Quadratic Convergence of the Tanh-sinh Quadrature Rule

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March 29, 2007

1 Introduction

In [5] and [2] the authors describe the remarkable effectiveness of the doubly exponential ‘tanh-sinh’ transformation for numerical integration—even for quite unruly integrands. Our intention in this note is to provide a theoretical underpinning when the integrand is analytic for the observed superlinear convergence of the method. Our analysis rests on the corresponding but somewhat easier analysis by Haber [1] of the less numerically effective ‘tanh’ substitution.

2 Preliminaries

The standard trapezoidal rule for numerical integration when the integrand is defined on the interval (−∞, ∞) is:

\[ \int_{-\infty}^{\infty} f(t) \, dt \approx h \sum_{n=-\infty}^{\infty} f(nh) \] (1)

On changing variables, we can approximate the definite integral via:

\[ \int_{-1}^{1} f(t) \, dt = \int_{-\infty}^{\infty} f(\psi(x)) \psi'(x) \, dx \approx h \sum_{n=-N}^{N} \psi'(nh)f(\psi(nh)). \] (2)

Here, \( \psi \) is any absolutely continuous monotonic increasing function mapping (−∞, ∞) onto (−1, 1), and without loss of generality, we assume the integrand is defined over (−1, 1).

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The **tanh-sinh rule** use the doubly-exponential transformation

\[ \psi(x) = \tanh \left( \frac{\pi}{2} \sinh(x) \right) \]

with

\[ \psi'(x) = \frac{\pi \cosh(x)}{2 \cosh^2 \left( \frac{\pi}{2} \sinh(x) \right)} \].

Correspondingly, \( \psi(x) := \tanh(x) \) gives rise to the scheme analyzed by Haber in [1].

The tanh-sinh scheme [8] is based on the frequent observation, rooted in the *Euler-Maclaurin summation formula* [6], that for certain bell-shaped integrands, a simple block-function approximation to the integral is much more accurate than one would normally expect. Various other efficient rules are described in [2, 5] (such as \( \psi(x) := \text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} \, dt \), which gives rise to “error function” or *erf quadrature* [9]) but since our experience is that the ‘tanh-sinh’ is almost always as-or-more effective, see [5, 6], we do not consider their analysis further herein. In practice tanh-sinh is almost invariably the best rule and is often the only effective rule when more than 50 or 100 digits are required. Figure 1 shows the three schemes we have introduced—erf and tanh(sinh) are visually very close while tanh is the outlier.

In [10], Sugihara studied the convergence properties of various transformations by introducing a number of function spaces. Each one of the spaces consists of functions which are analytic in a strip containing the real line and have a unique decay rate. The intention here is to use the functions in those spaces to represent the new integrands after different transformations.

**Definition 1** ([10], p.381) *For* \( d > 0 \), *define* \( D_d \) *to be the strip of length* \( 2d \) *containing the real axis:*

\[ D_d = \{ z \in \mathbb{C} \mid |\text{Im}z| < d \} , \]

Figure 1: \( \tanh(x) \), \( \text{erf}(x) \), \( \tanh(\sinh(x)) \) and their derivatives

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and let $B(D_d)$ be the set of functions $f$ analytic in $D_d$ such that
\[
\lim_{y \to d^{-}} \int_{-\infty}^{\infty} |f(x + iy)| + |f(x - iy)| \, dx < \infty
\]
and
\[
\int_{-d}^{d} |f(x + iy)| \, dy \to 0 \quad \text{as} \quad x \to \pm \infty.
\]
Furthermore, let $\omega(z)$ be a non-zero function defined on the strip $D_d$ that satisfies the conditions:
1. $\omega(z) \in B(D_d)$.
2. $\omega(z)$ takes real values on the real line.

We then define the functional space $H^\infty$ by:
\[
H^\infty(D_d, \omega) = \{ f : \mathbb{C} \to \mathbb{C} \mid f(z) \text{ is analytic in } D_d, \text{ and } \|f\|_{H^\infty} < \infty \},
\]
where the norm is defined as
\[
\|f\|_{H^\infty} = \sup_{z \in D_d} |f(z)/\omega(z)|.
\]

By the definition of the norm, we have
\[
|f(z)| \leq \|f\|_{H^\infty} |\omega(z)|.
\]
This means $f(z)$ and $\omega(z)$ have the same decay rate. The function $\omega(z)$ can be used to uniquely identify the spaces with different decay rates.

The following theorem by Sugihara [10] gives both the upper bounds for the error norm of the trapezoidal rule and lower bounds for the optimal quadrature formula in $H^\infty(D_d, \omega)$, where $\omega$ is either single exponentially or double exponentially convergent to zero at infinity.

**Theorem 1** ([10], Thm 3.1 and 3.2) Let the $\varepsilon^T_{N,h}(H^\infty(D_d, \omega))$ denote the error norm of the trapezoidal rule on the space $H^\infty(D_d, \omega)$:
\[
\varepsilon^T_{N,h}(H^\infty(D_d, \omega)) = \sup_{\|f\| \leq 1} \left| \int_{-\infty}^{\infty} f(x) \, dx - h \sum_{j=-n}^{n} f(jh) \right|.
\]

Let $\varepsilon^\min_{N}(H^\infty(D_d, \omega))$ denote the optimal $N$-points quadrature formula:
\[
\varepsilon^\min_{N}(H^\infty(D_d, \omega)) = \inf_{1 \leq l \leq N, a_j, c_{jk}} \left\{ \sup_{\|f\| \leq 1} \left| \int_{-\infty}^{\infty} f(x) \, dx - h \sum_{j=1}^{l} \sum_{k=0}^{m_{j-1}} c_{jk} f^{(k)}(a_j) \right| \right\}.
\]

where $a_j \in D_d, c_{jk} \in \mathbb{C}$ and $N = m_1 + m_2 + \cdots + m_l$. 3
1. If the decay rate of $\omega(z)$ is specified by

$$\alpha_1 \exp(-|\beta|x) \leq |\omega(x)| \leq \alpha_2 \exp(-|\beta|x)$$  \hspace{1cm} (5)

where $\alpha_1, \alpha_2, \beta > 0$ and $\rho \geq 1$, then the error norm of the trapezoidal rule can be estimated as

$$\epsilon_{T,N,h}^T (H^\infty(D_d, \omega)) \leq C_{d,\omega} \exp\left(-\pi d \beta N \pi^{-\frac{1}{\rho}}\right)$$  \hspace{1cm} (6)

where $C_{d,\omega}$ is a constant depending on $d$ and $\omega$, $N = 2n + 1$, and the step size $h$ is chosen as

$$h = \frac{(2\pi d)^{\frac{1}{\rho}} (\beta n)^{-\frac{1}{\rho}}}{\gamma}.$$  

Also, we have

$$\epsilon_{\text{min}}^N (H^\infty(D_d, \omega)) \geq C'_{d,\omega} N^{-\frac{1}{\rho}} \exp\left(-\left(\frac{2}{\rho + 1}\right)^{\frac{1}{\rho - 1}} 2\pi d \beta N \pi^{-\frac{1}{\rho}}\right).$$  \hspace{1cm} (7)

where $C'_{d,\omega}$ is another constant depending on $d$ and $\omega$.

