Maximality of Monotone Operators in General Banach Space

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Abstract

We establish maximality of the sum of two maximal monotone operators in general Banach space, assuming only the Rockafellar qualification assumption.

1 Introduction

The theory of maximal monotone operators in reflexive space is rather well understood and complete, as illustrated by Rockafellar’s foundational work in [18]. A good indication of our understanding fifteen-years ago can be gathered from Phelps’ exposition in [16] and [17]. Outside of reflexive space our knowledge is more fragmentary. This reflects on the difficulty of the subject (see its early roots in [10, 11]) and more especially on the paucity of tools available until recently—in reflexive space key use was made of the surjectivity of the duality map. The situation was ameliorated significantly by Simons’ monograph [22] which made more central the role of the Hahn-Banach theorem, and by the rediscovery of Fitzpatrick’s 1988 paper [9] beautifully exploited in Penot’s work [14] and that of Simons [22] and others.

As described in more detail in [4], this has allowed for a reduction of much monotone operator theory to convex analysis, culminating in very clean and elementary proofs of the maximality of the sum in reflexive space [4, 7, 25] under the weakest-known constraint qualification. It has also allowed for a flowering of results in non-reflexive space, most recently in conditions for the sum to be maximal using an approach pioneered by Voisei [27, 28, 5].

In this paper we build upon this foundation to supply various results establishing, in Theorems 39, that in any Banach space the sum of two maximal

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monotone operators $T$ and $M$ is maximal under Rockafellar’s original condition:

$$\emptyset \neq \text{dom } M \cap \text{int } (\text{dom } T). \quad \text{(1)}$$

A particularly pleasant consequence of this theorem is given in Corollary 42, namely the domain of every convex operator is semi-convex, that is it has a convex closure.

2 Notations and Preliminaries

Suppose $X$ is a Banach space and $X^*$ is its dual. We may view $X \times X^*$ paired with $X^* \times X$ using the coupling $\langle (y, y^*), (x^*, x) \rangle = \langle y, x^* \rangle + \langle x, y^* \rangle$ and the norm $\| (x, x^*) \|^2 = \| x \|^2 + \| x^* \|^2$. At times we will assume $X \times X^*$ is endowed with the product topology $s-bw^*(X, X^*)$ formed from the strong topology on $X$ and the bounded weak* topology on $X^*$ (see [13] page 150–154). We will pair this space with $X^* \times X$ which is endowed with product topology $bw^*-s(X^*, X)$ when $X^*$ is endowed with the bounded weak* topology and $X$ the strong topology. This is a valid pairing due to the fact that when $X$ is Banach the bounded weak* continuous linear functionals on $X^*$ are canonically isomorphic to $X$. At other times it will be advantageous to consider that $X^{**} \times X^*$ is the dual of $X^* \times X$.

The indicator function of a set $T \subseteq X \times X^*$ is denoted by $\delta_T$ and the Fenchel conjugate of a convex function $\mathcal{F} : X \times X^* \to \mathbb{R}$ is denoted by

$$\mathcal{F}^*(x^*, \alpha) := \sup_{(w, w^*) \in X \times X^*} \{ \langle w, x^* \rangle + \langle x^*, w^* \rangle - \mathcal{F}(w, w^*) \}.$$ 

When using the pairing alluded to above then we may consider $\mathcal{F}^*(x^*, x)$ := $\mathcal{F}^*(x^*, J_X(x))$ (tolerating the abuse of notation) which corresponds to the conjugate with respect to the space $X \times X^*$ paired with $X^* \times X$. The epigraph of $\mathcal{F}$ is the set $\text{epi} \mathcal{F} := \{(x, x^*, \alpha) \in X \times X^* \times \mathbb{R} | \alpha \geq \mathcal{F}(x, x^*) \}$. The support function of a set $A \subseteq X \times X^*$ is given by $\delta_A^* (x^*, x)$. When we pair the spaces $X \times X^*$ with $X^* \times X$, the second conjugate $\mathcal{F}^{**}(x, x^*) = \mathcal{F}(x, x^*)$ whenever $\mathcal{F}$ is a jointly strong-bounded weak* continuous, proper convex function. Alternatively, one can view $\mathcal{F}^{**}(x, x^*) := \mathcal{F}^{**}(J_X(x), x^*)$ being the restriction of $\mathcal{F}^{**} : X^{**} \times X^* \to \mathbb{R}$ to $J_X(X) \subseteq X^{**}$.

To simplify notation for a multi-function $T : X \rightrightarrows Y$ we denote its graph by $T := \{(x, y) \in X \times Y | y \in T(x) \}$. We say $T$ is a monotone set if it has the property that

$$\forall (x, x^*) \in T \quad \forall (y, y^*) \in T \quad \langle x - y, x^* - y^* \rangle \geq 0. \quad \text{(2)}$$

If $T$ does not possess a proper monotone extension then $T$ is said to be maximal monotone. We say $(y, y^*)$ is monotonically related to $T$ when $\forall (x, x^*) \in T$ we have $\langle x - y, x^* - y^* \rangle \geq 0$. When $T$ is maximal then $(y, y^*) \notin T$ implies the existence of $(x, x^*) \in T$ such that $\langle x - y, x^* - y^* \rangle < 0$. 

