Remarks on Strong Approximation of Continuous Functions

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ABSTRACT: In this note, some embedding relations among many important functional classes are considered. Results of Leindler [7] are extended and improved.

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1 Introduction

Recently, Leindler [3] defined a new class of sequences named as *sequences of rest bounded variation*, briefly denoted by $RBVS$, which sharing many good properties of decreasing sequences. The definition of $RBVS$ can be read as follows: A null sequences $c := \{c_n\}$ of nonnegative numbers is of rest bounded variation, or $c \in RBVS$, if $c_n \to 0$ and for any $m \in N$, it holds that

$$\sum_{n=m}^{\infty} |\Delta c_n| \leq K(c)c_m,$$

where $\Delta c_n = c_n - c_{n+1}$, and $K(c)$ denotes a positive constant only depending on $c$.

As an essential generalization of the monotonicity, $RBVS$ has been used to extend a great number of classic theorems in Fourier analysis. However, Leindler [5] found that $RBVS$ and $QMDS$\(^1\) are not comparable. In view of this, Le and Zhou [2] suggested the following new class of sequences to include both $RBVS$ and $QMDS$: Let $c := \{c_n\}$ be a nonnegative sequence tending to zero, if

$$\sum_{n=m}^{2m} |\Delta c_n| \leq K(c)c_m$$

holds for all $m = 1, 2, \cdots$, then we say $c \in GBVS$.

It was verified that both $RBVS$ and $QMDS$ are subsets of $GBVS$. Some classical theorems also have been generalized by using $GBVS$ to replace the conditions such as

\(^{1}$ $c = \{c_n\} \in QMDS$ means that there is an $\alpha \geq 0$ such that $c_n/n^\alpha$ is decreasing.
RBVs, QMDS and so on. In any sense, monotonicity and “rest bounbed variation” condition are all “one side” monotonicity condition, that is, a positive sequence \( b = \{b_n\} \) under any of these conditions satisfies \( b_n \leq Cb_k \) for \( n \geq k \): \( b_n \) can be controlled by one factor \( b_k \). For \( \{b_n\} \in GBVS \), one can calculate that, for \( k \leq n \leq 2k \),

\[
b_n = \sum_{j=k}^{n} \Delta b_j + b_k \leq \sum_{j=k}^{2k} |\Delta b_j| + b_k \leq K(b)b_k.
\]

Thus, \( GBVS \) can still be understood as a kind of “one side” monotonicity condition (in locally sense). Recently, we [12] suggest a new class of sequence named as \( NBVS \) to include \( GBVS \): Let \( c := \{c_n\} \) be a nonnegative sequence tending to zero, if

\[
\sum_{n=m}^{2m} |\Delta c_n| \leq K(c)(c_m + c_{2m})
\]

holds for all \( m = 1, 2, \cdots \), then we say \( c \in NBVS \).

Obviously, if \( \{b_n\} \in NBVS \), then

\[
b_n \leq \sum_{j=k}^{n-1} \Delta b_j + b_k \leq \sum_{j=k}^{2k} |\Delta b_j| + b_k \leq K(b_k + b_{2k}),
\]

for all \( k \leq n \leq 2k \). Generally speaking, the term \( b_{2k} \) can not be canceled, this can be actually regarded as a “two sided” monotonicity: \( b_n \) is controlled not only by \( b_k \) but also by \( b_{2k+1} \). Therefore, the essential point of \( NBV \) condition is to extend monotonicity from “one sided” to “two sided”.

Very recently, Leindler [6] further extended the definition of \( RBVS \), by introducing the so-called \( \gamma RBVS \), that is,

**Definition.** Let \( \gamma := \{\gamma_n\} \) be a positive sequences. If a null-sequences \( c := \{c_n\} \) of real numbers has the property

\[
\sum_{k=m}^{\infty} |\Delta c_k| \leq K(c)\gamma_m
\]

for all \( m \in N \), then we call the sequences \( c \) a \( \gamma RBVS \), briefly denoted by \( c \in \gamma RBVS \).

If \( \gamma \equiv c \), then \( cRBVS \equiv RBVS \). Moreover, \( \gamma RBVS \) may have infinitely many zeros and negative terms.