2. If the decay rate of $\omega(z)$ is specified by

$$\alpha_1 \exp(-\beta_1 \exp(\gamma|x|)) \leq |\omega(x)| \leq \alpha_2 \exp(-\beta_2 \exp(\gamma|x|))$$  \hspace{1cm} (8)

where $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma > 0$, then the error norm of the trapezoidal rule can be estimated as:

$$\epsilon_{T,N,h}^T (H^\infty(D_d, \omega)) \leq C_{d,\omega} \exp\left(-\frac{\pi d \gamma N}{\ln(\pi d \gamma N / \beta_2)}\right)$$  \hspace{1cm} (9)

where $C_{d,\omega}$ is a constant depending on $d$ and $\omega$, $N = 2n + 1$, and the step size $h$ is chosen as

$$h = \frac{\ln(2\pi d \gamma n / \beta_2)}{\gamma n}.$$  

Also, we have

$$\epsilon_{\text{min}}^N (H^\infty(D_d, \omega)) \geq C'_{d,\omega} \ln N \exp\left(-\frac{2\pi d \gamma N}{\ln(\pi d \gamma N / \beta_1)}\right).$$  \hspace{1cm} (10)

where $C'_{d,\omega}$ is another constant depending on $d$ and $\omega$.

If we compare the error bounds given above, it is clear that the upper bound of the trapezoidal rule is approximately equal to the lower bound of the optimal quadrature rule. The theorem actually implies that the trapezoidal rule is almost optimal among all possible quadrature rules over the space of $H^\infty(D_d, \omega)$ for single and double exponential decay $\omega$.

Now consider the tanh rule $\tanh(x/2)$, where

$$\omega = \frac{d}{dz} \tanh\left(\frac{z}{2}\right) = \frac{1}{2 \cosh^2(z/2)}.$$
The conditions of Theorem 1 holds with $\beta = \rho = 1$ in (5). By (6), the error of the tanh rule can be estimated as
\[
\varepsilon_{N,h}^T (H^\infty(D_d, \omega)) = O \left( \exp \left( -\sqrt{\pi d N} \right) \right).
\] (11)
Furthermore setting $d = \pi/2$, (11) gives
\[
\varepsilon_{N,h}^T (H^\infty(D_{\pi/2}, \omega)) = O \left( \exp \left( -\pi \sqrt{N/2} \right) \right),
\]
and the optimal step size should be chosen as $h = \pi \sqrt{2/N}$. This coincides with the result by Haber (see Theorem ?? and [1]).

For the error function
\[
\text{erf}(t) = \frac{2}{\sqrt{\pi}} \int_0^t \exp(-s^2)ds,
\]
the conditions of Theorem 1 hold with $\beta = \rho = 2$. Then (6) yields
\[
\varepsilon_{N,h}^T (H^\infty(D_{\pi/2}, \omega)) = O \left( \exp \left( -\pi d N^{2/3} \right) \right)
\]
where the step size $h = (2\pi d)^{1/2} n^{-2/3}$.

For the tanh-sinh rule, we have
\[
\omega = \frac{d}{dz} \tanh \left( \frac{\pi}{2} \sinh(z) \right) = \frac{\pi/2 \cdot \cosh(z)}{\cosh^2 \left( \frac{\pi}{2} \cdot \sinh(z) \right)},
\]
which satisfies the condition of Theorem 1 with $\beta_1 = \frac{\pi}{2}$ and $\beta_2 = \frac{\pi}{2} - \epsilon$, with $\epsilon$ a small positive number. The error can be estimated as
\[
\varepsilon_{N,h}^T (H^\infty(D_{\pi/2}, \omega)) = O \left( \exp \left( -\frac{\pi d N}{\ln(4\pi d n/ \pi - \epsilon)} \right) \right),
\]
where $h = \ln \left( 4\pi d n/(\pi - \epsilon) \right)/n$.

In this chapter, we will provide an alternative proof of the convergence property of the tanh-sinh scheme. The analysis rests on the corresponding but somewhat easier analysis by Haber [1] of the less numerically effective ‘tanh’ quadrature. A similar analysis may be undertaken for the ‘erf’ rule, but not as explicitly since the zeros of the error function will be estimated numerically. In addition, various of the summations required appear more delicate.

### 3 Hardy space

We will perform our analysis of the convergence of tanh-sinh rule in the Hardy space $H^2$, (see [3]).
Definition 2 ([3], p.2) For $0 < p < \infty$, **Hardy space** $H^p$ consists of the functions $f$, which are analytic on the unit disk and satisfies the growth condition

$$\|f\|_{H^p} := \sup_{0 \leq r < 1} \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right\} < \infty$$

where the quantity $\|f\|_{H^p}$ is called the Hardy norm.

Thus $H^2$ is the class of functions which are analytic on the unit disk (\{ $z : |z| < 1$\}) and whose Taylor coefficients satisfy

$$\sum_{n=0}^{\infty} |a_n|^2 < \infty.$$ 

A function $f$ is said to have nontangential limit $L$ at $e^{i\theta}$, if $f(z) \to L$ as $z \to e^{i\theta}$ inside the unit circle. Functions in $H^2$ have nontangential limits almost everywhere on the unit circle and belong to $L^2$ on the unit circle ([1], p.17 and p.21). The Hardy space $H^2$ becomes a Hilbert space on imposing the inner product

$$\langle f, g \rangle := \frac{1}{2\pi} \int_{|z|=1} f(z) \overline{g(z)} |dz|.$$ 

Definition 3 ([4, Def. 12.6.1]) Let $S$ denote a point set and $X$ a complete inner product space of functions on $S$. A function of two variables $z$ and $w$ in $S$, $K(z, w)$ is called a reproducing kernel for the space $X$ if

1. For each fixed $w \in S$, $K(z, w)$ is in $X$
2. For every function $f(z) \in X$ and for every point $w \in S$, the reproducing property
   $$f(w) = \langle f(z), K(z, w) \rangle_z.$$ (12)
   [The subscript $z$ indicates $w$ is fixed and the inner product is taken on the variable $z$.]

With respect to the above inner product, $H^2$ has an orthonormal basis given by \{1, $z$, $z^2$, \ldots, $z^n$, \ldots\} and a reproducing kernel

$$K(z, w) := \sum_{n=0}^{\infty} z^n \overline{w^n} = \frac{1}{1 - z \overline{w}}$$ (13)

Due to the existence of the reproducing kernel [4, Thm.12.6.1], the point functionals $P_z$ defined, for $|z| < 1$, by

$$P_z(f) := f(z)$$

are bounded linear functionals on $H^2$ satisfying

$$|P_z f|^2 \leq K(z, z) \|f\|^2.$$ (14)
3.1 A residue theorem

In the sequel we shall use the following lemma, a version of which is given without proof in [1]. We set

\[ \Psi(z) := \begin{cases} -\pi i, & \text{Im } z > 0 \\ \pi i, & \text{Im } z < 0 \end{cases} \]  

(15)

**Lemma 1** [1, Lemma 2] Suppose that \( \alpha \) and \( h \) are positive real numbers and \( f \) is a function satisfying the following conditions:

1. \( f \) is analytic on the closure of the strip \( S_{\alpha} = \{ -\alpha < \text{Im } z < \alpha \} \)
2. \( \lim_{a \to \pm \infty} \int_{a-\alpha}^{a+i\alpha} |f(z)|dz = 0 \)
3. \( \int_{-\infty}^{\infty} f(x)dx \) and \( \sum_{n=-\infty}^{\infty} f(nh) \) exists.

