Surprising Sinc Sums and Integrals

Robert Baillie*, D. Borwein† and Jonathan M. Borwein‡

January 5, 2007

Abstract

We show that a variety of trigonometric sums have unexpected closed forms by relating them to cognate integrals.

1 Motivation

Recall that sinc(\(x\)) := \(\sin(x)/x\) when \(x \neq 0\) and sinc(0) := 1. In [4] and [8], it was shown that, for \(N = 0, 1, 2, 3, 4, 5,\) and 6,

\[
\int_0^\infty \prod_{k=0}^{N} \text{sinc}\left(\frac{x}{2k+1}\right) \, dx = \frac{\pi}{2} \tag{1}
\]

but that for \(N = 7\), the integral is just slightly less than \(\pi/2:\)

\[
\int_0^\infty \text{sinc}(x) \text{sinc}\left(\frac{x}{3}\right) \cdots \text{sinc}\left(\frac{x}{15}\right) \, dx = \pi \left(\frac{1}{2} - \frac{6879714958723010531}{935615849440640907310521750000}\right). \tag{2}
\]

This surprising sequence is explained by Corollary 1 of Theorem 2 in [4] which we incorporate into Theorem 2 below.

It is also known (see, for example, [3] and [2]) that

\[
\int_0^\infty \text{sinc}(x) \, dx = \int_0^\infty \text{sinc}^2(x) \, dx = \frac{\pi}{2}, \quad \text{while} \quad \sum_{n=1}^\infty \text{sinc}(n) = \sum_{n=1}^\infty \text{sinc}^2(n) = \frac{\pi}{2} - \frac{1}{2}. \tag{3}
\]
Motivated by this connection between integrals and sums, the first author used Mathematica to experiment with sums that correspond to equation (1). It appeared that, for \(N = 0, 1, 2, 3, 4, 5, 6,\) and 7, the sums were also 1/2 less than the corresponding integrals:

\[
\sum_{n=1}^{\infty} \prod_{k=0}^{N} \text{sinc}\left(\frac{n}{2k+1}\right) = -\frac{1}{2} + \int_0^{\infty} \prod_{k=0}^{N} \text{sinc}\left(\frac{x}{2k+1}\right) \, dx.
\] (4)

It also seemed that, for \(N = 1, 2, 3, 4, 5,\) and 6 (but not 7),

\[
\sum_{n=1}^{\infty} \text{sinc}^N(n) = -\frac{1}{2} + \int_0^{\infty} \text{sinc}^N(x) \, dx.
\] (5)

Herein, we show that the theorems for integrals proven in [4] have analogues for sums. Our results below use basic Fourier analysis to explain the above sums, and others, and allow us to express many such sums in closed form.

## 2 When sums and integrals agree

The key to explaining these results is a form of Poisson summation [8]. Following Boas and Pollard [3] we suppose that \(G(x) = 0\) for \(x \notin (-2\pi, 2\pi),\) that \(G\) is square-Lebesgue integrable (and hence Lebesgue integrable) over the interval and is continuous on \((-\alpha, \alpha)\) for some \(\alpha \in (-2\pi, 2\pi),\) and that

\[
g(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iux} G(u) \, du.
\]

Then, by a standard result, see [11, 12], about Fourier transforms,

\[
G(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iux} g(x) \, dx \text{ for } u \in (-\alpha, \alpha),
\] (6)

provided \(g\) is Lebesgue integrable over \((-\infty, \infty).\) Hence, when \(n\) is an integer

\[
\sqrt{2\pi} \, g(n) = \int_{-2\pi}^{0} e^{-iux} G(u) \, du + \int_{0}^{2\pi} e^{-iux} G(u) \, du
\]

\[
= \int_{0}^{2\pi} e^{-iux} \left(G(u - 2\pi) + G(u)\right) \, du.
\] (7)

The numbers on the right of (7) are \(2\pi\) times the Fourier series coefficients for the interval \((0, 2\pi)\) of the function \(G(u - 2\pi) + G(u).\)