2
2.1 Representative Functions

Definition 1 The Fitzpatrick function associated with an operator $T : X \rightrightarrows X^*$ is defined by

$$F_T (y, y^*):= \sup_{(x, x^*) \in T} \{ \langle y, x^* \rangle + \langle x, y^* \rangle - \langle x, x^* \rangle \}$$

$$= \left\{ \langle y, y^* \rangle - \inf_{(x, x^*) \in T} \langle y - x, y^* - x^* \rangle \right\}$$

As $(y, y^*) \mapsto \langle y, x^* \rangle + \langle x, y^* \rangle - \langle x, x^* \rangle$ is continuous with respect to the strong-weak product topology we find that $F_T (y, y^*)$ is a jointly strong-weak lower semi–continuous proper convex function whenever $T$ is monotone. Alternatively one may observation that $(y, y^*) \mapsto \langle y, x^* \rangle + \langle x, y^* \rangle - \langle x, x^* \rangle$ is continuous with respect to the strong topology on $X \times X^*$ and state that $F_T (y, y^*)$ is a jointly strongly lower semi–continuous proper convex function on $X \times X^*$. As the strong and weak closures of a convex body coincide then we also may state that epi $F_T$ is a weakly closed, convex subset of $X \times X^*$. From this definition it is easily seen that for $y^* \in T (y)$ we have

$$F_T (y, y^*) = \langle y, y^* \rangle < +\infty,$$

(3)

when $T$ is monotone. Next, $F_T (y, y^*) \geq \langle y, y^* \rangle$ holds for all $(y, y^*)$ with equality when $y^* \in T (y)$ if and only if $T$ is maximal monotone, i.e.,

$$T = \{ (y, y^*) \mid F_T (y, y^*) = \langle y, y^* \rangle \}.$$

Define the ‘transpose’ operator $\dagger: \ (x^*, x) \rightarrow (x, x^*)$ and $c_T (\cdot, \cdot) := \delta_T (\cdot, \cdot) + \langle \cdot, \cdot \rangle$, with conjugate

$$c_T^* (y, y^*) := \sup_{(w, w^*) \in X \times X^*} \{ \langle y, w^* \rangle + \langle w, y^* \rangle - \langle w, w^* \rangle \}$$

$$= \sup_{(x, x^*) \in T} \{ \langle y, w^* \rangle + \langle w, y^* \rangle - \langle w, w^* \rangle \} = F_A (y, y^*).$$

The second conjugate of $c_T$ is of interest in that it leads to the Penot representative function:

$$P_T (y, y^*) := F_T^* (y, y^*) = (c_T)^{**} (y, y^*) = p_T (y, y^*)$$

where

$$p_T (y, y^*) = \inf \left\{ \sum_i \lambda_i \langle x_i, x_i^* \rangle \mid \sum_i \lambda_i \langle x_i, x_i^* \rangle = \langle y, y^*, 1 \rangle, \right\}$$

(4)

with the closure taken with respect to the topology on $X \times X^*$ consistent with the pairing. Clearly we have $P_T (y, y^*) = +\infty$ if $(y, y^*) \notin \overline{\text{co} T}$ and co $T \subseteq \text{dom} P_T$.

We recall that if core $\text{dom} T \neq \emptyset$ and $T$ is maximal then both $\overline{\text{dom} T}$ and int $\text{dom} T$ are convex [4]. In a reflexive space maximality is sufficient to ensure $\overline{\text{dom} T}$ is convex, a property referred to as semi-convexity. Recall that a
representative function of a monotone mapping $T$ on $X$ is a convex function $\mathcal{H}_T$ on $X \times X^*$ such that $\mathcal{H}_T(y, y^*) \geq \langle y, y^* \rangle$ for all $(y, y^*) \in X \times X^*$ with $\mathcal{H}_T(y, y^*) = \langle y, y^* \rangle$ when $y^* \in T(y)$. When $T$ is not specified we say a s-lbm* $(X, X^*)$ closed, proper convex function $f$ is representative when $f(y, y^*) \geq \langle y, y^* \rangle$ for all $(y, y^*) \in X \times X^*$. It has been noted by a number of authors, see [5], that when $T$ is maximal the largest representative function is $P_T$ and the smallest is $\mathcal{F}_T$ (and so $P_T \geq \mathcal{F}_T$ pointwise). Denote the restriction of a subset $U \subseteq X \times X^*$ to the space $X^* \times X$ by

$$R_{X^* \times X} (U) := U \cap (X^* \times X).$$

For a subset $V \subseteq X^* \times X$ denoted by $J_{X^* \times X}(V)$ the imbedding of $V$ into $X^* \times X^{**}$. Occasionally we will use the shorthand notation $J_X(X) = \bar{X}$ along with $J_X(x) = \bar{x}$ to denote the embedding into $X^{**}$. Fitzpatrick [9] showed that

$$P_T = R_{X^* \times X}(\text{Graph } \mathcal{F}^T_\beta) \uparrow \text{ and } \mathcal{F}_T = [R_{X^* \times X}(\text{Graph } c_T^\beta)] \uparrow$$

are representative functions when $T$ is maximal monotone, and in [4] it is shown that $P_T$ is representative when $T$ is just monotone.