Then

\[ \int_{-\infty}^{\infty} f(x)dx = \frac{1}{2\pi i} \int_{\partial S_{\alpha}} f(z)\Psi(z)dz. \]

Moreover,

\[ h \sum_{n=-\infty}^{\infty} f(nh) = \frac{1}{2\pi i} \int_{\partial S_{\alpha}} f(z) \pi \cot \frac{\pi z}{h}dz. \]

**Proof.** First, we consider integrating \( f \) on the boundary of rectangle \( A := \{ z : -\alpha < \text{Im } z < \alpha \text{ and } -a < \text{Re } z < a \} \) as shown in Figure 2. Since \( f \) is analytic on \( A \), we have

\[ \int_{C_1} f + \int_{-a}^{a} f = -\int_{C_{21}} f = -\int_{C_{41}} f \]

Figure 2: Contour for integration over the rectangle A
and
\[
\int_{C_3} f - \int_{-a}^a f = - \int_{C_{22}} f - \int_{C_{42}} f.
\]
Combining these two equations gives
\[
\int_{-a}^a f = \frac{1}{2} \left( - \int_{C_1} f + \int_{C_3} f - \int_{C_{21}} f - \int_{C_{41}} f + \int_{C_{22}} f + \int_{C_{42}} f \right).
\]
Letting \( a \to \infty \), by the condition 2. of the hypotheses, \( \int_{C_{21}} f, \int_{C_{41}} f, \int_{C_{22}} f \) and \( \int_{C_{42}} f \) will each converge to zero. Thus, we get
\[
\int_{-\infty}^\infty f(x)dx = \frac{1}{2\pi i} \int_{\partial S_\alpha} f(z)\Psi(z)dz.
\]
The function \( f(z)\pi \cot(\pi z/h) \) has poles at \( n h \) for integer \( n \) with residues \( hf(nh) \). According to the residue theorem, we have
\[
\frac{1}{2\pi i} \int_{C_1+C_2+C_3+C_4} f = h \sum_{n=-N}^{N} f(nh)
\]
where \( N \) is the largest integer such that \( Nh < a \) and \( C_2 = C_{21} + C_{22}, C_4 = C_{41} + C_{42} \). Letting \( a \to \infty \), we obtain
\[
h \sum_{n=-\infty}^{\infty} f(nh) = \frac{1}{2\pi i} \int_{\partial S_\alpha} f(z)\pi \cot \left( \frac{\pi z}{h} \right)dz,
\]
as claimed. \( \square \)

More generally, similar arguments will establish:

**Lemma 2** Suppose, in Lemma 1, that \( f \) has poles \( z_i \) with residues \( \text{Res}(z_i) \) inside \( S_\alpha \). Then
\[
\int_{-\infty}^\infty f(x)dx = \frac{1}{2\pi i} \int_{\partial S_\alpha} f(z)\Psi(z)dz + \pi i \sum_i \text{Res}(z_i^+) - \pi i \sum_i \text{Res}(z_i^-)
\]
and
\[
h \sum_{n=-\infty}^{\infty} f(nh) = \frac{1}{2\pi i} \int_{\partial S_\alpha} f(z)\pi \cot \left( \frac{\pi z}{h} \right)dz - \sum_i \text{Res}(z_i)\pi \cot \left( \frac{\pi z_i}{h} \right).
\]

Here \( z_i^+ \) and \( z_i^- \) represent the poles above and below the real line.
3.2 Our working notation

We will work with the following quantities for $h > 0$.

1. The integral: $I f := \int_{-1}^{1} f$.

2. The tanh-sinh approximation:

$$T_{h,N} f := h \sum_{n=-N}^{N} \frac{\pi \cosh(nh)}{2 \cosh^2 \left( \frac{\pi}{2} \sinh(nh) \right)} f \left( \tanh \left( \frac{\pi}{2} \sinh(nh) \right) \right).$$

3. The $N$-th approximation error:

$$E_{h,N} f := (I - T_{h,N}) f.$$

4. The approximation limit:

$$T_h f = \lim_{N \to \infty} T_{h,N} f,$$

which will be shown to exist in lemma 3.

5. The limit error:

$$E_h := (I - T_h) f.$$

Then $I$, $T_{h,N}$, $E_{h,N}$ and are bounded linear functionals on $H^2$. So are $T_h$, and $E_h$ once we show they exist.

4 The associated space $G^2$

Along with $H^2$, it is helpful to use the same change of variable as in (3) to define a corresponding space of functions, [4]. Precisely, we let $G^2$ be the set of functions of the form

$$\psi'(w)f(\psi(w)) \quad f \in H^2 \text{ and } \psi \text{ as in (3)}.$$

**Assumptions.** We assume $\psi$ maps region $A$ onto the unit disk, that functions in $G^2$ are analytic on $A$, and are defined almost everywhere on $\partial A$.

Letting $\hat{f}(w) := \psi'(w)f(\psi(w))$, we can induce an inner product in $G^2$ by

\begin{align*}
\langle \hat{f}, \hat{g} \rangle_{G^2} := \langle f, g \rangle_{H^2} &= \frac{1}{2\pi} \int_{|z|=1} f(z)\overline{g(z)} |dz| \\
&= \frac{1}{2\pi} \int_{\partial A} f(\psi(w))\overline{g(\psi(w))}|\psi'(w)| dw | \psi'(w)| \quad (18) \\
&= \frac{1}{2\pi} \int_{\partial A} \hat{f}(w)\overline{\hat{g}(w)} |dw/\psi'(w)|.
\end{align*}
Then $G^2$ is a Hilbert space and the mapping $f \mapsto \hat{f}$ is an isomorphism of $H^2$ onto $G^2$. Also, $G^2$ has an orthonormal basis with elements
\[ \phi_n(z) := \psi'(z)(\psi(z))^n \] (19)
and

**Lemma 3** Assume the reproducing kernel of $H^2$ is $K(z,w)$, then the reproducing kernel of $G^2$ is
\[ \hat{K}(z,w) = K(\psi(z),\psi(w)) \cdot \psi'(z) \cdot \overline{\psi'(w)} \] (20)
given the transformation $\psi(z)$ from $H^2$ to $G^2$.