Let us suppose further that \(G(u - 2\pi) + G(u)\) satisfies, at 0, a sufficient condition for the convergence of the Fourier series to its value at 0 (for example, it is enough to have the function of bounded variation in a neighborhood of 0, its value at 0 being the average
of its right-hand and left-hand limits). Then \( \sum_{n=-\infty}^{\infty} g(n)e^{int} \) is a Fourier series which converges at \( t = 0 \) to the value of the function that generates it; that is

\[
\sum_{n=-\infty}^{\infty} g(n) = \sqrt{2\pi} \left( G(-2\pi) + G(0) \right) = \sqrt{2\pi} G(0).
\]

Replacing \( G(0) \) by its value from (6), we get

\[
\sum_{n=-\infty}^{\infty} g(n) = \int_{-\infty}^{\infty} g(x) \, dx.
\]  

(8)

Applying this for \( a_k > 0 \) for \( k = 0, 1, 2, \ldots \) to

\[
g(x) := \prod_{k=0}^{N} \text{sinc}(a_k x),
\]

we obtain from (8)

\[
1 + 2 \sum_{n=1}^{\infty} \prod_{k=0}^{N} \text{sinc}(a_k n) = 2 \int_{0}^{\infty} \prod_{k=0}^{N} \text{sinc}(a_k x) \, dx,
\]  

(9)

provided

\[
A_N := \sum_{k=0}^{N} a_k < 2\pi \text{ with } N \geq 1.
\]  

(10)

The proviso is needed since (as shown for example in [4] and also in [9] p. 20 Entry 5.2) the Fourier transform \( G \) in this case is positive and continuous in the interval \((-A_N, A_N)\) and 0 outside it. That (9) can fail when (10) doesn’t hold is evidenced by taking \( a_0 = 2\pi \), in which case the left-hand side of (9) is equal to 1, but the right-hand side is usually not.

We emphasize, that the case \( N = 0 \) doesn’t follow from the above analysis since \( \text{sinc}(x) \) is not absolutely integrable over \((-\infty, \infty)\). It can be recaptured from \( N = 1 \) by a limit argument. Alternatively, among many known methods [8], we can use the well-known fact that

\[
\sum_{n=1}^{\infty} \frac{\sin(nt)}{n} = \frac{1}{2}(\pi - t) \text{ when } 0 < t < 2\pi.
\]

This equation with \( t = a_0 \) together with the familiar identity

\[
\int_{0}^{\infty} \frac{\sin(a_0 x)}{x} \, dx = \frac{\pi}{2}
\]
yields the case \( N = 0 \) of (9).

As is made clear in [3], this “sum=integral” paradigm is very general. However, as a perusal of [9] shows, there are not too many “natural” \( g \) for which \( G \) is as required.
Some nice examples are made in [3]; often they require massaging. For example, with $G(t) := (1 + e^{it})^\alpha$ for $|t| \leq \pi$ and zero otherwise, and with $\alpha > -1$, they obtain a result first established by Shisha and Pollard in [10]:

$$\sum_{n=-\infty}^{\infty} \left( \frac{\alpha}{n} \right) e^{int} = \int_{-\infty}^{\infty} \left( \frac{\alpha}{u} \right) e^{itu} \, du = (1 + e^{it})^\alpha$$

for $\alpha > -1, |t| < \pi$.

Additionally, however, in the case of sinc integrals, as in [4, Remark 1] and [8], the right-hand term in (9) is equal to $2^{-N}V_N\pi/a_0$ where $V_N$ is the – necessarily rational – volume of the part of the cube $[-1,1]^N$ between the parallel hyperplanes

$$a_1x_1 + a_2x_2 + \cdots + a_Nx_N = -a_0 \text{ and } a_1x_1 + a_2x_2 + \cdots + a_Nx_N = a_0.$$

**Theorem 1 (Sinc Sums)** One has

$$\frac{1}{2} + \sum_{n=1}^{\infty} \prod_{k=0}^{N} \text{sinc}(akn) = \int_{0}^{\infty} \prod_{k=0}^{N} \text{sinc}(akx) \, dx = \frac{\pi}{2a_0} \frac{V_N}{2^N} \leq \frac{\pi}{2a_0} \quad (11)$$

provided

$$A_N := \sum_{k=0}^{N} a_k < 2\pi. \quad (12)$$

Moreover (11) holds with equality provided additionally that

$$A_N < 2a_0. \quad (13)$$

Various extensions are possible when (12) or (13) fail. The following corollary follows immediately from (11) on making the substitution $x = \tau t$ in the integral.