**Lemma 2** ([4]) For any monotone mapping $T$ the function $P_T : X \times X^* \rightarrow \mathbb{R}$ is a representative convex function for $T$.

Because $f \leq \langle \cdot, \cdot \rangle + \delta_T = c_M$ we have $c_T^\beta(x^{**}, x^*) \leq f^*$. Thus, when $T$ is maximal

$$\langle x, x^* \rangle \leq \mathcal{F}_T(x, x^*) \leq \hat{f}^*(x^*, \hat{x}) = f^*(x^*, x).$$

That is, when $f$ is a representative function then the embedding of $\hat{f}^*(x^*, x^{**})$ into $X^* \times X$ is also a representative function. Hence

$$\begin{align*}
T : X \rightarrow X^* \text{ maximal} \\
f \text{ representative} \\
\Rightarrow \ \ f(x, x^*) \geq (x, x^*), \ \ \forall (x, x^*) \in X \times X^* \\
f^*(x^*, x) \geq \langle x, x^* \rangle, \ \ \forall (x, x^*) \in X \times X^* \\
\end{align*}$$

(5)

In [15, Prop. 1] and [3] the converse to this result was proved in a reflexive space. That is, when the inequalities on the right of (5) hold then $T := \{(x, x^*) \mid f(x, x^*) = (x, x^*)\}$ is a maximal monotone set. This provides a complete and simple characterisation of maximality in the context of reflexive spaces. This fact will be used a number of times in this paper. In passing we note that when $\mathcal{F}_T$ is a representative function then as $P_T$ is always representative then the above considerations apply. One should note that in this case $M := \{(x, x^*) \mid \mathcal{F}_T(x, x^*) = (x, x^*)\}$ is maximal when $X$ is reflexive but we may have $M \neq T$ (consider the case when only the closure of $T$ equals $M$). When $\mathcal{F}_T$ is a representative function we say that $T$ is almost maximal.

Another construction that arises in this paper occurs when we embed the graph of a monotone mapping $T \subseteq X \times X^*$ into $X^{**} \times X^*$. Then it is possible to obtain a maximal monotone extension $\hat{T}$ of the embedded $J_{X \times X^*} (T) = \hat{T} \subseteq X^{**} \times X^*$. An interesting question arises as to when the restriction $R_{X \times X^*}(\hat{T})$
coincides with $T$. When dealing with a maximal monotone operator $\overline{T} \subseteq X^{**} \times X^*$ it is possible to consider both $\mathcal{P}_T$ and $\mathcal{F}_T : X^{**} \times X^* \to \mathbb{R}$ as a representative functions which would then be consider to be $bw^*-s (X^{**}, X^*)$ closed, proper convex functions.

### 2.2 Structure of the Paper and Flow of Logic

We outline the general flow of logic that results in the proof of our main theorem regarding sums of monotone operators. The reader may find it useful to consult this brief outline as a road-map to the role and use of various results. There are a number of technical results required before a proof of the main sum theorem is possible.

In section 4 we make some critical observations regarding the maximality of certain extensions of a monotone set $T$ when it is embedded into the dual space $X^* \times X^{**}$. These results can be viewed as a generalization of the results [15, Prop. 1] to non-reflexive spaces. This allows us to prove Theorem 17 which provides sufficient conditions to ensure that the Fitzpatrick function $\mathcal{F}_T$ is a representative function.

The strategy leading to the sum Theorem 39, and consequences, is to prove that the conditions of Theorem 17 is satisfied by $T + M$ which will allow us to deduce $\mathcal{F}_{T+M}$ is a representative function. The proof is then completed as in [5] by showing that $T + M$ is maximal. In Sections 6 we work as follows:

1. To apply Theorem 17 we need to identify a representative function for $T + M$ to which the Theorem applies. Thus, we choose $h$ as in Theorem 31 and hence require (41) to hold for the choices of $f := \mathcal{F}_T$ and $g := \mathcal{F}_M$.

2. The second condition that requires validation is: $h^*(x^*, x^{**}) \geq \langle x^{**}, x^* \rangle$ for all $(x^*, x^{**}) \in X^* \times X^{**}$. This follows from Theorem 30, along with the equalities $f^* = \widehat{\mathcal{F}_T}^* \geq \langle \cdot, \cdot \rangle$ and $g^* = \widehat{\mathcal{F}_M}^* \geq \langle \cdot, \cdot \rangle$ valid on $X^* \times X^{**}$ (as observed in Corollary 12) under the assumption of the validity of qualification assumption (40).

3. The final condition needed in Theorem 17 is that there exists a family of finite-dimensional subspaces that cover $X$ such that $(T + M)_A := A^* \circ (T + M) \circ A$ is maximal for $A : Y \to X$ (the embedding of $Y$ in $X$).