**Proof.** Given the reproducing property (12) and the definition of the inner product (18), if $\hat{K}(z,w)$ denotes the reproducing kernel of $G^2$ then
\[
\langle \hat{f}(z), \hat{K}(z,w) \rangle_z = \frac{1}{2\pi} \int_{\partial A} \hat{f}(z) \overline{\hat{K}(z,w)} \frac{dz}{\psi'(z)}
\]
\[ = \hat{f}(w)
\]
\[ = \psi'(w) \cdot \langle f(z), K(z,\psi(w)) \rangle_z
\]
\[ = \psi'(w) \cdot \frac{1}{2\pi} \int_{\partial A} \hat{f}(z) K(\psi(z),\psi(w)) \psi'(z) \frac{dz}{\psi'(z)}
\]
\[ = \frac{1}{2\pi} \int_{\partial A} \hat{f}(z) K(\psi(z),\psi(w)) \psi'(z) \overline{\psi'(w)} \frac{dz}{\psi'(z)}
\]
Comparing the last integral with the one on the first line, we get
\[ \hat{K}(z,w) = K(\psi(z),\psi(w)) \cdot \psi'(z) \cdot \overline{\psi'(w)}. \]

Therefore for the tanh-sinh transformation, by (13), $G^2$ has a reproducing kernel
\[ \hat{K}(z,w) = \frac{1}{\pi^2} \left( 1 - \psi(z)\overline{\psi(w)} \right) \cdot \psi'(z) \cdot \overline{\psi'(w)}
\]
\[ = \frac{\pi^2}{4 \cosh \left( \frac{\pi}{2} \sinh(z) - \frac{\pi}{2} \sinh(w) \right)} \times \frac{\cosh(z) \cosh(w)}{\cosh \left( \frac{\pi}{2} \sinh(z) \right) \cosh \left( \frac{\pi}{2} \sinh(w) \right)}. \] (21)

Consequently, the point functionals on $G^2$, $\hat{P}_z$, defined for $z \in A$ by
\[ \hat{P}_z \hat{f} = \hat{f}(z) \]
are bounded linear functionals (see Thm. 12.6.1 of [4]) and satisfy
\[ |\hat{P}_z \hat{f}|^2 \leq \hat{K}(z,z) \| \hat{f} \|^2. \] (22)
4.1 Further working notation

We use similar notation in $G^2$ as in $H^2$:

\[ \hat{I} \hat{f} := \int_{-1}^{1} f(x) dx = \int_{-\infty}^{\infty} \hat{f}(u) du, \]

\[ \hat{T}_{h,N} \hat{f} := T_{h,N} f = h \left( \sum_{n=-N}^{N} \hat{f}(nh) \right), \]

\[ \hat{E}_{h,N} \hat{f} := \hat{I} \hat{f} - \hat{T}_{h,N} \hat{f}, \]

\[ \hat{T}_h \hat{f} := \lim_{N \to \infty} \hat{T}_{h,N} \hat{f} \quad \text{and} \quad \hat{E}_h \hat{f} := (\hat{I} - \hat{T}_h) \hat{f}. \]

**Lemma 4** For $h > 0$ the operators $\hat{T}_h$ and $\hat{E}_h$ are bounded linear functionals on $G^2$ (as are $\hat{T}_{h,N}$ and $\hat{E}_{h,N}$).

**Proof.** Define $|\hat{T}_{h,N} \hat{f}| = h \sum_{n=-N}^{N} |\hat{f}(nh)|$ The inequality (22) shows that

\[ |\hat{T}_{h,N} \hat{f}| \leq h \| \hat{f} \| \sum_{n=-N}^{N} \hat{K}(nh,nh)^{1/2} \]

\[ < h \| \hat{f} \| \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{1 - \tanh^2(\pi/2 \sinh(nh))}} \cdot \frac{\pi \cosh(nh)}{2 \cosh^{2}(\pi/2 \sinh(nh))} \]

\[ = h \| \hat{f} \| \sum_{n=-\infty}^{\infty} \frac{\pi \cosh(nh)}{2 \cosh(\pi/2 \sinh(nh))} \]

\[ \leq h\pi \| \hat{f} \| \sum_{n=-\infty}^{\infty} e^{n\pi - \pi/2} \sinh(nh)| \]

\[ \leq h\pi \| \hat{f} \| \sum_{n=-\infty}^{\infty} e^{-h|n|/2} = h \cdot \pi \cdot \coth(h/4) \| \hat{f} \|. \] (23)

where

\[ \| \hat{f} \| = \| \hat{f} \|^2 = (f,f) = \frac{1}{2\pi} \int_{|z|=1} f(z) \overline{f(z)} \, |dz| \]

\[ = \frac{1}{2\pi} \int_{|z|=1} |f(z)|^2 \, |dz|. \] (24)

Since $f \in H^2$ belongs to $L^2$ on the unit circle, (24) is finite. Equation (23) shows that $\hat{T}_{h,N} \hat{f}$ is absolutely convergent. Also,

\[ \hat{T}_h \hat{f} \leq h \cdot \pi \cdot \coth(h/4) \| \hat{f} \|. \]
Thus, $\hat{T}_h$ and also $\hat{E}_h$ are bounded linear operators on $G^2$.

If a space $X$ has a reproducing kernel, every bounded linear functional on $X$ has a very simple expression. In the next result, $L_z$ denotes a functional $L$ applied to $K$ with respect to variable $z$ while keeping $w$ constant.

**Theorem 2** ([4, Thm 12.6.6 and Cor 12.6.7]) Let $X$ has a reproducing kernel $K(z, w)$, and assume $L$ is a bounded linear functional defined on $X$, then

$$\|L\|^2 = L_w L_z K(z, w).$$

When $L$ is any of the previous functionals on $G^2$,

$$\|L\|^2 = L_z L(w) \hat{K}(z, w)$$

(25)

since all functionals considered are real.

Our goal is to estimate the approximation error $||\hat{E}_{h,N}|| = ||\hat{\sigma}_{h,N}||$ which will be computed by repeated use of Lemma 4 and (25).

### 5 Evaluation of the error norm $||\hat{E}_{h,N}||$

Let $\hat{\sigma}_{h,N} := \hat{T}_h - \hat{T}_{h,N}$. Then (25) implies that

$$\|\hat{E}_{h,N}\|^2 = \|\hat{E}_h + \hat{\sigma}_{h,N}\|^2$$

$$= \hat{E}_{h,(z)} \hat{E}_{h,(w)} \hat{K}(z, w) + \hat{E}_{h,(z)} \hat{\sigma}_{h,N,(w)} \hat{K}(z, w)$$

$$+ \hat{\sigma}_{h,N,(z)} \hat{E}_{h,(w)} \hat{K}(z, w) + \hat{\sigma}_{h,N,(z)} \hat{\sigma}_{h,N,(w)} \hat{K}(z, w).$$

(26)

In (26), the error is divided into four parts denoted by $e_1$, $e_2$, $e_3$, and $e_4$. In the following three subsections, we will evaluate these quantities term by term.