Before stating our main results, we need some notions and notations.

A sequence \( \gamma = \{\gamma_n\} \) of nonnegative numbers is called *almost increasing (decreasing) sequence* if there exists a constant \( K \geq 1 \) such that

\[
K\gamma_n \geq \gamma_m, \quad (\gamma_n \leq K\gamma_m), \ n \geq m.
\]
Briefly, write $\gamma \in AMIS(AMDS)$ if $\gamma$ is almost increasing (decreasing) sequence.

Let $f(x) \in C(T)$, that is, $f(x)$ is a continuous and $2\pi$–periodic function, and let

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

be its Fourier series. Denote the $n$th partial sum of (2) by $S_n(f, x)$.

Let $\omega(\delta)$ be a nondecreasing continuous function on $[0, 2\pi]$ having the following properties:

(i). $\omega(0) = 0$,

(ii). $\omega(\delta_1 + \delta_2) \leq \omega(\delta_1) + \omega(\delta_2)$ for any $0 < \delta_1 \leq \delta_2 \leq \delta_1 + \delta_2 \leq 2\pi$.

Then $\omega(\delta)$ is called a modulus of continuity. Define the modulus of smoothness of order $\beta (> 0)$ of the $f$ by

$$\omega_\beta(f, t) = \sup_{|h| \leq t} \left\| \sum_{k=0}^{\infty} (-1)^k \left( \frac{\beta}{k} \right) f(x + (\beta - k)h) \right\|,$$

where

$$\left( \frac{\beta}{k} \right) = \begin{cases} \frac{\beta(\beta-1)\cdots(\beta-k+1)}{k!}, & k \geq 1, \\ 1, & k = 0. \end{cases}$$

Write

$$h_n(f, \lambda, p, x) := \left( \Lambda_n^{-1} \sum_{k=1}^{n} \lambda_k |S_k(f, x) - f(x)|^p \right)^{1/p}$$

and its norm as

$$h_n(f, \lambda, p) := \|h_n(f, \lambda, p, x)\|,$$

where $\{\lambda_n\}$ be a sequence of positive numbers, $\Lambda_n = \sum_{k=1}^{n} \lambda_k$ and $\|\cdot\|$ is the usual supremum. Define

$$H(\lambda, p, r, \delta, \omega) := \left\{ f : h_n(f, \lambda, p) = O(n^{1-r-\delta} \omega(1/n)) \right\}, \quad 0 < \delta \leq 1.$$

$$W^r H_\beta^\omega := \left\{ f : \omega_\beta(f^{(r)}, \delta) = O(\omega(\delta)) \right\},$$

$$W^r H_{s, \beta}^\omega := \left\{ f : f = \sum_{n=1}^{\infty} b_n \sin nx, \{b_n\} \in RBVS \text{ and } \omega_\beta(f^{(r)}, t) = O(\omega(t)) \right\},$$

$$W^r H_{s, \beta}^{\omega, s} := \left\{ f : f = \sum_{n=1}^{\infty} b_n \sin nx, \{b_n\} \in NBVS \text{ and } \omega_\beta(f^{(r)}, t) = O(\omega(t)) \right\},$$

$$W^r H_{s, \Omega_r} := \left\{ f : f = \sum_{n=1}^{\infty} b_n \sin nx, f^{(r)} \in C(T), \{b_n\} \in \Omega_r RBVS \right\},$$

where $\Omega_r := \{\Omega_n := n^{-r} \omega(n^{-1})\}$.  

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Mazhar [8] established the following interesting result:

**Theorem M.** Let $p$ be a positive number, $r$ a nonnegative even integer, $0 < \delta \leq 1$ and $\omega$ a modulus of continuity. If the sequence
\[
\{ \lambda_n n^{1-pr} \omega(1/n)^p, \ n = 1, 2, \cdots \} \in AMIS,
\]
then
\[
W^r H^\omega_{\lambda,1} \subset H(\lambda, p, r, \delta, \omega),
\]
where $\{\lambda_n\}$ be a positive sequence such that\(^2\)
\[
n\lambda_n \sim \Lambda_n.
\]

