{k=0}^{n-1}|x-x_k|^{-\lambda}\right)^2}\right|^p\\
=:&\quad S_1+S_2+S_3+S_4+S_5+S_6. \qquad (20)
\end{aligned}$$

By (10), (11) and the obvious fact

$$\frac{1}{\sum\limits_{k=0}^{n-1}|x-x_k|^{-\lambda}}\le|x-x_j|^{\lambda}, \qquad (21)$$

we obtain that

$$\begin{aligned}
|S_1| &\le C_{\lambda,p}\left|\phi(x)|x-x_j|^{2\lambda}\sum_{k\neq j}|x-x_k|^{-\lambda-1}\frac{1}{x_{k+1}-x_k}\int_{x_k}^{x_{k+1}}|f(t)|dt\sum_{k\neq j}|x-x_k|^{-\lambda}\right|^p\\
&\le C_{\lambda,p}\left|n\sum_{k\neq j}(|j-k|+1)^{-\lambda-1}\frac{1}{x_{k+1}-x_k}\int_{x_k}^{x_{k+1}}|f(t)|dt\sum_{k\neq j}(|j-k|+1)^{-\lambda}\right|^p\\
&\le C_{\lambda,p}\left|n\sum_{k\neq j}(|j-k|+1)^{-\lambda-1}\frac{1}{x_{k+1}-x_k}\int_{x_k}^{x_{k+1}}|f(t)|dt\right|^p\\
&\le C_{\lambda,p}n^p\sum_{k\neq j}(|j-k|+1)^{-\lambda-1}\frac{1}{x_{k+1}-x_k}\int_{x_k}^{x_{k+1}}|f(t)|^pdt\\
&\le C_{\lambda,p}n^p\sum_{k\neq j}(|j-k|+1)^{-\lambda}\frac{1}{x_{k+1}-x_k}\int_{x_k}^{x_{k+1}}|f(t)|^pdt,
\end{aligned}$$

where in the fourth inequality, we used Hölder's inequality for $p > 1$. Therefore, by (9), we get

$$\begin{aligned}\int_{x_j}^{(x_j+x_{j+1})/2} |S_1|dx &\leq C_{\lambda,p} n^p \sum_{k\neq j} \frac{x_{j+1}-x_j}{x_{k+1}-x_k} (|j-k|+1)^{-\lambda} \int_{x_k}^{x_{k+1}} |f(t)|^p dt \\ &\leq C_{\lambda,p} n^p \sum_{k\neq j} (|j-k|+1)^{-\lambda+\beta-1} \int_{x_k}^{x_{k+1}} |f(t)|^p dt. \qquad (22)\end{aligned}$$

A similar discussion leads to

$$\int_{x_j}^{(x_j+x_{j+1})/2} |S_i|dx \leq C_{\lambda,p} n^p \sum_{k\neq j} (|j-k|+1)^{-\lambda+\beta-1} \int_{x_k}^{x_{k+1}} |f(t)|^p dt, \; i = 2,4,6. \qquad (23)$$

By (10), (11) and (21) again, we deduce that

$$\begin{aligned}|S_3| &\leq C_{\lambda,p} \left| \phi(x) \frac{1}{x_{j+1}-x_j} \int_{x_j}^{x_{j+1}} |f(t)|dt |x-x_j|^{\lambda-1} \sum_{k\neq j} |x-x_k|^{-\lambda} \right|^p \\ &\leq C_{\lambda,p} n^p \left| \frac{1}{x_{j+1}-x_j} \int_{x_j}^{x_{j+1}} |f(t)|dt \right|^p \\ &\leq C_{\lambda,p} n^p \frac{1}{x_{j+1}-x_j} \int_{x_j}^{x_{j+1}} |f(t)|^p dt.\end{aligned}$$

Hence,

$$\int_{x_j}^{(x_j+x_{j+1})/2} |S_3|dx \leq C_{\lambda,p} n^p \int_{x_j}^{x_{j+1}} |f(t)|^p dt. \qquad (24)$$

Similarly

$$\int_{x_j}^{(x_j+x_{j+1})/2} |S_5|dx \leq C_{\lambda,p} n^p \int_{x_j}^{x_{j+1}} |f(t)|^p dt. \qquad (25)$$

Altogether from (20), (22)-(25), we have

$$\begin{aligned}&\int_{x_j}^{(x_j+x_{j+1})/2} |\phi(x) N'_{n,\lambda}(f,x)|^p dx \\ &\leq C_{\lambda,p} n^p \left(\sum_{k\neq j} (|j-k|+1)^{-\lambda+\beta-1} \int_{x_k}^{x_{k+1}} |f(t)|^p d + \int_{x_j}^{x_{j+1}} |f(t)|^p dt \right). \qquad (26)\end{aligned}$$

A similar argument also yields that

$$\begin{aligned}&\int_{(x_j+x_{j+1})/2}^{x_{j+1}} |\phi(x) N'_{n,\lambda}(f,x)|^p dx \\ &\leq C_{\lambda,p} n^p \left(\sum_{k\neq j} (|j-k|+1)^{-\lambda+\beta-1} \int_{x_k}^{x_{k+1}} |f(t)|^p d + \int_{x_j}^{x_{j+1}} |f(t)|^p dt \right). \qquad (27)\end{aligned}$$

Now, we can deduce (19) from (26) and (27) by estimating

$$\begin{aligned}\|\phi(x)N'_{n,\lambda}(f,X,x)\|^p &\leq \sum_{j=0}^{n-1}\int_{x_j}^{x_{j+1}}|\phi(x)N'_{n,\lambda}(f,x)|^p dx\\ &\leq C_{\lambda,p}n^p\sum_{j=0}^{n-1}\sum_{k\neq j}(|j-k|+1)^{-\lambda+\beta-1}\int_{x_k}^{x_{k+1}}|f(t)|^p dt\\ &\quad +C_{\lambda,p}n^p\sum_{j=0}^{n-1}\int_{x_j}^{x_{j+1}}|f(t)|^p dt\\ &\leq C_{\lambda,\beta,p}n^p\left(\sum_{k\neq j}\int_{x_k}^{x_{k+1}}|f(t)|^p dt+\|f\|_p^p\right)\\ &\leq C_{\lambda,\beta,p}n^p\|f\|_p^p.\end{aligned}$$

Lemma 2.5 *Let $f'\in L^p_{[-1,1]}$, then*

$$\|\phi(x)N'_{n,\lambda}(f,X)\|_p\leq C_{\lambda,\beta,p}\left(\|\phi f'\|_p+\frac{1}{n^{\beta-1}}\|f'\|_p\right)$$

for $1\leq p<\infty$.