#### 5.1 Evaluation of $e_2$ and $e_3$

The second and third terms of (26) are actually equal. Indeed:

$$e_4 = \hat{\sigma}_{h,N,(z)} \hat{E}_{h,(w)} \hat{K}(z, w) = \hat{E}_{h,(z)} \hat{\sigma}_{h,N,(w)} \hat{K}(z, w)$$

$$= \hat{E}_{h,(z)} \left( h \sum_{|n|>N} \hat{K}(z, nh) \right)$$

$$= \hat{E}_{h,(z)} \left( h \sum_{|n|>N} F_{nh}(z) \right)$$

$$= h \sum_{|n|>N} \hat{E}_h F_{nh}$$

(27)
where
\[ F_{nh} := P_{nh(w)}K(z, w) \]
\[ = \frac{\pi^2 \cosh(z) \cosh(nh)}{4 \cosh \left( \frac{z}{2} \sinh(z) - \frac{nh}{2} \sinh(nh) \right) \cosh \left( \frac{z}{2} \sinh(z) \right) \cosh \left( \frac{nh}{2} \sinh(nh) \right)}. \tag{28} \]

To calculate (27), first we compute \( \hat{IF}_{nh} \)
\[ \hat{IF}_{nh} = \int_{-\infty}^{\infty} P_{nh(w)}K(z, w)dz \]
\[ = \int_{-\infty}^{\infty} \frac{\pi^2 \cosh(z) \cosh(nh)}{4 \cosh \left( \frac{z}{2} \sinh(z) - \frac{nh}{2} \sinh(nh) \right) \cosh \left( \frac{z}{2} \sinh(z) \right) \cosh \left( \frac{nh}{2} \sinh(nh) \right)} dz \]
\[ = \frac{\pi^2 \cosh(nh)}{4 \cosh \left( \frac{z}{2} \sinh(nh) \right)} \int_{-\infty}^{\infty} \cosh \left( \frac{z}{2} \sinh(z) - \frac{nh}{2} \sinh(nh) \right) \cosh \left( \frac{z}{2} \sinh(z) \right) dz. \]

Using the change of variable \( u := \frac{z}{2} \sinh(x) \), and letting \( a := \frac{z}{2} \sinh(nh) \), we get
\[ \hat{IF}_{nh} = \frac{\pi \cosh(nh)}{2 \cosh(a)} \int_{-\infty}^{\infty} \frac{1}{\cosh(a - a) \cosh(u)} du \]
\[ = \frac{\pi \cosh(nh)}{\cosh(a)} \int_{-\infty}^{\infty} \frac{1}{\cosh(a) + \cosh(2u - a)} du. \tag{29} \]

Applying the identity
\[ \frac{\sinh(a)}{\cosh(a) + \cosh(b)} = \frac{1}{1 + e^{b-a}} - \frac{1}{1 + e^{b+a}}, \tag{30} \]
we see that
\[ (29) = \frac{\pi \cosh(nh)}{\cosh(a) \sinh(a)} \int_{-\infty}^{\infty} \left( \frac{1}{1 + e^{2u-2a}} - \frac{1}{1 + e^{2u}} \right) du \]
\[ = \frac{\pi \cosh(nh)}{\cosh(a) \sinh(a)} \cdot a \]
\[ = \frac{\pi^2 \sinh(2nh)}{2 \sinh(\pi \sinh(nh))} = \hat{IF}_{nh}. \tag{31} \]

By a similar manipulation, we get \( \hat{T}_nF_{nh} = \]
\[ \frac{\pi^2 h \cosh(nh)}{\sinh(\pi \sinh(nh))} \sum_{r=-\infty}^{\infty} \left( \frac{1}{1 + e^{\pi \sinh(\pi h) - \pi \sinh(nh)}} - \frac{1}{1 + e^{\pi \sinh(nh)}} \right) \cosh(rh). \tag{32} \]

Therefore
\[ h \sum_{|n|>N} \hat{E}_h F_{nh} = \sum_{|n|>N} \frac{\pi^2 \cosh(nh)}{\sinh(\pi \sinh(nh))} \]
\[ \times \left\{ \sum_{r=-\infty}^{\infty} \left( \frac{1}{1 + e^{\pi \sinh(\pi h) - \pi \sinh(nh)}} - \frac{1}{1 + e^{\pi \sinh(nh)}} \right) h \cosh(rh) - \sinh(nh) \right\}. \tag{33} \]
Consider the summation
\[ S_n := \sum_{r=-\infty}^{\infty} \left( \frac{1}{1 + e^{\pi \sin(rh) - \pi \sin(nh)}} - \frac{1}{1 + e^{\pi \sin(rh)}} \right) h \cosh(rh), \]
and note that for \( h > 0 \)
\[ S_n = \left| e^{\pi \sin(nh)} - 1 \right| \sum_{r=-\infty}^{\infty} \frac{e^{\pi/2 \sinh(rh)}}{e^{\pi \sin(nh)} + e^{\pi \sin(rh)}} \frac{h \cosh(rh)}{2 \cosh(\pi/2 \sinh(rh))} \]
\[ = \sinh(\pi/2 \sinh(nh)) \sum_{r=-\infty}^{\infty} \frac{2 e^{\pi/2 \sinh(nh)} e^{\pi/2 \sinh(rh)}}{e^{\pi \sin(nh)} + e^{\pi \sin(rh)}} \frac{h \cosh(rh)}{\cosh(\pi/2 \sinh(rh))} \]
\[ \leq \sinh(\pi/2 \sinh(nh)) \sum_{r=-\infty}^{\infty} \frac{h \cosh(rh)}{\cosh(\pi/2 \sinh(rh))} \]
\[ = h \sinh(\pi/2 \sinh(nh)) \left\{ 1 + 2 \sum_{r=1}^{\infty} \frac{\cosh(rh)}{\cosh(\pi/2 \sinh(rh))} \right\}. \]

If we denote
\[ C(h) := 1 + 2 \sum_{r=1}^{\infty} \frac{\cosh(rh)}{\cosh(\pi/2 \sinh(rh))}, \]
then
\[ S_n \leq h C(h) \sinh(\pi/2 \sinh(nh)) \]
and by (33)
\[ h \sum_{|n| > N} \bar{E}_n F_{nh} \leq \sum_{|n| > N} \frac{\pi^2 \cosh(nh)}{\sinh(\pi \sinh(nh))} \left\{ hC(h) \sinh(\pi/2 \sinh(nh)) + \sinh(nh) \right\} \]
\[ = \frac{\pi^2}{2} \left( \sum_{|n| > N} \frac{hC(h) \cosh(nh)}{\cosh(\pi \sinh(nh)/2)} + \sum_{|n| > N} \frac{\sinh(2nh)}{\sinh(\pi \sinh(nh))} \right). \]

Since \( \frac{\cosh(x)}{\cosh(\pi/2 \sinh(x))} \) is a decreasing function, we have
\[ \sum_{n > N} \frac{\cosh(nh)}{\cosh(\pi/2 \sinh(nh))} \leq \frac{1}{\hbar} \int_{\pi/2 \sinh(N-1)}^{\infty} \frac{\cosh(x)}{\cosh(\pi/2 \sinh(x))} \, dx \]
\[ = \frac{2}{\pi \hbar} \int_{\pi/2 \sinh(N-1)}^{\infty} \text{sech}(u) \, du \]
\[ = \frac{2}{\pi \hbar} \left( \frac{\pi}{2} - \arctan \left( \frac{\sinh(\pi(N-1))}{\pi} \right) \right) \]

Also, because
\[ \lim_{x \to \infty} x \left( \frac{\pi}{2} - \arctan(x) \right) = \frac{1}{2} \]
when \( x \) is large enough,

\[
O \left( \frac{\pi}{2} - \arctan (x) \right) = O \left( \frac{1}{x} \right)
\]

thus

\[
\sum_{n > N} \frac{\cosh(nh)}{\cosh(\pi/2 \sinh(nh))} = O \left( \frac{1}{h} e^{-\frac{N-1}{h}} \right) \tag{36}
\]