Leindler [7] further generalized Theorem M to the following

**Theorem L.** Let $p$ be a positive number, $r$ a nonnegative even integer, $0 < \delta \leq 1$ and $\omega$ a modulus of continuity. If the sequence
\[
\{ \Lambda_n n^{-pr} \omega(1/n)^p, \ n = 1, 2, \cdots \} \in AMIS,
\]
then (3) still holds with $\{\lambda_n\}$ be a positive sequence such that
\[
\Lambda_{2n} \leq K \Lambda_n.
\]

Taking $\lambda_n = n^{\beta-1}$, $\beta > 0$ and $\delta = 1$, then both Theorem M and Theorem L yield to the well known result of Leindler [3]. However, since $H(\lambda, p, r, \omega) := H(\lambda, p, r, 1, \omega) \subset H(\lambda, p, r, \delta, \omega)$ for $0 < \delta < 1$, then the really interesting case of Theorem M and Theorem L is just the case $\delta = 1$. If $\delta > 1$, then $H(\lambda, p, r, \delta, \omega) \subset H(\lambda, p, r, \omega)$. However, generally speaking, (3) can not be hold yet. To see it, assume that $\lambda_n = n^{p+pr}$, $f^*(x) := \sin x + \sin 2x$, $\delta = 1 + \frac{2}{p}$, then $\lambda_n$ satisfies both conditions of Theorem M and conditions of Theorem L. It is obvious that $f^* \in W^r H^\omega_{\lambda} \subset H(\lambda, p, r, \omega)$ with $\omega(t) = t$. However, a direct calculation yields that
\[
h_n(f, \lambda, p) \sim n^{-1-r-1/p},
\]
which implies that $f^* \notin H(\lambda, p, r, \delta, \omega)$.

\(^2\)Denote $a_n \sim b_n$ if there exists a constant $K$ independent on $n$, such that $K^{-1} a_n \leq b_n \leq K a_n$.  

4
2 Main Results

First, we further generalize Theorem L to the following

**Theorem 1.** Let $p$ be a positive number, $r$ a nonnegative integer (not necessary even!), $\omega$ a modulus of continuity. If $\{\Lambda_n n^{-\nu} \omega(1/n)^p\}$ is almost increasing and $\{\lambda_n\}$ be a positive sequence satisfying (6), then

\[
W_r^{H_{\omega}} \subset W_r^{H_{\omega}} \subset W_r^{H_{\Omega_r}} \subset H(\lambda, p, r, \omega).
\]

Since $\omega_{\beta}(f^{(r)}(t)) = O(\omega(t))$ is not any more an assumption in $W_r^{H_{\Omega_r}}$, then $W_r^{H_{\Omega_r}}$ is not necessary a subclass of $W_r^{H_{\beta}}$. The following result shows that, we can make $W_r^{H_{\Omega_r}} \subset W_r^{H_{\beta}}$ by adding some additional conditions on $\omega(t)$.

**Theorem 2.** If the sequence $\{\omega(n^{-1})\}$ satisfies the following

(A). \[ n^{-\beta} \sum_{k=1}^{n} k^{\beta-1} \omega\left(\frac{1}{k}\right) = O\left(\omega\left(\frac{1}{n}\right)\right). \]

Then for any nonnegative even integer $r$, it holds that

\[
W_r^{H_{\Omega_r}} \subset W_r^{H_{\beta}}.
\]

If the sequence $\{\omega(n^{-1})\}$ satisfies condition (A) and the following condition (B)

(B). \[ \sum_{k=n+1}^{\infty} k^{-1} \omega\left(\frac{1}{k}\right) = O\left(\omega\left(\frac{1}{n}\right)\right). \]

Then for any nonnegative odd integer $r$, it holds that

\[
W_r^{H_{\Omega_r}} \subset W_r^{H_{\beta}}.
\]

A sequence $\eta := \{\eta_n\}$ of positive numbers is *quasi $\beta-$power-monotone increasing (decreasing)* if there exists a constant $K := K(\beta, \eta) \geq 1$ such that

\[
K n^{\beta} \eta_n \geq m^{\beta} \eta_m \quad (n^{\beta} \eta_n \leq K m^{\beta} \eta_m)
\]

holds for any $n \geq m$.