Proof. Since $\sum\limits_{k=0}^{n-1}A_k(x)\equiv 1$, then $\sum\limits_{k=0}^{n-1}A'_k(x)\equiv 0$. For any $x\in(x_j,(x_j+x_{j+1})/2],\ 0\leq j\leq n-1$, in a similar way to Lemma 2.4, it follows that

$$\begin{aligned}N'_{n,\lambda}(f,X,x) &= \sum_{k=0}^{n-1}A'_k(x)\frac{1}{x_{k+1}-x_k}\int_{x_k}^{x_{k+1}}(f(t)-f(x))dt\\ &= -\sum_{k=0}^{n-1}A'_k(x)\frac{1}{x_{k+1}-x_k}\int_{x_k}^{x_{k+1}}\int_t^x f'(u)dudt\\ &= \lambda\left(\sum_{k=0}^{n-1}|x-x_k|^{-\lambda}\right)^{-2}\left\{\left(\sum_{k=0}^{j}|x-x_k|^{-\lambda-1}\frac{1}{x_{k+1}-x_k}\int_{x_k}^{x_{k+1}}\int_t^x f'(u)dudt-\right.\right.\\ &\quad\left.\sum_{k=j+1}^{n-1}|x-x_k|^{-\lambda-1}\frac{1}{x_{k+1}-x_k}\int_{x_k}^{x_{k+1}}\int_t^x f'(u)dudt\right)\sum_{k=0}^{n-1}|x-x_k|^{-\lambda}\\ &\quad-\sum_{k=0}^{n-1}|x-x_k|^{-\lambda}\frac{1}{x_{k+1}-x_k}\int_{x_k}^{x_{k+1}}\int_t^x f'(u)dudt\\ &\quad\left.\times\left(\sum_{k=0}^{j}|x-x_k|^{-\lambda-1}-\sum_{k=j+1}^{n-1}|x-x_k|^{-\lambda-1}\right)\right\}.\end{aligned}$$

Hence

$$|\phi(x)N'_{n,\lambda}(f,X,x)|^p\leq C_{\lambda,p}\left|\phi(x)\frac{\sum\limits_{k\neq j}|x-x_k|^{-\lambda-1}\frac{1}{x_{k+1}-x_k}\int_{x_k}^{x_{k+1}}\left|\int_t^x f'(u)du\right|dt\sum\limits_{k\neq j}|x-x_k|^{-\lambda}}{\left(\sum\limits_{k=0}^{n-1}|x-x_k|^{-\lambda}\right)^2}\right|^p$$

$$
\begin{aligned}
&+C_{\lambda,p}\left|\phi(x)\frac{\sum\limits_{k\neq j}|x-x_k|^{-\lambda}\frac{1}{x_{k+1}-x_k}\int_{x_k}^{x_{k+1}}\left|\int_t^x f'(u)du\right|dt\sum\limits_{k\neq j}|x-x_k|^{-\lambda-1}}{\left(\sum\limits_{k=0}^{n-1}|x-x_k|^{-\lambda}\right)^2}\right|^p\\
&+C_{\lambda,p}\left|\phi(x)\frac{\frac{1}{x_{j+1}-x_j}\int_{x_j}^{x_{j+1}}\left|\int_t^x f'(u)du\right|dt|x-x_j|^{-\lambda-1}\sum\limits_{k\neq j}|x-x_k|^{-\lambda}}{\left(\sum\limits_{k=0}^{n-1}|x-x_k|^{-\lambda}\right)^2}\right|^p\\
&+C_{\lambda,p}\left|\phi(x)\frac{\sum\limits_{k\neq j}|x-x_k|^{-\lambda-1}\frac{1}{x_{k+1}-x_k}\int_{x_k}^{x_{k+1}}\left|\int_t^x f'(u)du\right|dt|x-x_j|^{-\lambda}}{\left(\sum\limits_{k=0}^{n-1}|x-x_k|^{-\lambda}\right)^2}\right|^p\\
&+C_{\lambda,p}\left|\phi(x)\frac{\frac{1}{x_{j+1}-x_j}\int_{x_j}^{x_{j+1}}\left|\int_t^x f'(u)du\right|dt|x-x_j|^{-\lambda}\sum\limits_{k\neq j}|x-x_k|^{-\lambda-1}}{\left(\sum\limits_{k=0}^{n-1}|x-x_k|^{-\lambda}\right)^2}\right|^p\\
&+C_{\lambda,p}\left|\phi(x)\frac{\sum\limits_{k\neq j}|x-x_k|^{-\lambda}\frac{1}{x_{k+1}-x_k}\int_{x_k}^{x_{k+1}}\left|\int_t^x f'(u)du\right|dt|x-x_j|^{-\lambda-1}}{\left(\sum\limits_{k=0}^{n-1}|x-x_k|^{-\lambda}\right)^2}\right|^p\\
=:&\quad T_1+T_2+T_3+T_4+T_5+T_6. \qquad (28)
\end{aligned}
$$