Also, the second summation in (35) is less than the first one, which gives

\[
e2 = e3 = h \sum_{|n| > N} \hat{E}_n F_{nh} = O \left( C(h) e^{-\frac{N-1}{h}} \right) \tag{37}
\]

5.2 Evaluation of \( e_1 \)

In order to compute \( e_1 = \hat{E}_{h,(z)} \hat{E}_{h,(w)} \hat{K}(z,w) \), we have recourse to the results of Section 3.1. In this case, the reproducing kernel \( \hat{K}(z,\bar{w}) \) has poles at \( w = i \cdot \arcsin(2n + 1) \) with residue \( \pi \cosh(z)/\sinh(\pi \sinh(z)) \) and poles at \( w = \text{arcsinh}(\sinh(z) + (2n + 1)i) \) with reside \( -\pi \cosh(z)/\sinh(\pi \sinh(z)) \). Since we are dealing with real functions, \( \hat{K}(z,\bar{w}) = \hat{K}(z,w) \), as a function of \( \bar{w} \), satisfies the conditions of Lemmas 1 and 2, if we take \( \alpha \leq i \arcsin(1) \).

Therefore,

\[
\hat{E}_{h,(w)} \hat{K}(z,w) = \frac{1}{2\pi i} \int_{\partial S_{\alpha}} \hat{K}(z,\bar{w}) \Phi(w) \, dw
\]

\[
= \frac{1}{2\pi i} \int_{\partial S_{\alpha}} \frac{\pi^2 \cosh(z) \cosh(w) \Phi(w)}{4 \cosh \left( \frac{\pi}{2} \sinh(z) - \frac{\pi}{2} \sinh(w) \right) \cosh \left( \frac{\pi}{2} \sinh(z) \right) \cosh \left( \frac{\pi}{2} \sinh(w) \right)} \, dw,
\]

where

\[
\Phi(z) := \Phi(z;h) = \Psi(z) - \pi \cot \frac{\pi z}{h} = \begin{cases} 
\frac{-2\pi i}{1-\exp(-2\pi iz/h)}, & \text{Im}(z) > 0 \\
\frac{2\pi i}{1-\exp(2\pi iz/h)}, & \text{Im}(z) < 0.
\end{cases}
\tag{38}
\]

Here \( \Psi(x) \) is defined as in (15). If we let \( \alpha \to \infty \) while keeping \( \partial S_{\alpha} \) away from the poles,(16) gives

\[
\int_{-\infty}^{\infty} f(x) dx
\]

\[
= \frac{1}{2\pi i} \int_{\partial S_{\alpha}} f(z) \Phi(z) \, dz - \frac{\pi \cosh(z)}{\sinh(\pi \sinh(z))} \sum_{n=-\infty}^{\infty} \Psi(i \arcsin(2n + 1))
\]

\[
+ \frac{\pi \cosh(z)}{\sinh(\pi \sinh(z))} \sum_{n=-\infty}^{\infty} \Psi(\text{arcsinh}(\sinh(z) + (2n + 1)i))
\]

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and equation (17) gives
\[ h \sum_{n=-\infty}^{\infty} f(nh) = \frac{1}{2\pi i} \int_{\partial S_0} f(z) \frac{\pi z}{h} dz - \frac{\pi \cosh(z)}{\sinh(\pi \sinh(z))} \sum_{n=-\infty}^{\infty} \pi \cot \left( \frac{\pi \cdot \arcsinh(2n+1)}{h} \right) \]
\[ + \frac{\pi \cosh(z)}{\sinh(\pi \sinh(z))} \sum_{n=-\infty}^{\infty} \pi \cot \left( \frac{\pi \cdot \arcsinh(z + (2n+1)i)}{h} \right) \]

We thus have
\[ \hat{E}_{h,(w)} \hat{K}(z,w) = \frac{1}{2\pi i} \int_{\partial S_0} \hat{K}(z,w) \Phi(w) dw - \frac{\pi \cosh(z)}{\sinh(\pi \sinh(z))} \sum_{n=-\infty}^{\infty} \Phi(i \arcsin(2n+1)) \]
\[ + \frac{\pi \cosh(z)}{\sinh(\pi \sinh(z))} \sum_{n=-\infty}^{\infty} \Phi(\arcsinh(z + (2n+1)i)) \]

Since \(|\Phi(x+iy)| \sim 2\pi \exp(-2\pi|y|/h)|\), the integral part will go to zero as \(\alpha \to \infty\).
Also, since \(\Phi(-w) = -\Phi(w)\),
\[ \frac{\pi \cosh(z)}{\sinh(\pi \sinh(z))} \sum_{n=-\infty}^{\infty} \Phi(i \arcsin(2n+1)) = \frac{\pi \cosh(z)}{\sinh(\pi \sinh(z))} \lim_{N \to \infty} \sum_{n=-N}^{N} \Phi(i \arcsin(2n+1)) = 0. \]

Therefore
\[ \hat{E}_{h,(w)} \hat{K}(z,w) = \frac{\pi \cosh(z)}{\sinh(\pi \sinh(z))} \sum_{n=-\infty}^{\infty} \Phi(\arcsinh(z + (2n+1)i)). \quad (39) \]

Letting
\[ H(z) := \hat{E}_{h,(w)} \hat{K}(z,w), \]
it is easy to see that \(H(z) \in C^\infty\). Also, since we have shown \(\hat{E}_h\) is a bounded linear functional and
\[ \hat{E}_h = \hat{E}_{h,(z)} H(z) = \int_{-\infty}^{\infty} H(z) dz - h \sum_{n=-\infty}^{\infty} H(nh). \]

Thus, \(\sum_{n=-\infty}^{\infty} H(nh)\) exists for all \(h > 0\) and \(H(z)\) is integrable because the integrand is independent of \(h\). Taking \(h \to 0\), the summation part will converge...
to zero so that the integral is finite. Therefore, $H(z)$ satisfies the conditions of the Poisson summation formula (see the proof of [7, 243–4]). It follows that

$$\|\hat{E}_h\|^2 = \hat{E}_h H = -2 \sum_{r=1}^{\infty} \int_{-\infty}^{\infty} H(x) \cos(2\pi rx/h) \, dx. \tag{40}$$

Set

$$H_n(x) := \Phi(\text{arcsinh} (\sinh(z) + (2n-1)i)) + \Phi(\text{arcsinh} (\sinh(z) - (2n-1)i))$$

then

$$H(x) = \sum_{n=1}^{\infty} \frac{\pi \cosh(x)}{\sinh(\pi \sinh(x))} H_n(x). \tag{41}$$