From Theorem 2, we have

**Corollary.** If the sequences $\{\omega(n^{-1})\}$ is simultaneously quasi $\varepsilon-$power-monotone decreasing with some $\varepsilon > 0$ and $(1-\delta)-$power-monotone increasing with some $1-\min(1, \beta) < \delta < 1$, then for any nonnegative integer $r$, it holds that

\[
W_r^{H_{\Omega_r}} \subset H_\beta^{r}.
\]
3 Proof of Result.

We need some lemmas.

**Lemma 1.** Let $p$ be a positive number, $r$ a nonnegative integer, and $\{b_n\} \in \Omega_r RBVS$. If $\{\Lambda_n n^{-pr} \omega(1/n)^p\}$ is almost increasing and $\{\lambda_n\}$ be a positive sequence satisfying (6), then

$$f_0(x) := \sum_{n=1}^{\infty} b_n \sin nx \in H(\lambda, p, r, \omega).$$

It is the special case of $\delta = 1$ of Lemma 3 in [7].

**Lemma 2.** ([9]) Let $\beta > 0$, and $f(x)$ be a continuous function, then

$$E_n(f) \leq K \omega_{\beta^2} \left( f, \frac{1}{n} \right) \leq Kn^{-\beta} \sum_{k=1}^{n} k^{\beta-1} E_k(f).$$

(8)

$$\omega_{\alpha+\beta}(f, \delta) \leq K \omega_{\beta}(f, \delta), \text{ for } \alpha \geq 0.$$  

(9)

**Lemma 3.** ([10]) Let $f(x)$ be a continuous function, and $f(x)$ has a Fourier series of the form

$$\sum_{n=1}^{\infty} b_n \sin nx, b_n \geq 0,$$

then

$$n^{-\beta} \sum_{k=1}^{n} k^{\beta} b_k \leq K \omega_{\beta} \left( f, \frac{1}{n} \right), \beta \neq 2l, l = 1, 2, \cdots.$$  

**Lemma 4.** ([10]) Let $f(x)$ be a continuous function, and $f(x)$ has a Fourier series of the form

$$\sum_{n=1}^{\infty} a_n \cos nx, a_n \geq 0,$$

then

$$n^{-\beta} \sum_{k=1}^{n} k^{\beta} b_k \leq K \omega_{\beta} \left( f, \frac{1}{n} \right), \beta \neq 2l - 1, l = 1, 2, \cdots.$$  

**Lemma 5.** If $\{b_n\} \in \Omega_r RBVS$, then for any $0 \leq \alpha \leq r$, $\{n^\alpha b_n\} \in \Omega_{r-\alpha} RBVS.$

**Proof.** By the definition of $\Omega_r RBVS$, we have for $\{b_n\} \in \Omega_r RBVS$,

$$|b_n| \leq \sum_{k=n}^{\infty} |\Delta b_k| \leq K(b)n^{r-1}\omega(1/n).$$

(10)
Thus,
\[
\sum_{k=n}^{2n} |\Delta(k^\alpha b_k)| \leq \sum_{k=n}^{2n} k^\alpha |\Delta b_k| + \sum_{k=n}^{2n} ((k + 1)^\alpha - k^\alpha)|b_{k+1}|
\]
\[
\leq Kn^\alpha \sum_{k=n}^{2n} |\Delta b_k| + Kn^{-r-1} \omega \left(\frac{1}{n}\right) \sum_{k=n}^{2n} ((k + 1)^\alpha - k^\alpha)
\]
\[
\leq Kn^\alpha |b_k| + Kn^{-r+\alpha-1} \omega \left(\frac{1}{n}\right)
\]
\[
\leq Kn^{-r+\alpha-1} \omega \left(\frac{1}{n}\right).
\]
Therefore,
\[
\sum_{k=n}^{\infty} |\Delta(k^\alpha b_k)| \leq \sum_{j=0}^{\infty} \sum_{k=2^jn}^{2^{j+1}n} |(\Delta k^\alpha b_k)| \leq K \sum_{j=0}^{\infty} \left(2^jn\right)^{-r+\alpha-1} \omega \left(\frac{1}{2^jn}\right)
\]
\[
\leq Kn^{-r+\alpha-1} \omega \left(\frac{1}{n}\right).
\]
Lemma 5 is over.