**Corollary 1** Let $\tau$ be any positive number such that $0 < \tau A_N < 2\pi$. Then

$$\frac{\tau}{2} + \tau \sum_{n=1}^{\infty} \prod_{k=0}^{N} \text{sinc}(\tau akn) = \int_{0}^{\infty} \prod_{k=0}^{N} \text{sinc}(akx) \, dx = \frac{\pi}{2a_0} \frac{V_N}{2^N} \leq \frac{\pi}{2a_0}, \quad (14)$$

is independent of $\tau$ in the given interval.

When (13) fails but holds for $A_{N-1}$, as proven in [4, Cor. 1] we may specify the volume change:

**Theorem 2 (First Bite)** Suppose that $2a_k \geq a_N$ for $k = 0, 1, \ldots, N - 1$ and that $A_{N-1} \leq 2a_0 < A_N$, and $0 < \tau A_N < 2\pi$. Then

$$\frac{\tau}{2} + \tau \sum_{n=1}^{\infty} \prod_{k=0}^{r} \text{sinc}(\tau akn) = \int_{0}^{\infty} \prod_{k=0}^{r} \text{sinc}(akx) \, dx = \frac{\pi}{2a_0} \quad \text{for } r = 0, 1, \ldots, N - 1, \quad (15)$$

while

$$\frac{\tau}{2} + \tau \sum_{n=1}^{\infty} \prod_{k=0}^{N} \text{sinc}(\tau akn) = \int_{0}^{\infty} \prod_{k=0}^{N} \text{sinc}(akx) \, dx = \frac{\pi}{2a_0} \left( 1 - \frac{(A_N - 2a_0)^N}{2^{N-1}N! \prod_{k=1}^{N} a_k} \right). \quad (16)$$
3 Examples and extensions

We may now explain the original discoveries:

Example 1 Let $N$ be an integer and for $k = 0, 1, \ldots, N$, let $a_k := 1/(2k + 1)$. If $N$ is in the range $1 \leq N \leq 6$, then

$$A_N := \sum_{k=0}^{N} a_k < 2a_0 \quad \text{and} \quad A_N < 2\pi.$$

Hence, for each of these $N$, conditions (12) and (13) of Theorem 1 hold and so we can apply that theorem to get

$$\sum_{n=1}^{\infty} \prod_{k=0}^{N} \text{sinc} \left( \frac{n}{2k+1} \right) = -\frac{1}{2} + \int_{0}^{\infty} \prod_{k=0}^{N} \text{sinc} \left( \frac{x}{2k+1} \right) \, dx = \frac{\pi}{2} - \frac{1}{2}. $$

Now for $N = 7$, condition (13) fails because

$$A_N = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{13} + \frac{1}{15} > 2a_0 = 2 > A_{N-1} = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{13}. $$

However, the conditions of Theorem 2 are met, namely

$$A_{N-1} = \frac{88069}{45045} < 2a_0 < A_N = \frac{91072}{45045} < 2\pi,$$

and for each $k = 0, 1, \ldots, N - 1$, we have $2a_k > a_N$. Therefore, we can take $\tau = 1$ and apply equation (16) of Theorem 2 to get

$$\sum_{n=1}^{\infty} \prod_{k=0}^{7} \text{sinc} \left( \frac{n}{2k+1} \right) = -\frac{1}{2} + \int_{0}^{\infty} \prod_{k=0}^{7} \text{sinc} \left( \frac{x}{2k+1} \right) \, dx

= -\frac{1}{2} + \frac{\pi}{2} \left( 1 - \left( \frac{91072}{2^67!}\frac{2}{3} \cdot \frac{1}{5} \cdot \frac{1}{7} \cdot \frac{1}{9} \cdot \frac{1}{11} \cdot \frac{1}{13} \right) \right) = \frac{\pi}{2} \left( 1 - \frac{6879714958723010531}{467807924720320453655260875000} \right) - \frac{1}{2}. $$

QED

Example 2 Let $a_0 := 1$ and—to inject a little number theory—let $a_1, a_2, \ldots, a_9$ be the reciprocals of the odd primes $3, 5, 7, 11, \ldots, 29$. Then the conditions of Theorem 2 are satisfied, so

$$\frac{1}{2} + \sum_{n=1}^{\infty} \text{sinc}(n) \text{sinc}(n/3) \text{sinc}(n/5) \cdots \text{sinc}(n/23) \text{sinc}(n/29)