Define the Hardy-Littlewood maximum function $M(f,x)$ by

$$M(f,x) := \sup_{-1\le t\le 1}\frac{1}{x-t}\int_t^x |f(u)|du.$$

It is well known that

$$\|M(f)\|_p \le C_p\|f\|_p,\ p>1. \qquad (29)$$

If $j=0$, then by (7), (10), (11), (21), we get

$$
\begin{aligned}
|T_1| &\le C_{\lambda,p}\left|\phi(x)|x-x_j|^{2\lambda}\sum_{k\neq 0}|x-x_k|^{-\lambda-1}\frac{1}{x_{k+1}-x_k}\int_{x_k}^{x_{k+1}}\left|\int_t^x f'(u)du\right|dt\sum_{k\neq j}|x-x_k|^{-\lambda}\right|^p\\
&\le C_{\lambda,p}\left|n\sum_{k\neq 0}(k+1)^{-\lambda-1}\frac{1}{x_{k+1}-x_k}\int_{x_k}^{x_{k+1}}|x-t|\left|\frac{1}{x-t}\int_t^x|f'(u)|du\right|dt\right|^p\\
&\le C_{\lambda,p}n^p|M(f',x)|^p\left|\sum_{k\neq 0}(k+1)^{-\lambda-1}\frac{1}{x_{k+1}-x_k}\int_{x_k}^{x_{k+1}}|x-t|dt\right|^p\\
&\le C_{\lambda,\beta,p}n^p|M(f',x)|^p\left|\sum_{k\neq 0}(k+1)^{-\lambda-1}|x_{k+1}-x|\right|^p\\
&\le C_{\lambda,\beta,p}n^{-\beta p+p}|M(f',x)|^p\left|\sum_{k\neq 0}(k+1)^{-\lambda-1+\beta}\right|^p\\
&\le C_{\lambda,\beta,p}n^{-\beta p+p}|M(f',x)|^p,
\end{aligned}
$$

which implies for $p>1$ that (by (29))

$$\begin{aligned}\int_{-1}^{(-1+x_1)/2}|T_1|dx &\le C_{\lambda,\beta,p}n^{-\beta p+p}\int_{-1}^{(-1+x_1)/2}|M(f',x)|^p dx\\ &\le C_{\lambda,\beta,p}n^{-\beta p+p}\|f'\|_p^p. \end{aligned}\tag{30}$$

Set

$$t(j,k):=\begin{cases} x_{k+1}, & j\le k,\\ x_k, & j>k,\end{cases}$$

and

$$x(j,k):=\begin{cases} x_j, & j\le k,\\ x_{j+1}, & j>k\end{cases}$$

for $j,k=0,1,\cdots,n-1$. For $p=1$, we have

$$|T_1|\le C_\lambda n\sum_{k\ne 0}(k+1)^{-\lambda-1}\left|\int_{x(0,k)}^{t(0,k)}|f'(u)|du\right|,$$

which indicates that

$$\begin{aligned}\int_{-1}^{(-1+x_1)/2}|T_1|dx &\le C_{\lambda,\beta}n\frac{1+x_1}{2}\sum_{k\ne 0}(k+1)^{-\lambda-1}\left|\int_{x(0,k)}^{t(0,k)}|f'(u)|du\right|\\ &\le C_{\lambda,\beta}n^{-\beta+1}\sum_{k\ne 0}(k+1)^{-\lambda-1}\|f'\|_1\\ &\le C_{\lambda,\beta}n^{-\beta+1}\|f'\|_1. \end{aligned}\tag{31}$$

Combining (30) and (31), we obtain that

$$\int_{-1}^{(-1+x_1)/2}|T_1|dx\le C_{\lambda,\beta,p}n^{-\beta p+p}\|f'\|_p^p,\quad 1\le p<\infty.$$

Similar arguments lead to that[2]

$$\int_{-1}^{(-1+x_1)/2}|T_i|dx\le C_{\lambda,\beta,p}n^{-\beta p+p}\|f'\|_p^p,\quad i=2,3,4,5,6,$$

and

$$\int_{(1+x_{n-1})/2}^{1}|T_i|dx\le C_{\lambda,\beta,p}n^{-\beta p+p}\|f'\|_p^p,\quad i=1,2,3,4,5,6.$$

Thus,

$$\int_{-1}^{(-1+x_1)/2}|\phi(x)N'_{n,\lambda}(f,X,x)|^p dx\le C_{\lambda,\beta,p}n^{-\beta p+p}\|f'\|_p^p,\quad i=1,2,3,4,5,6,\tag{32}$$

and

$$\int_{(1+x_{n-1})/2}^{1}|\phi(x)N'_{n,\lambda}(f,X,x)|^p dx\le C_{\lambda,\beta,p}n^{-\beta p+p}\|f'\|_p^p,\quad i=1,2,3,4,5,6.\tag{33}$$

[2] Note that $\lambda>\beta+1$.

Assume that $x \in [x_j, (x_j + x_{j+1})/2],\ 1 \le j \le n-2$. By (6), (10), (11) and (21), we get for $p > 1$ that