**Proof of Theorem 1.** The first relation can be deduced directly from the definitions of $W^r H_{S,\beta}$ and $W^r H_{S,\beta}^{w,*}$. Now we prove the relation
\[
W^r H_{S,\beta}^{w,*} \subset W^r H_{S,\Omega_r}.
\]

**Case 1.** $r$ is an even integer. If
\[
f(x) = \sum_{n=1}^{\infty} b_n \sin nx
\]
belongs to the class $W^r H_{S,\beta}^{w,*}$, then
\[
f^{(r)}(x) = \pm \sum_{n=1}^{\infty} n^r b_n \sin nx
\]
belongs to the class $H_{\beta}^w$. By Lemma 3, it yields that
\[
m^{-\beta} \sum_{n=1}^{4m} n^{r+\beta} b_n \leq K \omega_\beta \left(f^{(r)}, 1/m\right) \leq K \omega(1/m), \quad \beta \neq 2l, l = 1, 2, \cdots.
\]
On the other hand, by (9),
\[
m^{-\beta-1} \sum_{n=1}^{4m} n^{r+\beta+1} b_n \leq K \omega_{\beta+1} \left(f^{(r)}, 1/m\right) \leq K \omega_\beta \left(f^{(r)}, 1/m\right) \leq K \omega(1/m), \quad \beta = 2l, l = 1, 2, \cdots.
\]
So, we always have
\[ m^{-\beta-1} \sum_{n=1}^{4m} r^n b_n \leq K\omega(1/m), \beta > 0. \] (12)

Since \( \{b_n\} \in NBVS \), it follows that
\[ b_n \leq \sum_{i=k}^{n-1} |\Delta b_i| + |b_k| \leq K(b_k + b_{2k}), \lfloor n/2 \rfloor + 1 \leq k \leq n - 1, \]
hence, by (12), we deduce that
\[ b_n + b_{2n} \leq Kn^{-1} \left( \sum_{k=[n/2]+1}^{n-1} (b_k + b_{2k}) + \sum_{k=n+1}^{2n-1} (b_k + b_{2k}) \right) \]
\[ \leq Kn^{-1} \sum_{k=[n/2]+1}^{2n} (b_k + b_{2k}) \leq Kn^{-1} \sum_{k=[n/2]+1}^{4n} b_k \]
\[ \leq Kn^{-r-\beta-2} \sum_{k=[n/2]+1}^{4n} k^{r+\beta+1} b_k \leq Kn^{-r-1} \omega \left( \frac{1}{n} \right). \]

Therefore, by the definition of \( NBVS \) again, we see that
\[ \sum_{k=n}^{2n} |\Delta b_k| \leq K(b_n + b_{2n}) \leq Kn^{-r-1} \omega \left( \frac{1}{n} \right), \]
hence
\[ \sum_{k=n}^{\infty} |\Delta b_k| = \sum_{j=0}^{\infty} \sum_{k=2^j n}^{2^{j+1} n} |\Delta b_k| \leq K \sum_{j=0}^{\infty} (b_{2^j n} + b_{2^{j+1} n}) \]
\[ \leq K \sum_{j=0}^{\infty} (2^j n)^{-r-1} \omega(1/2^j n) \leq K n^{-r-1} \omega \left( \frac{1}{n} \right), \]
which implies that \( \{b_n\} \in \Omega_r RBVS \), and thus we finish (11) for any nonnegative even integer \( r \).

**Case 2.** \( r \) is an odd nonnegative integer. In this case, we have
\[ f^{(r)}(x) = \pm \sum_{k=1}^{\infty} n^r b_n \cos nx \in H^r_{\beta}. \]

By Lemma 4 and a similar way as that of case 1, we still have (12). Thus, we obtain (11) by repeating the proof of case 1.