= \int_{0}^{\infty} \text{sinc}(x) \text{sinc}(x/3) \text{sinc}(x/5) \cdots \text{sinc}(x/23) \text{sinc}(x/29) \, dx = \frac{\pi}{2} \left( 1 - \frac{6879714958723010531}{467807924720320453655260875000} \right) - \frac{1}{2}. $$
\[
\pi \left( \frac{1}{2} - \frac{395973516133305543036508251110417065491501430638069299939389760351849}{435103498335938196749586813353939508898811560744584793771403209434374849817500} \right) \\
\sim 0.499999990899 \pi.
\]

**Example 3** Let \(1 \leq N \leq 6\), and take all \(a_0 = a_1 = \cdots = a_{N-1} = 1\). Then condition (12) with \(N\) replaced by \(N-1\) is satisfied, so equation (11) of Theorem 1 tells us that for each \(N = 1, 2, 3, 4, 5\) and 6, we have
\[
\sum_{n=1}^{\infty} \text{sinc}^N(n) = -\frac{1}{2} + \int_0^{\infty} \text{sinc}^N(x) \, dx
\]
Moreover, (see [4, Remark 1] and [8]) for each \(N \geq 1\) the integral is a computable rational multiple of \(\pi\). If \(N = 7\), then \(A_{N-1} = 7 > 2\pi\), so (12) with \(N\) replaced by \(N-1\) is no longer satisfied and, in this case, as Example 4 shows, the sum and the integral do not differ by 1/2. Indeed, for \(N \geq 7\), the sums have an entirely different quality: they are polynomials in \(\pi\) of degree \(N\).

\(\text{QED}\)

We continue this discussion in the next counter-example for which we define, for \(N = 1, 2, \ldots\),
\[
i_N := \int_0^{\infty} \text{sinc}^N(x) \, dx, \quad s_N := \sum_{n=1}^{\infty} \text{sinc}^N(n).
\]

**Example 4** (a) Formulae for general \(i_N\) are known (see [4]), in particular we have \(i_7 = (5887/23040)\pi\). This is no surprise. However, Mathematica gives as \(s_7\) a linear combination of eight poly-logarithms. The ‘FullSimplify’ command takes a while and finally gives the next result:
\[
s_7 = -\frac{1}{2} + \frac{43141}{15360} \pi - \frac{16807}{3840} \pi^2 + \frac{2401}{768} \pi^3 - \frac{343}{288} \pi^4 + \frac{49}{192} \pi^5 - \frac{7}{240} \pi^6 + \frac{1}{720} \pi^7. \quad (17)
\]
Although (12) fails, we can explain this sum anyway. Note that
\[
s_7 - i_7 = -\frac{1}{2} + \frac{117649}{46080} \pi - \frac{16807}{3840} \pi^2 + \frac{2401}{768} \pi^3 - \frac{343}{288} \pi^4 + \frac{49}{192} \pi^5 - \frac{7}{240} \pi^6 + \frac{1}{720} \pi^7. \quad (18)
\]
Mathematica also gives:
\[
s_8 - i_8 = -\frac{1}{2} + \frac{1024}{315} \pi - \frac{256}{45} \pi^2 + \frac{64}{15} \pi^3 - \frac{16}{9} \pi^4 + \frac{4}{9} \pi^5 - \frac{1}{15} \pi^6 + \frac{1}{180} \pi^7 - \frac{1}{5040} \pi^8. \quad (19)
\]
(b) For \(N \leq 6\), \(s_N\) is 1/2 less than a rational multiple of \(\pi\). The sudden change to a polynomial in \(\pi\) of degree \(N\) is explained by the use of trigonometric identities and known Bernoulli polynomial evaluations of Fourier series. In general, we have the following two identities whose proof we leave to the reader:
\[
\sin^{2N+1}(n) = \frac{1}{2^{2N}} \sum_{k=1}^{N+1} (-1)^{k+1} \binom{2N+1}{N-k+1} \sin((2k-1)n) \quad (20)
\]
and
\[ \sin^{2N}(n) = \frac{1}{2^{2N-1}} \left( \frac{1}{2} \binom{2N}{N} + \sum_{k=1}^{N} (-1)^k \binom{2N}{N-k} \cos(2kn) \right). \quad (21) \]