Letting $A(z, n)$ and $B(z, n)$ denote the real part and imaginary part of $\text{arcsinh}(\sinh(z) + (2n-1)i)$,

we have

$$A(z, n) = \text{sgn}(z) \cdot \ln \left( \frac{1}{2} \sqrt{\sinh(z)^2 + 4n^2 + \frac{1}{2} \sinh(z)^2 + (2n-2)^2} + \sqrt{\left( \frac{1}{2} \sqrt{\sinh(z)^2 + 4n^2 + \frac{1}{2} \sinh(z)^2 + (2n-2)^2} \right)^2 - 1} \right)$$

$$B(z, n) = \arcsin \left( \frac{1}{2} \sqrt{\sinh(z)^2 + 4n^2 - 1/2 \sinh(z)^2 + (2n-2)^2} \right)$$

where

$$\text{sgn}(z) = \begin{cases} 1, & \text{Re}(z) > 0 \\ -1, & \text{Re}(z) < 0. \end{cases} \tag{42}$$

Therefore

$$\Phi(\text{arcsinh}(\sinh(z) + (2n-1)i)) = \Phi(A(z, n) + B(z, n)i)$$

$$\Phi(\text{arcsinh}(\sinh(z) - (2n-1)i)) = \Phi(A(z, n) - B(z, n)i)$$

and

$$H_n(z) = 2\pi i \left( \frac{-1}{1 - \exp(2\pi B(z, n)/h)} + \frac{1}{1 - \exp(2\pi B(z, n)/h + 2\pi A(z, n)/h)} \right)$$

$$= \frac{-4\pi \sin(\frac{2\pi A(z, n)}{h}) \exp(\frac{2\pi B(z, n)/h}{h})}{1 - 2 \exp(\frac{2\pi B(z, n)/h}{h}) \cos(\frac{2\pi A(z, n)/h}{h}) + \exp(\frac{4\pi B(z, n)/h}{h})} \tag{43}$$

By (40) and (41),

$$\|\hat{E}_h\|^2 = -2 \sum_{r=1}^{\infty} \int_{-\infty}^{\infty} \sum_{n=1}^{\infty} \frac{\pi \cosh(x)}{\sinh(\pi \sinh(x))} H_n(x) \cos(2\pi rx/h) \, dx$$

$$= -2 \sum_{r=1}^{\infty} \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} \frac{\pi \cosh(x)}{\sinh(\pi \sinh(x))} H_n(x) \cos(2\pi rx/h) \, dx \tag{44}$$
Using a more elaborate (and computer algebra assisted) version of the argument given by [1] for tanh, we can estimate $e_1$ by

$$e_1 = \|\hat{E}_h\|^2 = O\left(e^{-A/h}\right)$$  \hfill (45)

where $A$ is a positive constant. This entails integrating the integral component of (44) by parts twice to obtain

$$\int_{-\infty}^{\infty} H_n(x) \frac{\pi \cosh(x)}{\sinh(\pi \sinh(x))} \cos(2\pi r x/h) dx
= -\left(\frac{h}{2\pi r}\right)^2 \int_{-\infty}^{\infty} \left(H_n(x) \frac{\pi \cosh(x)}{\sinh(\pi \sinh(x))}\right)^{\prime\prime} \cos(2\pi r x/h) \ dx. \quad (46)$$

which is

$$O\left(\frac{e^{-A/h}}{r^2n^2}\right)$$

Our numerical experiments indicate that $A$ is some constant bounded below by $5/2$ and above by 10.

### 5.3 Evaluation of $e_4$

We have:

$$e_4 = \hat{\sigma}_{h,N,(z)} \hat{\sigma}_{h,N,(w)} \hat{K}(z,w)$$

$$= \frac{h^2\pi^2}{4} \sum_{|r|,|n|>N} \cosh(nh) \cosh(rh) \cosh\left(\frac{\pi}{2} \sinh(nh) \right) \cosh\left(\frac{\pi}{2} \sinh(rh) \right)$$

which can be written as

$$\frac{h^2\pi^2}{2} \sum_{r,n>N} \cosh(nh) \cosh(rh) \cosh\left(\frac{\pi}{2} \sinh(nh) \right) \cosh\left(\frac{\pi}{2} \sinh(rh) \right)$$

$$+ \frac{h^2\pi^2}{2} \sum_{r,-n>N} \cosh(nh) \cosh(rh) \cosh\left(\frac{\pi}{2} \sinh(nh) \right) \cosh\left(\frac{\pi}{2} \sinh(rh) \right)$$  \hfill (47)

Then, because of (36), equation (48) is equal to

$$\frac{h^2\pi^2}{2} \sum_{r,n>N} \cosh(nh) \cosh(rh) \cosh\left(\frac{\pi}{2} \sinh(nh) + \frac{\pi}{2} \sinh(rh) \right) \cosh\left(\frac{\pi}{2} \sinh(nh) \right)$$

$$\leq \frac{h^2\pi^2}{2} \left(\sum_{n>N} \cosh(nh) \cosh(\pi/2 \sinh(nh))\right)^2 = O\left(e^{-2n^{-1}}\right)$$  \hfill (49)
Also, (47) can be written as

\[
\frac{h^2 \pi^2}{2} \sum_{n>N} \frac{\cosh^2(nh)}{\cosh^2(\pi/2 \sinh(nh))} + 2 \sum_{n\geq r>N} \frac{\cosh(nh) \cosh(rh)}{\cosh(\frac{\pi}{2} \sinh(nh) - \frac{\pi}{2} \sinh(rh)) \cosh(\frac{\pi}{2} \sinh(nh)) \cosh(\frac{\pi}{2} \sinh(rh))}.
\]

The first term above is less than

\[
O \left( e^{-\frac{2N-1}{h}} \right)
\]

and the second term is less than the first. Equation (49) and (50) together give

\[
e_4 = \hat{\sigma}_{h,N}(z) \hat{\sigma}_{h,N}(w) \hat{K}(z,w) = O \left( e^{-\frac{2N-1}{h}} \right).
\]

6 The main results

Combining (37), (45) and (51) we get

Theorem 3 (Tanh-sinh convergence.) (a) For \( f \in H^2 \) and \( \psi(x) = \tanh(\frac{\pi}{2} \sinh(x)) \), the error bound can be evaluated as:

\[
||\hat{E}_{h,N}||^2 = ||E_{h,N}||^2 = O \left( e^{-A/h} \right) + O \left( C(h) e^{-\frac{2N-1}{\pi}} \right)
\]

where the order constant is independent of both \( N \) and \( h \). Here as before

\[
C(h) := 1 + 2 \sum_{r=1}^{\infty} \frac{\cosh(rh)}{\cosh(\pi/2 \sinh(rh))} \leq 1 + \frac{4}{\pi h}
\]

(b) This method exhibits quadratic convergence as we let \( N \to \infty \) and \( h \to 0^+ \), while keeping \( Nh \) a constant.