$$\begin{aligned}
|T_1| &\le C_{\lambda,p} n^p \left| \sum_{k\neq j} (|j-k|+1)^{-\lambda-1} \frac{1}{x - t(j,k)} \int_{t(j,k)}^{x} |x - t(j,k)||f'(u)| du \right|^p \\
&\le C_{\lambda,\beta,p} \left| \sum_{k\neq 0,j,n} (|j-k|+1)^{-\lambda-1+\beta} \frac{1}{x - t(j,k)} \int_{t(j,k)}^{x} |\phi(u) f'(u)| du \right|^p \\
&\quad + C_{\lambda,p} n^p \left| (j+1)^{-\lambda-1} (1+x) \frac{1}{1+x} \int_{-1}^{x} |f'(u)| du \right|^p \\
&\quad + C_{\lambda,p} n^p \left| (|n-j|+1)^{-\lambda-1} (1-x) \frac{1}{1-x} \int_{x}^{1} |f'(u)| du \right|^p \\
&\le C_{\lambda,\beta,p} |M(\phi f', x)|^p \left| \sum_{k\neq j} (|j-k|+1)^{-\lambda-1+\beta} \right|^p \\
&\quad + C_{\lambda,\beta,p} n^p \left| (j+1)^{-\lambda-1} (|1+x_1| + |x - x_1|) \right|^p |M(f', x)|^p \\
&\quad + C_{\lambda,\beta,p} n^p \left| (|n-j|+1)^{-\lambda-1} (|1-x_{n-1}| + |x - x_{n-1}|) \right|^p |M(f', x)|^p \\
&\le C_{\lambda,\beta,p} |M(\phi f', x)|^p + C_{\lambda,\beta,p} n^p \left((j+1)^{-\lambda-1} \left(\frac{1}{n^\beta} + \frac{j+1}{n} \phi(x_1) \right) \right)^p |M(f', x)|^p \\
&\quad + C_{\lambda,\beta,p} n^p \left((|n-j|+1)^{-\lambda-1} \left(\frac{1}{n^\beta} + \frac{|n-j|+1}{n} \phi(x_{n-1}) \right) \right)^p |M(f', x)|^p \\
&\le C_{\lambda,\beta,p} |M(\phi f', x)|^p + C_{\lambda,\beta,p} n^{-\beta p + p} |M(f', x)|^p.
\end{aligned}$$

Similar procedures also lead to the same estimates for other $T_i,\ i = 2,3,4,5,6$. Altogether, we get for $p > 1$ that

$$\begin{aligned}
&\int_{x_j}^{(x_j+x_{j+1})/2} |\phi(x) N'_{n,\lambda}(f, X, x)|^p dx \\
&\le C_{\lambda,\beta,p} \int_{x_j}^{(x_j+x_{j+1})/2} |M(\phi f', x)|^p dx + C_{\lambda,\beta,p} n^{-\beta p + p} \int_{x_j}^{(x_j+x_{j+1})/2} |M(f', x)|^p dx.
\end{aligned} \tag{34}$$

For $p = 1$, we have

$$\begin{aligned}
|T_1| &\le C_{\lambda,\beta} n \sum_{k\neq j} (|j-k|+1)^{-\lambda-1} \frac{1}{|x - t(j,k)|} \left| \int_{x}^{t(j,k)} |f'(u)||x - t(j,k)| du \right| \\
&\le C_{\lambda,\beta} \sum_{k\neq j} (|j-k|+1)^{-\lambda-1+\beta} \frac{1}{|x - t(j,k)|} \left| \int_{x}^{t(j,k)} |\phi(u) f'(u)| du \right| \\
&\le C_{\lambda,\beta} \sum_{k\neq j} (|j-k|+1)^{-\lambda-1+\beta} \frac{1}{|x(j,k) - t(j,k)|} \left| \int_{x(j,k)}^{t(j,k)} |\phi(u) f'(u)| du \right|.
\end{aligned} \tag{35}$$

We need the following inequality to continue our work:

$$\frac{x_{j+1} - x_j}{|x(j,k) - t(j,k)|} \le C_\beta (|j-k|+1)^{-1},\ 1 \le j \le n-2. \tag{36}$$

We only prove (36) for the case $j < k$, while another case can be treated exactly in the same way. It follows from (6) and (11) for $j \ge 1$ that

$$|x_{j+1} - x_j| \le C_\beta \frac{\phi(x_j)}{n}, \tag{37}$$

and

$$|x_{k+1}-x_j| \geq C_\beta \frac{\phi(x_j)}{n}(|j-k|+1). \tag{38}$$

Combining (37)-(38), we obtain (36).

Now, by (35) and (36), we can drive that

$$\begin{aligned}\int_{x_j}^{(x_j+x_{j+1})/2}|T_1|dx &\leq C_{\lambda,\beta}\sum_{k\neq j}(|j-k|+1)^{-\lambda-1+\beta}\frac{x_{j+1}-x_j}{|x(j,k)-t(j,k)|}\left|\int_{x(j,k)}^{t(j,k)}|\phi(u)f'(u)|du\right| \\ &\leq C_{\lambda,\beta}\sum_{k\neq j}(|j-k|+1)^{-\lambda-2+\beta}\left|\int_{x(j,k)}^{t(j,k)}|\phi(u)f'(u)|du\right| \\ &\leq C_{\lambda,\beta}\sum_{k=0}^{n-1}(|j-k|+1)^{-\lambda+\beta-1}\left|\int_{x(j,k)}^{t(j,k)}|\phi(u)f'(u)|du\right|.\end{aligned}$$

Similarly, we can prove the above estimate to hold when T_1 is replaced by T_i for $1 \leq i \leq 6$. Therefore,

$$\begin{aligned}&\int_{x_j}^{(x_j+x_{j+1})/2}|\phi(x)N'_{n,\lambda}(f,X,x)|dx \\ &\leq C_{\lambda,\beta,p}\sum_{k=0}^{n-1}(|j-k|+1)^{-\lambda+\beta-1}\left|\int_{x(j,k)}^{t(j,k)}|\phi(u)f'(u)|du\right|.\end{aligned} \tag{39}$$