In particular, to compute \( s_7 \), we start with
\[ \sin^7(n) = \frac{35}{64} \sin(n) - \frac{21}{64} \sin(3n) + \frac{7}{64} \sin(5n) - \frac{1}{64} \sin(7n). \quad (22) \]

Now, for \( 0 \leq x \leq 2\pi \),
\[ \sum_{n=1}^{\infty} \frac{\sin(n x)}{n^{2N+1}} = \frac{(-1)^{N-1}}{2} (2\pi)^{2N+1} \phi_{2N+1} \left( \frac{x}{2\pi} \right), \quad (23) \]
and
\[ \sum_{n=1}^{\infty} \frac{\cos(n x)}{n^{2N}} = \frac{(-1)^{N-1}}{2} (2\pi)^{2N} \phi_{2N} \left( \frac{x}{2\pi} \right), \quad (24) \]

where \( \phi_N(x) \) is the \( N \)-th Bernoulli polynomial, normalized so that the high order coefficient is \( 1/N! \), see [11, p. 430]. We divide (22) by \( n^7 \) and sum over \( n \). Then, we would like to use (23) four times with \( N = 3 \) and \( x = 1, 3, 5, 7 \). But there is a hitch: (23) is not valid for \( x = 7 \) because \( x > 2\pi \). So instead of 7 we use \( 7 - 2\pi \). It is this value, \( 7 - 2\pi \), substituted into the Bernoulli polynomial, that causes \( s_7 \) to be a 7th degree polynomial in \( \pi \). For \( s_{13} \), for example, we would have to use \( x = 1, 3, 5, 7 - 2\pi, 9 - 2\pi, 11 - 2\pi, \) and \( 13 - 4\pi \). For \( N \geq 7 \), we would end up with an \( N \)th degree polynomial in \( \pi \).

(c) With more effort this process yields a closed form for each such sum. First, for \( N = 7 \) we have observed that
\[ -64 \sin^7(n) = \sin(7n) - 7 \sin(5n) + 2 \sin(3n) - 35 \sin(n), \quad (25) \]
and that
\[ \sum_{n=1}^{\infty} \frac{\sin(n x)}{n^7} = 64\pi^7 \phi_7 \left( \frac{x}{2\pi} \right) \quad \text{for} \ 0 \leq x \leq 2\pi, \quad (26) \]
where
\[ \phi_7(x) := \frac{1}{30240} x - \frac{1}{4320} x^3 + \frac{1}{1440} x^5 - \frac{1}{5040} x^7 - \frac{1}{1440} x^9 \quad (27) \]
is the Bernoulli polynomial of order seven. Note that in (25), 7 is the only coefficient that falls outside the interval \( (0, 2\pi) \). Thence, applying (26) to (25) with 7 replaced by \( 7 - 2\pi \), yields both (17) and (18). The same procedure, with versions of (26) and (25) using cosines in place of sines, yields (19). An interesting additional computation shows that
\[ s_7 + \frac{1}{2} - i7 = 64\pi^7 \left\{ \phi_7 \left( \frac{7 - 2\pi}{2\pi} \right) - \phi_7 \left( \frac{7}{2\pi} \right) \right\}. \quad (28) \]
In other words the difference between \( s_7 + 1/2 \) and \( i_7 \) resides in the one term in (25) with coefficient outside the interval \((0, 2\pi)\).

(d) Let use the fractional part

\[ \{z\}_{2\pi} := \frac{z}{2\pi} - \left\lfloor \frac{z}{2\pi} \right\rfloor. \]

Combining (20), (21), (23) and (24), we ultimately obtain pretty closed-forms for each \( s_M \).

For \( M \) odd:

\[
 s_M = \frac{(-1)^{M+1}}{M!} \pi^M \sum_{k=1}^{M+1} (-1)^{k+1} \left( \frac{M}{M-1} - k + 1 \right) \phi_M \left( \{2k-1\}_{2\pi} \right). 
\]

For \( M \) even:

\[
 s_M = \frac{(-1)^{M/2}}{M!} \pi^M \sum_{k=0}^{M} (-1)^{k+1} \left( \frac{M}{M} - k \right) \delta_{k,0} + 1 \phi_M \left( \{2k\}_{2\pi} \right),
\]

where, as usual, \( \delta_{k,0} = 1 \) when \( k = 0 \), and 0 otherwise. Remarkably, these formulae are rational multiples of \( \pi \) exactly for \( M \leq 6 \) and thereafter are of degree \( M \). QED.