Finally, the last relation in (7) is the corollary of Lemma 1.
Proof of Theorem 2. If \( r \) is an nonnegative even integer, then, for any \( f(x) := \sum_{n=1}^{\infty} b_n \sin nx \in W^r H_{S,\Omega_r} \), it holds that

\[
f^{(r)}(x) = \pm \sum_{n=1}^{\infty} n^r b_n \sin nx.
\]

Therefore,

\[
\Delta(x) := |f^{(r)}(x) - S_n(f^{(r)}, x)| = \left| \sum_{k=n+1}^{\infty} k^r b_k \sin kx \right|.
\]

It is clear that \( \Delta(0) = \Delta(\pi) = 0 \). Now, we consider the case \( x \in (0, \pi) \). Write \( N = \lfloor 1/x \rfloor \), and

\[
\Delta(x) \leq \left| \sum_{k=n+1}^{N} k^r b_k \sin kx \right| + \left| \sum_{k=N}^{\infty} k^r b_k \sin kx \right| := J_1(x) + J_2(x). \tag{13}
\]

Noting that \( \{b_n\} \in \Omega_r RBVS \), by (10), we get

\[
J_1(x) \leq K x \sum_{k=n}^{N-1} k^{r+1} |b_k| \leq K \omega \left( \frac{1}{n} \right) x(N-1) \leq K \omega \left( \frac{1}{n} \right). \tag{14}
\]

By (10), Lemma 5 and Abel’s transformation, we have

\[
J_2(x) \leq \sum_{k=N}^{\infty} |\Delta(k^r b_k)| D_k(x) + N^r |b_N| |D_{N-1}(x)|
\leq K x^{-1} N^{-1} \omega \left( \frac{1}{N} \right) \leq K \omega \left( \frac{1}{n} \right). \tag{15}
\]

Altogether (13), (14), (15), we obtain

\[
E_n(f^{(r)}) \leq \| f^{(r)} - S_n(f^{(r)}) \| \leq K \omega \left( \frac{1}{n} \right). \tag{16}
\]

Thus, by (8), (16) and condition (A),

\[
\omega_{\beta} \left( f^{(r)}, \frac{1}{n} \right) \leq K n^{-\beta} \sum_{k=1}^{n} k^{\beta-1} E_k(f^{(r)}) \leq K n^{-\beta} \sum_{k=1}^{n} k^{\beta-1} \omega \left( \frac{1}{k} \right) \leq K \omega \left( \frac{1}{n} \right),
\]

which implies that Theorem 2 holds for even integer \( r \).

If \( r \) is an odd nonnegative integer, then by condition (B), we can directly to achieve that

\[
E_n(f^{(r)}) \leq \| f^{(r)} - S_n(f^{(r)}) \| \leq \sum_{k=n+1}^{\infty} k^r |b_k|
\]

\[3\text{If } N \leq n+1, \text{ then a similar discussion can be made directly to } \sum_{k=n+1}^{\infty} k^r b_k \sin kx.]
\[ \leq K \sum_{k=n+1}^{\infty} k^{-1} \omega \left( \frac{1}{k} \right) \leq K \omega \left( \frac{1}{n} \right), \]

hence, we still have

\[ \omega_{\beta} \left( f^{(r)}, \frac{1}{n} \right) \leq Kn^{-\beta} \sum_{k=1}^{n} k^{\beta-1} E_k \left( f^{(r)} \right) \leq K \omega \left( \frac{1}{n} \right) \]

for any odd integer \( r \).

**Proof of Corollary.** We only need to verify that both (A) and (B) are satisfied under the assumption of corollary. In fact, from the fact that \( \omega \left( \frac{1}{n} \right) \) is \( (1-\delta)- \)power-monotone increasing with some \( 1 - \min(1, \beta) < \delta < 1 \), we get

\[ n^{-\beta} \sum_{k=1}^{n} k^{\beta-1} \omega \left( \frac{1}{k} \right) = n^{-\beta} \sum_{k=1}^{n} k^{\beta+\delta-2} \omega \left( \frac{1}{k} \right) k^{1-\delta} \leq K \omega \left( \frac{1}{n} \right). \]

On the other hand, if \( \omega \left( \frac{1}{n} \right) \) is \( \varepsilon \)-power-monotone decreasing for some \( \varepsilon > 0 \), then

\[ \sum_{k=n+1}^{\infty} k^{-1} \omega \left( \frac{1}{k} \right) = \sum_{k=n+1}^{\infty} k^{-1-\varepsilon} \omega \left( \frac{1}{k} \right) k^{\varepsilon} \leq K \omega \left( \frac{1}{n} \right). \]

**References**


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