The same arguments also lead to that

$$\int_{(-1+x_1)/2}^{x_1}|\phi(x)N'_{n,\lambda}(f,X,x)|^p dx \leq C_{\lambda,\beta,p}n^{-\beta p+p}\|f'\|_p^p, \quad i=1,2,3,4,5,6, \tag{40}$$

$$\int_{x_{n-1}}^{(1+x_{n-1})/2}|\phi(x)N'_{n,\lambda}(f,X,x)|^p dx \leq C_{\lambda,\beta,p}n^{-\beta p+p}\|f'\|_p^p, \quad i=1,2,3,4,5,6, \tag{41}$$

$$\begin{aligned}&\int_{(x_j+x_{j+1})/2}^{x_{j+1}}|\phi(x)N'_{n,\lambda}(f,X,x)|^p dx \\ &\leq C_{\lambda,\beta,p}\int_{(x_j+x_{j+1})/2}^{x_{j+1}}|M(\phi f',x)|^p dx + C_{\lambda,\beta,p}n^{-\beta p+p}\int_{(x_j+x_{j+1})/2}^{x_{j+1}}|M(f',x)|^p dx, p>1,\end{aligned} \tag{42}$$

and

$$\begin{aligned}&\int_{(x_j+x_{j+1})/2}^{x_{j+1}}|\phi(x)N'_{n,\lambda}(f,X,x)|dx \\ &\leq C_{\lambda,\beta,p}\sum_{k=0}^{n-1}(|j-k|+1)^{-\lambda+\beta-1}\left|\int_{x(j,k)}^{t(j,k)}|\phi(u)f'(u)|du\right|.\end{aligned} \tag{43}$$

Hence, by (30), (32)-(34) and (39)-(43), we have for $p>1$ that

$$\begin{aligned}
\|\phi N'_{n,\lambda}(f,X)\|_p^p &\leq \int_{-1}^{x_1}|\phi(x)N'_{n,\lambda}(f,X,x)|^p dx+\int_{x_{n-1}}^{1}|\phi(x)N'_{n,\lambda}(f,X,x)|^p dx \\
&\quad +\sum_{j=1}^{n-2}\int_{x_j}^{x_{j+1}}|\phi(x)N'_{n,\lambda}(f,X,x)|^p dx \\
&\leq C_{\lambda,\beta,p}\left(n^{-\beta p+p}\|f'\|_p^p+\sum_{j=1}^{n-2}\int_{x_j}^{x_{j+1}}|M(\phi f',x)|^p dx\right) \\
&\leq C_{\lambda,\beta,p}\left(n^{-\beta p+p}\|f'\|_p^p+\|M(\phi f')\|_p^p\right) \\
&\leq C_{\lambda,\beta,p}\left(n^{-\beta p+p}\|f'\|_p^p+\|\phi f'\|_p^p\right), \qquad (44)
\end{aligned}$$

and for $p=1$ that

$$\begin{aligned}
\|\phi N'_{n,\lambda}(f,X)\|_1 &\leq C_{\lambda,\beta}\sum_{j=1}^{n-2}\sum_{k=0}^{n-1}(|j-k|+1)^{-\lambda+\beta-1}\left|\int_{x(j,k)}^{t(j,k)}|\phi(u)f'(u)|du\right| \\
&\quad +C_{\lambda,\beta}n^{-\beta+1}\|f'\|_1 \\
&\leq C_{\lambda,\beta}\sum_{m=0}^{n-1}\sum_{|j-k|=m}(m+1)^{-\lambda+\beta-1}\left|\int_{x(j,k)}^{t(j,k)}|\phi(u)f'(u)|du\right| \\
&\quad +C_{\lambda,\beta}n^{-\beta+1}\|f'\|_1 \\
&\leq C_{\lambda,\beta}\left(\sum_{m=0}^{n-1}(m+1)^{-\lambda+\beta}\|\phi f'\|_1+n^{-\beta+1}\|f'\|_1\right) \\
&\leq C_{\lambda,\beta}\left(\|\phi f'\|_1+n^{-\beta+1}\|f'\|_1\right). \qquad (45)
\end{aligned}$$

We complete Lemma 2.5 by (44) and (45).

By [10: Theorem 2.1.1] and [10: Theorem 3.1.2], we have

Lemma 2.6 *Suppose that* $f\in L^p_{[-1,1]},\ 1\leq p<\infty$. *Define*

$$\bar{K}_\phi(f,t)_p:=\inf_{g\in A.C.}\left\{\|f-g\|_p+t\|\phi g'\|_p+t^\beta\|g'\|_p\right\},$$

and

$$K_\phi(f,t)_p:=\inf_{g\in A.C.}\left\{\|f-g\|_p+t\|\phi g'\|_p\right\}.$$

Then there exists a constant t_0 *such that*

$$\bar{K}_\phi(f,t)_p\sim K_\phi(f,t)_p\sim\omega_\phi(f,t)_p \qquad \text{for } 0<t<t_0.$$

Lemma 2.7 ([1]) *Suppose that* $\rho(t)$ *increases on* $[0,a],\ 0<\alpha<1,\ r>\alpha$. *If for any* $h,t\in[0,a]$, *the inequality*

$$\rho(h)\leq M(t^\alpha+(h/t)^r\rho(t))$$

holds, then

$$\rho(t)\leq M_2t^\alpha, \quad t\in[0,a],$$

where M_2 *is a positive constant only depending on* M,α,r *and* a.

3 Proof of Theorems.

Proof of Theorem 1.5 By Lemma 2.6, for any $f \in L^p_{[-1,1]},\ 1 \le p < \infty$, we can choose a g such that

$$\|f-g\|_p \le C\omega_\phi\left(f, \frac{1}{n}\right)_p, \tag{46}$$

$$\|g'\|_p \le Cn^\beta\omega_\phi\left(f, \frac{1}{n}\right)_p, \tag{47}$$

$$\|\phi g'\|_p \le nC\omega_\phi\left(f, \frac{1}{n}\right)_p. \tag{48}$$

By the definition of $N_{n,\lambda}(f,X)$, we have

$$\begin{aligned}\|g-N_{n,\lambda}(g,X)\|_p^p \quad \le \quad & \int_{-1}^{x_1}\left|\sum_{k=0}^{n-1}A_k(x)\frac{1}{x_{k+1}-x_k}\int_{x_k}^{x_{k+1}}\left|\int_t^x|g'(u)|du\right|dt\right|^p dx \\ & +\int_{x_{n-1}}^{1}\left|\sum_{k=0}^{n-1}A_k(x)\frac{1}{x_{k+1}-x_k}\int_{x_k}^{x_{k+1}}\left|\int_t^x|g'(u)|du\right|dt\right|^p dx \\ & +\sum_{j=1}^{n-2}\int_{x_j}^{x_{j+1}}\left|\sum_{k=0}^{n-1}A_k(x)\frac{1}{x_{k+1}-x_k}\int_{x_k}^{x_{k+1}}\left|\int_t^x|g'(u)|du\right|dt\right|^p dx \\ =: \quad & I_1+I_2+I_3.\end{aligned} \tag{49}$$