4. As $Y$ is finite-dimensional, showing maximality of $(T + M)_A$ reduces to prescription of a representative function, we denote by $H$, that has the properties $H \geq \langle \cdot, \cdot \rangle$ and $H^* \geq \langle \cdot, \cdot \rangle$ on $Y \times Y^*$ (see [15, Prop. 1]). Consequently we define $H$ as in Proposition 32 and note that we have the intermediate conjugation formula as described in Proposition 32.

The inequality $H \geq \langle \cdot, \cdot \rangle$ is immediate from Theorem 30 and the fact that $h$ is a representative function for $(T + M)$, by Theorem 31. Thus, all that remains to show $H$ is a representative function of a maximal monotone operator is to show that $H^* \geq \langle \cdot, \cdot \rangle$ on $Y \times Y^*$. The idea is then to only impose an interiority condition on dom $T$ and leave dom $M$ general. Then
we apply the duality given in Proposition 32, from which we can argue as in Proposition 33 to establish the inequality $H^* \geq \langle \cdot, \cdot \rangle$ on $Y \times Y^*$ (which requires Theorem 19). This establishes that $H$ is indeed a representative function of a maximal monotone operator on $Y$, which we denote by $Q_A$.

5. Finally, we are finished once we can identify $H$ as a representative function of $(T + M)_A$. The trouble is that now one cannot assume that the infimum in the definition of $H$ is achieved. The best one can do is to argue as in Proposition 34 and so to obtain an inclusion inside $\text{int} \, \text{dom} \, T \cap \hat{Y}$.

6. This inclusion needs to be extended to the $\text{dom} \, Q_A$ where $Q_A$ is the maximal monotone set defined by $H$. Here Corollary 28 comes to the rescue allowing us to take limits and add normal cones to reconstruct $Q_A$ from information within $\text{ri} \, \text{dom} \, Q_A$. This requires us to make sure the normal cones match and that the domain closures of $Q_A$ and $(T + M)_A$ are the same. This is shown in a number of Lemmas in section 5.

All this is required because the obvious shortcut is to impose an interiority condition on both $\text{dom} \, T$ and $\text{dom} \, M$, as in [5], which would mean Rockafellar’s qualification assumption is lost.

3 Some Topological Issues

We will need to use the pairing of the spaces $s$-$bw^+ (Z, Z^*)$ paired with $bw^+$-$s (Z^* \times Z)$. When $Z = X^*$ we obtain the following pairing $s$-$bw^+ (X^*, X^{**})$ paired with $bw^+$-$s (X^{**} \times X^*)$. In [13] the bounded weak* topology is defined inductively via the closure property that $C \subseteq Z^*$ is $bw^*$-closed if and only if $C \cap U$ is weak* closed for every weak* compact $U \subseteq Z^*$. When dealing with the embedded $J_{X \times X^*} (T) = \hat{T} \subseteq X^{**} \times X^*$ both $P_{\hat{T}}$ and $F_{\hat{T}} : X^{**} \times X^* \to \mathbb{R}$ would then be consider to be $bw^*$-$s (X^{**}, X^*)$ closed, proper convex functions. We will have a need to consider the relationship between $P_{\hat{T}}$ and the the “natural” conjugate

$$(x^*, x^{**}) \mapsto \hat{F}_{\hat{T}}^* (x^*, x^{**})$$

of $F_T (x, x^*) = \sup_{(z, z^*) \in T} \left\{ \langle x, x^* \rangle - \inf_{(z, z^*) \in T} \langle x - z, x^* - z^* \rangle \right\}$.

Some care must be taken when using the bounded weak* topology. It is stronger than the weak* topology of $Z^*$ and is characterised by uniform convergence of linear functionals on compact subsets of $Z$. A net $\{x^*_\beta\} \subseteq Z^*$ which weak* convergent to $x^*$ and is bound normed is also $bw^*$-convergent to $x^*$. In general the converse is false. For convex sets we have the Krein–Smulian theorem that states that in a Banach space $Z$ a convex subset $C$ of $Z^*$ is weak*-closed if and only if it is $bw^*$-closed. That is, a convex set $C$ is $bw^*$-closed exact when it contains all the limits of all its bounded and weak* convergent nets.
This is a consequence of the observations that $Z$ is complete if and only if every $bw^*$-continuous linear function on $Z^*$ is actually weak* continuous.

Denote by $bw^* \times s$ the product topology formed by using the $bw^*$-topology on $X^{**}$ and the strong topology on $X^*$. Then a convex set $C \subseteq X^{**} \times X^*$ is $bw^* \times s$-closed exactly when it contains all limits $(x^{**}, x^*)$ of bounded nets $\{ (x^*_\beta, x_\beta^*) \}$ (i.e. $\exists M > 0$ such that $\| (x^*_\beta, x_\beta^*) \| \leq M$) with $\{ x^{**}_\beta \}$ converging weak* to $x^{**}$ and $\{ x_\beta^* \}$ strongly convergent to $x^*$. As $x^*_\beta \to x^*$ strongly, the norm boundedness of $\{ x^*_\beta \}$ is actually superfluous. The embedding of a function $f : X \times X^* \to \mathbb{R}$ into $X^{**} \times X^*$ is given by

$$\hat{f}(x^{**}, x^*) = \begin{cases} f(\hat{x}, x^*) & \text{if } x^{**} = \hat{x} \in \hat{X} \\ +\infty & \text{otherwise} \end{cases}$$

Note that $\text{epi} \hat{f} = \text{epi} \hat{f}$.