Many variations on the previous themes are possible. For example, one may insert powers of cosine as in [4, Thm. 3], although it does not seem possible to extend Theorem 1 to this case. In simple cases it is, however, easy to proceed as follows:

**Example 5** Let us denote by

\[
i_{i,j} := \sum_{n=1}^{\infty} \text{sinc}(n)^i \cos(n)^j.
\]

We discovered experimentally that \( s_{1,1} = s_{1,2} = s_{2,1} = s_{3,1} = s_{2,2} = \pi/4 - 1/2 \) and that in each case the corresponding integral equals \( \pi/4 \). Likewise \( s_{1,3} = s_{2,3} = s_{3,3} = 3\pi/16 - 1/2 \) while the corresponding integrals are equal to \( 3\pi/16 \). Except for \( s_{1,2}, s_{2,2} \) and \( s_{2,3} \), the identity \( \text{sinc}(n) \cos(n) = \text{sinc}(2n) \) allows us to apply Theorem 1. In the remaining three cases, we may use the method of Example 4 to prove the discovered results, but a good explanation has eluded us. QED

4 An Extremal property

We finish with a useful Siegel-type lower bound, [7, Exercise 8.4], giving an extremal property of the sinc\(^k\) integrals. This has applications to giving an upper bound on the size of integral solutions to integer linear equations. In [1] it was intimated the proof was easy; it appears not to be so.
Theorem 3 (Lower Bound) Suppose $a_0 \geq a_k > 0$ for $k = 1, 2, \ldots, n$. Then
\[
\int_0^\infty \prod_{k=0}^n \operatorname{sinc}(a_k x) \, dx \geq \int_0^\infty \operatorname{sinc}^{n+1}(a_0 x) \, dx.
\] (29)

In view of Corollary 1 we then have the following:

Corollary 2 Suppose $a_0 \geq a_k > 0$ for $k = 1, 2, \ldots, n$ and $0 < \tau A_n < 2\pi$. Then
\[
\frac{\tau}{2} + \tau \sum_{r=1}^\infty \prod_{k=0}^n \operatorname{sinc}(\tau a_k r) = \int_0^\infty \prod_{k=0}^n \operatorname{sinc}(a_k x) \, dx \geq \int_0^\infty \operatorname{sinc}^{n+1}(a_0 x) \, dx.
\] (30)

Proof of Theorem 3. Let
\[
\tau_n := \int_0^\infty \prod_{k=0}^n \operatorname{sinc}(a_k x) \, dx, \quad \mu_n := \int_0^\infty \operatorname{sinc}^{n+1}(a_0 x) \, dx,
\]
and, for $a > 0$, let
\[
\chi_a(x) := 1 \text{ if } |x| < a; \quad \frac{1}{2} \text{ if } |x| = a; \quad 0 \text{ if } |x| > a.
\]

Further, let
\[
F_0 := \frac{1}{a_0} \sqrt{\frac{\pi}{2}} \chi_{a_0}, \quad F_n := (\sqrt{2\pi})^{1-n} f_1 * f_2 * \cdots * f_n, \quad \text{where } f_n := \frac{1}{a_n} \sqrt{\frac{\pi}{2}} \chi_{a_n},
\]

and * indicates convolution, i.e.,
\[
f_j * f_k(x) := \int_{-\infty}^\infty f_j(x-t)f_k(t) \, dt.
\]

Then (see [4], and [9] p. 20 Entry 5.2) $F_0$ is the Fourier transform of $\operatorname{sinc}(a_0 x)$ and, for $n \geq 1$, $F_n$ is the Fourier transform of $\prod_{k=1}^n \operatorname{sinc}(a_k x)$. In addition, for $n \geq 1$, $F_n(x)$ is an even function which vanishes on $(-\infty, -\sigma_n) \cup (\sigma_n, \infty)$ and is positive on $(-\sigma_n, \sigma_n)$, where $\sigma_n := A_n - a_0 = a_1 + a_2 + \cdots + a_n$. Furthermore, for $n \geq 1$, $F_n(x)$ is monotone non-increasing on $(0, \infty)$. Hence, by a version of Parseval’s theorem (see [4]),
\[
\tau_n = \int_0^\infty F_n(x) F_0(x) \, dx = \frac{1}{a_0} \sqrt{\frac{\pi}{2}} \int_0^{\min(\sigma_n, a_0)} \sqrt{2\pi} F_n(x) \, dx \quad \text{for } n \geq 1.
\] (31)