First assume that $p>1$. By (17) and (7), we deduce that

$$\begin{aligned} I_1 \quad \le \quad & C_\beta\int_{-1}^{x_1}\left|\sum_{k=1}^{n-1}(k+1)^{-\lambda}\left|\int_x^{t(1,k)}|g'(u)|du\right|\right|^p dx + C_\beta\int_{-1}^{x_1}\left|\int_{-1}^{x_1}|g'(u)|du\right|^p dx \\ \le \quad & C_\beta\int_{-1}^{x_1}\left|\sum_{k=1}^{n-1}(k+1)^{-\lambda}\left|\frac{1}{x-t(1,k)}\int_x^{t(1,k)}|g'(u)|du|x-t(1,k)|\right|\right|^p dx \\ & +C_\beta(1+x_1)^p\int_{-1}^{x_1}|g'(x)|^p dx \\ \le \quad & C_\beta n^{-\beta p}\int_{-1}^{x_1}|M(g',x)|^p\left|\sum_{k=0}^{n-1}(k+1)^{-\lambda+\beta}\right|^p dx + C_\beta n^{-\beta p}\int_{-1}^{x_1}|g'(x)|^p dx \\ \le \quad & C_{\lambda,\beta}n^{-\beta p}\left(\int_{-1}^{1}|M(g',x)|^p dx+\int_{-1}^{x_1}|g'(x)|^p dx\right) \le C_{\lambda,\beta,p}n^{-\beta p}\|g'\|_p^p.\end{aligned} \tag{50}$$

Similarly, we produce that

$$I_2 \le C_{\lambda,\beta,p}n^{-\beta p}\|g'\|_p^p. \tag{51}$$

By (6) and (17), we have

$$\begin{aligned} I_3 \quad \le \quad & C_\beta\sum_{j=1}^{n-2}\int_{x_j}^{x_{j+1}}\left|\sum_{k=0}^{n-1}(|j-k|+1)^{-\lambda}\left|\frac{1}{x-t(j,k)}\int_x^{t(j,k)}|g'(u)||x-t(j,k)|du\right|\right|^p dx \\ \le \quad & C_\beta n^{-p}\sum_{j=1}^{n-2}\int_{x_j}^{x_{j+1}}\left|\sum_{k=0}^{n-1}(|j-k|+1)^{-\lambda+\beta}\left|\frac{1}{x-t(j,k)}\int_x^{t(j,k)}|\phi(u)g'(u)|du\right|\right|^p dx \\ \le \quad & C_{\lambda,\beta}n^{-p}\sum_{j=1}^{n-2}\int_{x_j}^{x_{j+1}}|M(\phi g',x)|^p dx \\ \le \quad & C_{\lambda,\beta}n^{-p}\|M(\phi g')\|_p^p \le C_{\lambda,\beta,p}n^{-p}\|\phi g'\|_p^p.\end{aligned} \tag{52}$$

Altogether from (49)-(52), we obtain for $p > 1$ that

$$\|g - N_{n,\lambda}(g, X)\|_p^p \leq C_{\lambda,\beta,p}\left(n^{-p}\|\phi g'\|_p^p + n^{-\beta p}\|g'\|_p^p\right). \tag{53}$$

From Lemma 2.3, (46)-(48) and (53), we get

$$\begin{aligned}\|f - N_{n,\lambda}(f, X)\|_p &\leq \|N_{n,\lambda}(f, X) - N_{n,\lambda}(g, X)\|_p \\ &\quad + \|g - N_{n,\lambda}(g, X)\|_p + \|f - g\|_p \\ &\leq C_{\lambda,\beta,p}\left(\|f - g\|_p + n^{-1}\|\phi g'\|_p + n^{-\beta}\|g'\|_p\right) \\ &\leq C_{\lambda,\beta,p}\omega_\phi\left(f, \frac{1}{n}\right).\end{aligned}$$

Thus we have proved (2) for $p > 1$.

Next, we consider the case $p = 1$. We surely have

$$\begin{aligned}I_1 &\leq C_\beta \int_{-1}^{x_1}\left|\sum_{k=0}^{n-1}(k+1)^{-\lambda}\left|\int_0^1 |g'(u)|du\right|\right| dx \\ &\leq C_\beta n^{-\beta}\sum_{k=0}^{n-1}(k+1)^{-\lambda}\int_0^1 |g'(u)|du \\ &\leq C_{\lambda,\beta} n^{-\beta}\|g'\|_1.\end{aligned} \tag{54}$$