**Lemma 3** Suppose $f : X \times X^* \to \mathbb{R}$ is a proper convex function and let $\text{epi} \hat{f} = \text{epi} f$ where $\text{epi} f$ is the embedding of $\text{epi} f$ into $X^{**} \times X^* \times \mathbb{R}$. Then $f(x, x^*) \geq \langle x, x^* \rangle$ for all $(x, x^*) \in X \times X^*$ implies

$$\hat{f}(x^{**}, x^*) \geq \langle x^{**}, x^* \rangle \quad \text{for all } (x^{**}, x^*) \in X^{**} \times X^*.$$

**Proof.** First note that as $f$ is convex $f$ is convex and thus $(x^{**}, x^*, \alpha) \in \text{epi} f$ is convex exactly when there exists a net $\{ (\hat{x}_\beta, x^*_\beta, \alpha_\beta) \} \in \text{epi} f$ with $\| (\hat{x}_\beta, x^*_\beta) \| \leq M$, for some $M > 0$, and with $\{ (\hat{x}_\beta, x^*_\beta, \alpha_\beta) \} \to \text{epi} f$ $(x^{**}, x^*, \alpha)$. Thus we have

$$\hat{f}(x^{**}, x^*) = \inf \{ (\hat{x}_\beta, x^*_\beta) \to \text{epi} f \} \liminf_{\beta} f(\hat{x}_\beta, x^*_\beta) \quad (7)$$

Take $\{ (\hat{x}_\beta, x^*_\beta) \} \to \text{epi} f$ with $\| \hat{x}_\beta \| \leq M$, then

$$|\langle \hat{x}_\beta, x^*_\beta \rangle - \langle x^{**}, x^* \rangle | = |\langle \hat{x}_\beta - x^{**}, x^* \rangle + \langle x^{**}, x^*_\beta - x^* \rangle |$$

$$\leq |\langle \hat{x}_\beta - x^{**}, x^* \rangle | + M \| x^*_\beta - x^* \| \to 0$$

and as $\{ (\hat{x}_\beta, x^*_\beta, \alpha_\beta) \} \in \text{epi} f$ we have $\alpha_\beta \geq f(\hat{x}_\beta, x^*_\beta) \geq \langle \hat{x}_\beta, x^*_\beta \rangle$. On taking limits it follows that $\alpha \geq \langle x^{**}, x^* \rangle$ and so

$$\hat{f}(x^{**}, x^*) \geq \langle x^{**}, x^* \rangle.$$

**Remark 4** We cannot claim that $(x^{**}, x^*) \to (x^{**}, x^*)$ is $bw^* \times s$ continuous. Indeed placing $f(x^{**}, x^*) \equiv \langle x^{**}, x^* \rangle$ we see that neither $\text{epi} f$ nor $\text{epi} (-f)$ are convex. Indeed it can be shown that $(x^{**}, x^*) \to (x^{**}, x^*)$ is $bw^* \times s$ continuous only when $X$ is finite dimensional as noted in a private communication with...
If \( \hat{\alpha} \) is given by conjugate into the dual space (\( \hat{s} \) by definitions consistent in that the dual of \( f \) may be smaller than the \( \hat{f} \)) with respect to the weak* \( w^* \times s \)-convergent, norm bounded nets. When \( f \) is a representative function on \( X \times X^* \) of a monotone set

\[
M_f := \{(x, x^*) | f(x, x^*) = \langle x, x^* \rangle \}
\]

we can view \( \hat{f} \) as a representative function of a monotone set in \( X^{**} \times X^* \) given by

\[
M_{\hat{f}} := \{(x^{**}, x^*) | \hat{f}(x^{**}, x^*) = \langle x^{**}, x^* \rangle \}.
\]

It is clear that \( M_{\hat{f}} \) contains the set \( \overline{M_f} \), were we denote for any set \( S \subseteq X^{**} \times X^* \)

\[
\overline{S} := \{(x^{**}, x^*) | \exists K > 0 \text{ and a net } (x^{**}_\beta, x^*_\beta) \in S \text{ with } \| (x^{**}_\beta, x^*_\beta) \| \leq K \text{ s.t. } (x^{**}_\beta, x^*_\beta) \rightarrow (x^{**}, x^*) \}
\]

Note that the set \( \overline{M_f} \) may be smaller than the \( \hat{bw}^* \times s \) closure of \( M_f \) as the set \( M_f \) is a nonconvex set in \( X \times X^* \).

**Remark 5** If \( X \) is a Banach space then \( \overline{X} = X^{**} \). This follows from the well known fact that the weak* closure of \( \hat{X} \) equals \( X^{**} \). As \( \overline{X} \) is convex and thus a \( bw^* \)-closed subset of \( X^{**} \) it must be also weak* closed (by the Krein–Smulian theorem). Thus \( \overline{X} = \left( \overline{\overline{X}}^\prime \right)^\prime = \overline{X} = X^{**} \), giving equality.