Observe that, for $n \geq 2$,
\[
F_n = \frac{1}{\sqrt{2\pi}} F_{n-1} * f_n,
\]

and hence that, for $y > 0$,
\[
\int_0^y F_n(v) \, dv = \frac{1}{\sqrt{2\pi}} \int_0^y dv \int_{-\infty}^\infty F_{n-1}(v-t) f_n(t) \, dt = \frac{1}{2a_n} \int_0^y dv \int_{-a_n}^{a_n} F_{n-1}(v-t) \, dt = \frac{1}{2a_n} \int_{-a_n}^{a_n} dt \int_0^y F_{n-1}(v-t) \, dv = \frac{1}{2a_n} \int_{-a_n}^{a_n} dt \int_{-t}^{y-t} F_{n-1}(u) \, du.
\]
Thus, we determine that
\[ \int_0^y F_n(v) \, dv = \int_0^y F_{n-1}(u) \, du + I_1(a_n) + I_2(a_n), \]  
(32)

where, for \( x > 0 \),
\[ I_1(x) := \frac{1}{2x} \int_{-x}^{x} dt \int_{t}^{0} F_{n-1}(u) \, du \quad \text{and} \quad I_2(x) := \frac{1}{2x} \int_{-x}^{x} dt \int_{y-t}^{y} F_{n-1}(u) \, du. \]

Now \( I_1(x) = 0 \) since \( \int_{-x}^{0} F_{n-1}(u) \, du \) is an odd function of \( t \), and for \( y \geq x \),
\[ I_2(x) = \frac{1}{2x} \int_{0}^{x} dt \int_{y-t}^{y} F_{n-1}(u) \, du + \frac{1}{2x} \int_{-x}^{0} dt \int_{y}^{y-t} F_{n-1}(u) \, du = \frac{1}{2x} \int_{0}^{x} \phi(t) \, dt, \]  
(33)

where
\[ \phi(t) := \int_{y}^{y+t} F_{n-1}(u) \, du - \int_{y-t}^{y} F_{n-1}(u) \, du \leq 0 \quad \text{for} \quad 0 \leq t \leq y \]  
(34)

since \( F_{n-1}(u) \) is monotonic non-increasing for \( u \geq 0 \). Observe that \( \phi'(t) = F_{n-1}(y + t) - F_{n-1}(y - t) \leq 0 \) for \( 0 \leq t \leq y \), apart from at most two exceptional values of \( t \) when \( n = 2 \). Hence
\[ I_2'(x) = \frac{1}{x^2} \int_{0}^{x} (\phi(x) - \phi(t)) \, dt = \frac{1}{x^2} \int_{0}^{x} dt \int_{t}^{x} \phi'(u) \, du \leq 0, \]

and so
\[ I_2(x) \text{ is non-increasing for } 0 \leq x \leq y. \]  
(35)

Our aim is to prove that \( \tau_n \geq \mu_n \). Since, by Theorem 1, this inequality automatically holds when \( a_0 \geq \sigma_n \), we assume that \( a_0 < \sigma_n \). Note that in case \( n = 1 \) the hypothesis \( a_0 \geq \sigma_1 = \sigma \) immediately implies the desired inequality. Assume therefore that \( n \geq 2 \) in the rest of the proof. Suppose \( a_0, a_1, \ldots, a_n \) are not all equal, and re-index them so that \( a_0 \) remains fixed and \( a_n < a_{n-1} \leq a_0 \). If \( a_n \) is increased to \( a_{n-1} \), it follows from (35) with \( x = a_n \) and \( y = a_0 \) that \( I_2(a_n) \) is not increased and hence, by (31), and (32) with \( y = a_0 \), that \( \tau_n \) is not increased. Continuing in this way, we can coalesce all the \( a_k \)'s into the common value \( a_0 \) without increasing the value of \( \tau_n \). This final value of \( \tau_n \) is, of course, \( \mu_n \), and so the original \( \tau_n \geq \mu_n \), as desired.

Perhaps a somewhat analogous version of Theorem 3 holds for sums?

References