Similarly,

$$I_2 \leq C_{\lambda,\beta} n^{-\beta}\|g'\|_1. \tag{55}$$

For I_3, we see that

$$\begin{aligned}I_3 &\leq C_\beta n^{-1}\sum_{j=1}^{n-2}\int_{x_j}^{x_{j+1}}\sum_{k=0}^{n-1}(|j-k|+1)^{-\lambda+\beta}\left|\frac{1}{x - t(j,k)}\int_x^{t(j,k)} |\phi(u)g'(u)|du\right| dx \\ &\leq C_\beta n^{-1}\sum_{j=1}^{n-2}\int_{x_j}^{x_{j+1}}\sum_{k=0}^{n-1}(|j-k|+1)^{-\lambda+\beta}\left|\frac{1}{x(j,k) - t(j,k)}\int_{x(j,k)}^{t(j,k)} |\phi(u)g'(u)|du\right| dx \\ &\leq C_\beta n^{-1}\sum_{j=1}^{n-2}\sum_{k=0}^{n-1}(|j-k|+1)^{-\lambda+\beta}\frac{x_{j+1} - x_j}{|x(j,k) - t(j,k)|}\left|\int_{x(j,k)}^{t(j,k)} |\phi(u)g'(u)|du\right| \\ &\leq C_\beta n^{-1}\sum_{j=1}^{n-2}\sum_{k=0}^{n-1}(|j-k|+1)^{-\lambda+\beta-1}\left|\int_{x(j,k)}^{t(j,k)} |\phi(u)g'(u)|du\right| \qquad \text{(by (36))} \\ &\leq C_\beta n^{-1}\sum_{m=1}^{n-2}\sum_{|j-k|=m}(m+1)^{-\lambda+\beta-1}\left|\int_{x(j,k)}^{t(j,k)} |\phi(u)g'(u)|du\right| \\ &\leq C_\beta n^{-1}\sum_{m=1}^{n-2}(m+1)^{-\lambda+\beta}\|\phi(u)g'(u)\|_1 \leq C_{\lambda,\beta} n^{-1}\|\phi(u)g'(u)\|_1.\end{aligned} \tag{56}$$

Altogether, the estimates (54)-(56), with the same argument as that of $p > 1$, yield (2) for $p = 1$.

Now we turn to prove (3). By Lemmas 2.4-2.6, we can choose a proper $g \in A.C.,\ g' \in L^p_{[-1,1]}$, such that

$$\begin{aligned}\|\phi N'_{n,\lambda}(f, X)\|_p &\leq \|\phi N'_{n,\lambda}(f - g, X)\|_p + \|\phi N'_{n,\lambda}(g, X)\|_p \\ &\leq C_{\lambda,\beta,p}\left(n\|f - g\|_p + \|\phi g'\|_p + n^{-\beta+1}\|\phi g'\|_p\right) \\ &\leq C_{\lambda,\beta,p} n\left(\|f - g\|_p + n^{-1}\|\phi g'\|_p + n^{-\beta}\|\phi g'\|_p\right) \\ &\leq C_{\lambda,\beta,p} n\bar{K}_\phi\left(f, \frac{1}{n}\right)_p \leq C_{\lambda,\beta,p} n\omega_\phi\left(f, \frac{1}{n}\right)_p.\end{aligned} \tag{57}$$

By Lemma 2.6, (2) and (57), we obtain that

$$\begin{aligned}\omega_\phi\left(f,\frac{1}{n}\right)_p &\le CK_\phi\left(f,\frac{1}{n}\right)_p\\ &\le C\left(\|f-N_{n,\lambda}(f,X)\|_p+n^{-1}\|\phi N'_{n,\lambda}(f,X)\|_p\right)\\ &\le C_{\lambda,\beta,p}\omega_\phi\left(f,\frac{1}{n}\right)_p.\end{aligned}$$

Hence, we finish (3).

By using (2) again, if $\omega_\phi\left(f,t\right)_p=O(t^\alpha)$, it is easy to obtain that

$$\|f-N_{n,\lambda}(f,X)\|_p=O(n^{-\alpha}).$$

We verify the converse inequality. By the definition of $K_\phi(f,t)_p$, Lemma 2.4 and Lemma 2.5 yield that[3]

$$\begin{aligned}K_\phi\left(f,\frac{1}{n}\right)_p &\le \|f-N_{k,\lambda}(f,X)\|_p+\frac{1}{n}\|\phi N'_{k,\lambda}(f,X)\|_p\\ &\le \|f-N_{k,\lambda}(f,X)\|_p+\frac{1}{n}\left(\|\phi N'_{k,\lambda}(f-g,X)\|_p+\|\phi N'_{k,\lambda}(g,X)\|_p\right)\\ &\le \|f-N_{k,\lambda}(f,X)\|_p+\frac{k}{n}\left(\|f-g\|_p+\frac{1}{k}\|\phi g'\|_p+\frac{1}{k^\beta}\|g'\|_p\right)\\ &\le \|f-N_{k,\lambda}(f,X)\|_p+C_{\lambda,\beta,p}\frac{k}{n}\bar{K}_\phi\left(f,\frac{1}{k}\right)_p\\ &\le \|f-N_{k,\lambda}(f,X)\|_p+C_{\lambda,\beta,p}\frac{k}{n}K_\phi\left(f,\frac{1}{k}\right)_p.\end{aligned}$$

Hence, if

$$\|f-N_{n,\lambda}(f,X)\|_p=O(n^{-\alpha}),$$

then

$$K_\phi\left(f,\frac{1}{n}\right)_p\le C_{\lambda,\beta,p}k^{-\beta}+C_{\lambda,\beta,p}\frac{k}{n}K_\phi\left(f,\frac{1}{k}\right)_p.$$

By the well known result of Berens and Lorentz [2], we have finished (4).Finally, (5) can be deduced by the same way as [5] or [10].