Recall that for a representative function \( f \) on \( X \times X^* \) we denote by \( \hat{f}^* \) its conjugate into the dual space \( (X \times X^*)^* \) and its conjugate between the paired spaces \( s-bw^* (X \times X^*) \) and \( bw^* - s (X^{**} \times X^*) \) by \( \hat{f} \). Note that this notation is consistent in that the dual of \( f \) within \( s-bw^* (X \times X^{**}) \) and \( bw^* - s (X^{**} \times X^*) \) which is given by

\[
(\hat{f})^*(x^*, x^{**}) = \sup_{(y^*, y^{**}) \in X^* \times X^{**}} \left\{ \langle y^*, x^{**} \rangle + \langle y^{**}, x^* \rangle - \hat{f}(y^*, y^{**}) \right\} \\
= \sup_{(y^*, y) \in X^* \times \overline{X}} \left\{ \langle y^*, x^{**} \rangle + \langle y, x^* \rangle - \hat{f}(y^*, y) \right\} \\
= \sup_{(y^*, y) \in X^* \times X^*} \left\{ \langle y^*, x^{**} \rangle + \langle y, x^* \rangle - f(y^*, y) \right\} = \hat{f}^*(x^*, x^{**})
\]

and by definitions \( \hat{f}^*(x^*, x) = f^*(x^*, x) \). As the spaces are paired as \( s \times bw^*(X^* \times X^{**}) \) with \( bw^* \times s (X^{**} \times X^*) \) we have the same conjugate for \( \hat{f} \) and its \( bw^* \times s \) closure i.e.

\[
\left\{ \overline{f} \right\}^* (x^*, x^{**}) = \left\{ \hat{f} \right\}^* (x^*, x^{**}).
\]

Using (8) it follows that

\[
\left\{ \overline{f} \right\}^* (x^*, x^{**}) = \left\{ \hat{f} \right\}^* (x^*, x^{**}) = \hat{f}^* (x^*, x^{**})
\]
where the final conjugate is the "natural conjugate" of $f$ into the dual space $(X \times X^*)^*$.

We may imbed $M$ into $X^{**} \times X^*$ and denote the Fitzpatrick function of $\hat{M} = J_{X \times X^*} (M)$ by

$$F_{\hat{M}} (x^{**}, x^*) = \sup_{(\hat{y}, y^*) \in \hat{M}} \{\langle \hat{y}, x^* \rangle + \langle x^{**}, y^* \rangle - \langle \hat{y}, y^* \rangle \}.$$

Note that

$$F_{\hat{M}} (\hat{x}, x^*) = \sup_{(y, y^*) \in M} \{\langle y, x^* \rangle + \langle \hat{x}, y^* \rangle - \langle \hat{y}, y^* \rangle \} = F_M (x, x^*) = \hat{F}_M (\hat{x}, x^*)$$

and when $M$ is maximal on $X$ we have

$$F_{\hat{M}} (\hat{x}, x^*) = F_M (x, x^*) \geq \langle \hat{x}, x^* \rangle.$$

Similarly the Penot representative function of the embedded monotone mapping is given by $P_{\hat{M}}$, we have the natural conjugate given by

$$\hat{F}_M^* (x^*, \hat{x}) = (P_M)^! (x^*, \hat{x}) \geq \langle \hat{x}, x^* \rangle, \text{ for all } (x^*, \hat{x}) \in X^* \times \hat{X} \quad (9)$$

where $P_M$ denotes the Penot representative function on $X \times X^*$.

**Remark 6** We can argue that there is some consistency in these associations and we give an indication of this line of logic in this remark. As $\text{epi} \hat{f}^*$ is closed in the weak$^*$ topology of $(X \times X^* \times \mathbb{R})^* = X^* \times X^{**} \times \mathbb{R}$ then by definition the restriction

$$\text{epi} (f^*) = R_{X^* \times X} \left[ \text{epi} \left( \hat{f}^* \right) \right] \subseteq X^* \times X \times \mathbb{R}$$

must then be closed with respect to the product of the the weak$^*$ topology on $X^*$ and the weak topology on $X$. As $\text{epi} (f^*)^!$ is a convex body this means that weak closure may be equated with strong closure implying $w^* \times s$-closure of within $X^* \times X$. As noted above the Krein–Smulian theorem enables us to then deduce that $(x^*, x) \mapsto f (x^*, x)$ is $bw^* \times s$ lower semi–continuous (as expected).

The embedding of paired spaces.
Denote
\[ M_f := \{(x, x^*) \in X \times X^* \mid f(x, x^*) = \langle x, x^* \rangle \} \]
\[ M_{f^*} := \{(x^*, x) \in X^* \times X \mid f^*(x^*, x) = \langle x^*, x \rangle \} \quad \text{and} \]
\[ M_{\hat{f}} := \{(x^*, x^{**}) \in X^* \times X^{**} \mid \hat{f}^*(x^*, x^{**}) = \langle x^*, x^{**} \rangle \}. \]

**Definition 7** Suppose \( T \) is a monotone operator from \( X \) to \( X^* \). Then \( T \) is of type negative infimum or (NI) if \( \inf_{(y, y^*) \in M} \langle x^{**} - y, x^* - y^* \rangle \leq 0 \) for all \( (x^{**}, x^*) \in X^{**} \times X^* \).