Proof. (a) We estimate

\[
||\hat{E}_{h,N}||^2 = ||E_{h,N}||^2 = e_1 + e_2 + e_3 + e_4 = O \left( e^{-A/h} \right) + O \left( C(h) e^{-\frac{2N-1}{\pi}} \right).
\]

To obtain the estimate for \( C(h) \) observe that

\[
hC(h) \leq h + 2 \int_{0}^{\infty} \frac{\cosh(t)}{\cosh(\pi/2 \sinh(t))} dt = h + \frac{4}{\pi} \int_{0}^{\infty} \text{sech} \left( \frac{\pi}{2} x \right) dx
\]

since the first integrand is decreasing. The latter integral evaluates to 1.
(b) In this case the order term

\[ O\left(e^{-A/h}\right) \]

shows quadratic convergence as \( h \to 0 \), because when \( h \) is halved, \( e^{-A/h} \) will be squared. Similarly,

\[ O\left(C(h)e^{-\frac{N-1}{\pi}}\right) \]

exhibits exponential convergence as \( N \to \infty \).

\[ \Box \]

7 Analysis for Error Function and Other Transformations

As shown in Figure 1, tanh, tanh-sinh and error function have very similar properties: absolutely continuous monotonic increasing functions mapping \((-\infty, \infty)\) onto \((-1, 1)\), infinitely differentiable and every order of their derivatives quickly vanishes at infinity. It’s not surprising that a similar approach can be applied to analyze the convergent property of error function and more generally, to any transformation with the property above.

For example, the error quadrature uses the transformation:

\[ \psi(x) = \text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} \, dt \]  \hspace{1cm} (53)

with

\[ \psi'(x) = \frac{2}{\sqrt{\pi}} e^{-x^2} \]

As in section 4, one can introduce a corresponding space \( G_2 \), which contains the set of functions of the form

\[ \psi'(w)f(\psi(w)) \quad f \in H^2 \text{ and } \psi \text{ as in (53).} \]

\( G_2 \) then has a reproducing kernel

\[ \hat{K}(z, w) = \frac{4}{\pi (1 - \text{erf}(z)\text{erf}(w))} e^{-z^2} e^{-w^2} \]

The approximation error \( \| \hat{E}_{h,N} \| \) can still be divided into 4 parts as in (26) and

\[ e_2 = e_3 = h \sum_{|n| > N} \hat{E}_h F_{uh} \]

\[ e_4 = \frac{4}{\pi} \sum_{|r|,|n| > N} \frac{e^{-r^2 h^2} e^{-n^2 h^2}}{1 - \text{erf}(r h)\text{erf}(n h)} \]
where

\[ F_{nh} := P_{nh,w} \tilde{K}(z, w) \]

\[ = \frac{4}{\pi} \frac{e^{-z^2}e^{-n^2h^2}}{1 - \text{erf}(z)\text{erf}(nh)}. \]

In order to evaluate \( e_1 \), one need to find out the poles of

\[ \frac{1}{1 - \text{erf}(z)\text{erf}(w)}. \]

Also, the evaluation of \( e_2 \) and \( e_3 \) involves the careful estimation of

\[ \sum_{n=a}^{\infty} \sum_{r=b}^{\infty} \frac{e^{-t^2h^2}}{1 - \text{erf}(rh)\text{erf}(nh)}. \]

Here we will not give further analysis. For any other quadrature rules with similar properties, the similar approach can be taken.

8 Numerical Convergence and Examples

We conclude with a few numerical illustrations and a brief discussion of the optimality of the coefficient of the infinite trapezoidal rule.

8.1 Numerical results

We exhibit results for the following set of test functions, earlier studied in [5]:

1. \( \int_{0}^{1} t \log(1 + t) \, dt = \frac{1}{4} \)
2. \( \int_{0}^{1} t^2 \arctan t \, dt = (\pi - 2 + 2 \log 2)/12 \)
3. \( \int_{0}^{\pi/2} e^t \cos t \, dt = (e^{\pi/2} - 1)/2 \)
4. \( \int_{0}^{1} \frac{\arctan(\sqrt{\pi t})}{(1+t^2)^{\pi/2}} \, dt = 5\pi^2/96 \)
5. \( \int_{0}^{1} \sqrt{t} \log t \, dt = -4/9 \)
6. \( \int_{0}^{1} \sqrt{1-t^2} \, dt = \frac{\pi}{4} \)
7. \( \int_{0}^{1} \frac{\sqrt{t}}{\sqrt{1-t^2}} \, dt = 2\sqrt{\pi} \Gamma(3/4)/\Gamma(1/4) = \beta(1/2, 3/4)/2 \)
8. \( \int_{0}^{1} \log^2 t \, dt = 2 \)
9. \( \int_{0}^{\pi/2} \log(\cos t) \, dt = -\pi \log(2)/2 \)
10. \( \int_{0}^{\pi/2} \tan t \, dt = \pi \sqrt{2}/2 \)
11. \( \int_0^\infty \frac{1}{1+t^2} \, dt = \pi/2 \)

12. \( \int_0^\infty \frac{e^{-x}}{\sqrt{t}} \, dt = \sqrt{\pi} \)

13. \( \int_0^\infty e^{-t/2} \, dt = \sqrt{\pi/2} \)

14. \( \int_0^\infty e^{-t} \cos t \, dt = 1/2 \)

Note that the first four of these integrals involve well-behaved continuous functions on finite intervals. The integrands in problems 5 and 6 have an infinite derivative at an endpoint. In 7 through 10, there is an integrable singularity at an endpoint. In 11 through 13, the interval of integration is infinite, requiring a transformation such as \( s = 1/(t + 1) \). Problem 14 is an oscillatory function on an infinite interval.

Table 1 gives the actual error resulting from using the summation (2) as an approximation to the specified integral, rounded to the nearest power of ten. The precision used in these calculations was 1000 digits, so no results are shown with errors less than \( 10^{-1000} \). The “level” shown in the table is an index of the interval \( h \), according to the rule \( h = 2^{-m} \), where \( m \) is the level. In other words, at each successive level, the interval \( h \) is halved.

It can be seen from the results in the table that the tanh-sinh scheme achieves approximately exponential convergence, in the sense that the number of correct digits approximately doubles with each level. In some cases, such as Problems 3, 5 and 8, the convergence is slightly better than a doubling with each level; in other cases, such as Problems 12, 13 and 14, the rate is slightly less. It is remarkable that Problems 1, 6, 7, 9 and 10 exhibit virtually identical convergence behavior.

### 8.2 Non-optimality of the limiting trapezoidal rule

Finally, we examine the optimality of the coefficient of the infinite trapezoidal rule \( \hat{T}_h \) with \( \psi(x) \) as defined in (3). Let \( A := \{ \ldots , a_{-1}, a_0, a_1 \ldots \} \) be the sequence of coefficients for \( \hat{T}_h \) such that

\[
\hat{T}_{h;A} \hat{f} = \sum_{n=-\infty}^{\infty} a_n \hat{f}(nh)
\]

for every \( \hat{f} \in C^2 \), and

\[
\hat{E}_{h;A} = \hat{I} - \hat{T}_{h;A}
\]

We may ask whether \( A_h \) where \( a_n \equiv h \) is optimal in the sense that

\[
\| \hat{E}_{h;A_h} \| = \inf \| \hat{E}_{h;A} \|
\]

for all \( A \)'s for which \( \hat{T}_A \) is a bounded linear functional. As shown in [1], \( A_h \) is optimal if and only if \( \hat{T}_h \) integrates \( F_{nh} \) exactly for every integer \( n \). But from the analysis in Section 5.1, we know \( \hat{E}_h F_{nh} \neq 0 \); consequently \( A_h \) is not optimal.
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Table 1: Tanh-sinh errors at level $m$ for the 14 test problems
References