Proof of Theorem 1.6 First, we verify that $(\mathbf{A})\Rightarrow(\mathbf{B})$. From (2) and the Condition ($\mathbf{A}$), it follows that

$$\omega_\phi\left(f,\frac{1}{n}\right)_p\ge C_{\lambda,\beta,p}n^{-\alpha}. \tag{58}$$

For any $\delta\in(0,1)$, take n such that $1/n\le\delta<1/(n-1)$. The properties of modulus of continuity and (58) lead to that

$$\omega_\phi(f,\delta)_p\ge\omega_\phi\left(f,\frac{1}{n}\right)_p\ge C_{\lambda,\beta,p}n^{-\alpha}\ge C_{\lambda,\beta,p}\delta^\alpha. \tag{59}$$

We verify the converse inequality. For $0<h\le\delta$, working as [10: (2.4.3), (2.4.4)], we obtain that

$$\begin{aligned}\left\|\Delta_{h\phi}f\right\|_p &\le \left\|\Delta_{h\phi}(f-N_{n,\lambda}(f,X))\right\|_p+\left\|\Delta_{h\phi}N_{n,\lambda}(f,X)\right\|_p\\ &\le C\left(\|f-N_{n,\lambda}(f,X)\|_p+h\|\phi N'_{n,\lambda}(f,X)\|_p\right).\end{aligned}$$

[3] Choose $g\in A.C.$ such that $\|f-g\|_p+k^{-1}\|\phi g'\|_p+k^{-\beta}\|g'\|_p\le C\bar{K}_\phi\left(f,k^{-1}\right)_p$.

Hence, it follows from Condition ($\mathbf{A}$) and (57) that

$$\left\|\Delta_{h\phi(x)}f\right\|_p \leq C_{\lambda,\beta,p}\left(n^{-\alpha}+\delta n\omega_\phi\left(f,\frac{1}{n}\right)_p\right),$$

which implies

$$\omega_\phi(f,\delta)_p \leq C_{\lambda,\beta,p}\left(n^{-\alpha}+\delta n\omega_\phi\left(f,\frac{1}{n}\right)_p\right).$$

The above inequality with Lemma 2.7 thus yields that

$$\omega_\phi(f,\delta)_p \leq C_{\lambda,\beta,p}\delta^\alpha. \tag{60}$$

Combining (59) and (60), we finally achieve that

$$\omega_\phi(f,\delta)_p \sim \delta^\alpha.$$

The proof of $(\mathbf{B}) \Rightarrow (\mathbf{A})$ can be proceeded exactly in the same way as that of [16], we omit the details here.

Remark. There is a lack of symmetry in the definition of $N_{n,\lambda}(f,x)$ in contrast to the operators defined by Della Vecchia [7], for $f(x)=f(-x)$ does not imply $N_{n,\lambda}(f,-x)=N_{n,\lambda}(f,x)$. This lack could be repaired in obvious way, for example by $\frac{1}{2}\left(N_{n,\lambda}(f,x)+\widetilde{N}_{n,\lambda}(f,x)\right)$, where

$$\widetilde{N}_{n,\lambda}(f,x)=\left(\sum_{k=1}^{n}|x-x_k|^{-\lambda}\right)^{-1}\left(\sum_{k=1}^{n}|x-x_k|^{-\lambda}\frac{1}{x_k-x_{k-1}}\int_{x_{k-1}}^{x_k}f(t)dt\right),$$

has analogous properties as $N_{n,\lambda}(f,x)$.

Acknowledgement. The above remark is given by the anonymous referee. We would like to express our appreciation to him/her for his/her valuable comments.

References

[1] M. Becker, R. J. Nessel, An Elementary Approach to Inverse Approximation Theorems, J. Approx. Theory. **23**, 99-103(1978)
[2] H. Berens, G. G. Lorentz, InverseTheorems for Bernstein Polynomials, Indiana Univ. Math. J. **21**, 693-708(1972).
[3] G. Criscuolo, G. Mastroianni, Estimate of the Shepard Interpolatory Procedure, Acta Math. Hungar. **61**, 79-91(1993).
[4] B. Della Vecchia, Direct and Converse Results by Rational Operators, Constr. Approx. **12**, 271-285(1996).
[5] B. Della Vecchia, Weighted Approximation by Rational Operators, Result. Math. **43**, 79-87(2003).
[6] B. Della Vecchia, Direct and Converse Results for the Weighted Rational Approximation of Functions with Inner Singularities, J. Approx. Theory **126**, 16-35(2004).
[7] B. Della Vecchia, G. Mastroianni, Pointwise Simultaneous Approximation by Rational Operators, J. Approx. Theory **65**, 140-150(1991).
[8] B. Della Vecchia, G. Mastroianni, J. Szabados, Weighted Uniform Approximation on the Semiaxis by Rational Operators, J. Inequal. Appl. **4**, 241-264(1999).
[9] B. Della Vecchia, G. Mastroianni, V. Totik, Saturation of the Shepard Operators, Approx. Theory Appl., **6**, 76-84(1990).
[10] Z. Ditzian, V. Totik, Moduli of Smoothness, Springer Verlag, 1987.
[11] G. Mastroianni, J. Szabados, Baláz-Shepard Operators on Infinite Intervals, II, J. Approx. Theory **90**, 1-8(1997).
[12] G. Somorjai, On a Saturation Problem, Acta Math. Hungar. **32**, 377-381(1976).
[13] J. Szabados, Direct and Converse Approximation Theorems for the Shepard Operators, Approx. Theory Appl. **7**, 63-76(1991).
[14] P. Vértesi, Saturation of the Shepard operators, Acta Math Hungar. **72**, 307-317(1996).
[15] W. Xiao, S. P. Zhou, A Jackson Type Estimate for Shepard Operators in L^p Spaces for $p\geq 1$, Acta Math Hungar. **95**, 217-224(2002).
[16] W. Xiao, S. P. Zhou, The Equivalence of the Shepard Operators in L^p Spaces for $p\geq 1$, Analysis **21**, 265-275(2001).
[17] W. Xiao, S. P. Zhou, L. Y. Zhu, L^p-approximation of Shepard Operators for $p\geq 1$, J. Zhejiang Univ. Sci. Ed. **27**, 364-371(2000), in Chinese.
[18] T. F. Xie, R. J. Zhang, S. P. Zhou, Three Conjectures on Shepard Operators, J. Approx. Theory **93**, 399-414(1998).
[19] X. L. Zhou, The saturation class of Shepard operators, Acta Math Hungar. **80**, 293-310(1998).