**Lemma 8** Let \( M \) be a monotone set in \( X \times X^* \).

1. Then
\[
\mathcal{P}_{\tilde{M}}(x^{**}, x^*) := \mathcal{F}_{\tilde{M}}^*(x^{**}, x^*) \geq \mathcal{F}_{\tilde{M}}^*(x^{**}, x^*) \quad \text{for all} \quad (x^{**}, x^*). \quad (10)
\]

2. We have \( \mathcal{P}_{\tilde{M}} = \overline{P_M}^{bw \times s} \) (the closure within \( X^{**} \times X^* \) using the bounded weak* closure in \( X^{**} \) and strong closure in \( X^* \)) where \( p_M \) is defined as in (4). In particular
\[
\tilde{M} \subseteq \{ (y^{**}, y^*) \mid \mathcal{P}_{\tilde{M}}(y^{**}, y^*) = \langle y^{**}, y^* \rangle \} := M_P \subseteq \text{dom} \mathcal{P}_{\tilde{M}} \subseteq \overline{\text{co} M}^{bw \times s}. \quad (11)
\]

3. We have
\[
\mathcal{F}_{\tilde{M}}^*(x^{**}, x^*) \geq \mathcal{F}_{\tilde{M}}^*(x^{**}, x^*) \quad \text{for all} \quad (x^{**}, x^*) \quad (12)
\]

Thus \( \mathcal{F}_{\tilde{M}}^*(\hat{x}, x^*) \geq \langle \hat{x}, x^* \rangle \) for all \( (x, x^*) \in X \times X^* \), when \( M \) is maximal and \( \mathcal{F}_{\tilde{M}}^*(x^{**}, x^*) \geq \langle x^{**}, x^* \rangle \) for all \( (x^{**}, x^*) \in X^{**} \times X^* \), when \( M \) is (NI).

4. Whenever \( f^* \) is a representative function the set \( M_{f^*} \) is monotone. Similarly when \( \hat{f}^* \) is a representative function the set \( M_{\hat{f}} \) is monotone on \( X^* \times X^{**} \).

**Proof.** The first equality of (10) is well known. The second follows from definitions. Indeed for \( (x, x^*) \in J_{X \times X^*}(M) \subseteq X^{**} \times X^* \) we have
\[
\mathcal{F}_{\tilde{M}}^*(x^{**}, x^*) := c_{J_{X \times X^*}(M)}(x^{**}, x^*) = \sup_{(\hat{y}, y^*) \in J_{X \times X^*}(M)} \{ \langle \hat{y}, x^* \rangle + \langle x^{**}, y^* \rangle - \langle \hat{y}, y^* \rangle \}
\]
and so
\[
\mathcal{P}_{\tilde{M}}(x^{**}, x^*) := \mathcal{F}_{\tilde{M}}^*(x^*, x^{**}) = c_{J_{X \times X^*}(M)}(x^*, x^{**}).
\]
This later function is the smallest \( bw \times s \) closed convex function that interpolates \( \langle \cdot, \cdot \rangle \) on \( J_{X \times X^*}(M) \) that is \( \overline{P_M}^{bw \times s} \). Consequently \( \text{dom} \overline{P_M}^{bw \times s} \subseteq \overline{\text{co} M}^{bw \times s} \). Clearly \( \tilde{M} \subseteq M_P \) (as \( \mathcal{P}_{\tilde{M}}(\hat{y}, y^*) \) is a representative function for \( M \) on \( X \times X^* \)).
Because
\[ M_P = \{(y^{**}, y^*) \mid P_M (y^{**}, y^*) - \langle y^{**}, y^* \rangle \leq 0\} \]
and \((y^{**}, y^*) \mapsto P_M (y^{**}, y^*)\) is \(bw^* \times s\) lower semi–continuous \(M_P = M_{\tilde{M}}\) which implies (11).

By definition, for all \((y, y^*) \in M\) we have
\[ \tilde{F}_M^{**} (x^{**}, x^*) \geq \langle x^{**}, y^* \rangle + \langle x^*, y \rangle \]
and so
\[ \tilde{F}_M^{**} (x^{**}, x^*) + \langle y, y^* \rangle \geq \langle x^{**}, y^* \rangle + \langle x^*, y \rangle. \]

On taking \(\sum i \lambda_i (y_i, y^*_i, 1) = (y, y^*, 1)\), with \((y_i, y^*_i) \in M\) then we find
\[ \tilde{F}_M^{**} (x^{**}, x^*) + \lambda P_M (\tilde{y}, y^*) \geq \langle x^{**}, y^* \rangle + \langle x^*, \tilde{y} \rangle. \]
Taking the \(bw^* \times s\) closure within \(X^{**} \times X\) we obtain for all \((y^{**}, y^*) \in X^{**} \times X^*\)
\[ \tilde{F}_M^{**} (x^{**}, x^*) + P_M (y^{**}, y^*) \geq \langle x^{**}, y^* \rangle + \langle x^*, y^{**} \rangle. \]
Consequently
\[ \tilde{F}_M^{**} (x^{**}, x^*) \geq P_M^{**} (x^{**}, x^*) = F_{\tilde{M}} (x^{**}, x^*) \]
establishing the inequality in (12). So when \(M\) is maximal we have \(\tilde{F}_M^{**} (x, x^*) \geq F_{\tilde{M}} (\tilde{x}, x^*) = F_M (x, x^*) \geq \langle x, x^* \rangle\) for all \((x, x^*)\). We have \(M\) of type (NI) exactly when \(F_{\tilde{M}}\) is a representative function on \(X^{**} \times X^*\). The last assertion follows from now standard calculations as in [14].

Since the \(bw^* \times s\)–closure of the convex set \(\text{epi} \; \tilde{f}\) is characterised by convergence of weak\(^*\)–strongly convergent bounded nets this type of closure operation is applicable to other quantities that characterise \(\tilde{f}\).

**Lemma 9** Suppose \(f : X \times X^* \to \mathbb{R}\) is proper, \(s \times bw^*\)–closed convex function. Then for all \((\bar{y}, y^*) \in \text{dom} \; \tilde{f} \subseteq \bar{X} \times X^*\) we have \(\tilde{f} (\tilde{y}, y^*) = f (y, y^*)\).

**Proof.** This is deduced from the following fact (that we prove next)
\[ R_{X \times X^*} \left[ \text{epi} \; \tilde{f} \right] = \text{epi} \; f. \] (13)
As \(f\) is \(s \times bw^*\)–closed and proper then \(\text{epi} \; f\) is also a strongly closed convex subset of \(X \times X^* \times \mathbb{R}\). As \(\text{epi} \; f\) is convex it must then be also a weakly closed subset of \(X \times X^* \times \mathbb{R}\) (under the association of \(X \times X^* \times \mathbb{R}\) with \(X^* \times X^{**} \times \mathbb{R}\)). A jointly weakly closed subset \(\text{epi} \; f\) is also a weak–strongly closed subset of \(X \times (X^* \times \mathbb{R})\). Now if we take \((\bar{y}_\beta, y^*_\beta) \to_{w^* \times s} (\tilde{y}, y^*)\) (within \(X^{**} \times X^*\) with \(\|y_\beta\| \leq K\)) then for all \(v^* \in X^*\) and \(\delta > 0\) we have \((\bar{y}_\beta, y^*_\beta) \in \{(z, z^*) \mid \langle z - y, v^* \rangle < \delta \text{ and } z^* \in B_\delta (y^*)\}\), eventually. As these sets form a subbasis for the weak–strong topology on \(X \times X^*\) we have \((y_\beta, y^*_\beta) \to_{w^* \times s} (y, y^*)\).
and the weak×strongly closedness of epi \( f \) demands that all accumulation points of \((y_\beta, y_\beta^*, f(y_\beta, y_\beta^*))\) must lie inside epi \( f \). Consequently,

\[
\liminf_{(y_\beta, y_\beta^*) \to y^* \text{ s.t. } \|y^*_\beta\| \leq K_y} \tilde{f}(y_\beta, y_\beta^*) \geq \tilde{f}(y, y^*)
\]

or equivalently (as epi \( \tilde{f} \) is convex),

\[
\tilde{f}(y_\beta, y_\beta^*) \geq \tilde{f}(y, y^*)
\]

The reverse inequality is obvious and so \( \tilde{f}(y, y^*) = f(y, y^*) \).

**Proposition 10** Suppose \( f : X \times X^* \to \mathbb{R} \) is proper, \( s \times bw^* \)-closed convex function. Then

\[
\text{Graph } \partial \tilde{f} = \text{Graph } (\partial f)^{-1}.
\]

**Proof.** We only need show that Graph \( \partial \tilde{f} = \text{Graph } \partial \{ \tilde{f} \} = \text{Graph } \partial \{ \tilde{f} \} = \text{Graph } \{ \tilde{f} \}^{-1} \) and \( \{ \tilde{f} \}^* = \tilde{f}^* \). Let \( (\tilde{x}_\beta, x^*_\beta) \to \beta_\beta \times s (y^*_\beta, x^*_\beta) \) and \( (\tilde{y}_\beta, y^*_\beta) \to \beta_\beta \times s (y^*_\beta, y^*_\beta) \) be such that there exists \( K_x, K_y > 0 \) with \( \|y^*_\beta\| \leq K_y \) and \( \|\tilde{y}_\beta\| \leq K_x \). Then

\[
\begin{align*}
|\langle \tilde{x}_\beta - x^*_\beta, y^*_\beta \rangle | & \leq |\langle \tilde{x}_\beta - x^*_\beta, y^*_\beta \rangle | + |\langle \tilde{x}_\beta - x^*_\beta, y^*_\beta \rangle | + |\langle \tilde{x}_\beta - x^*_\beta, y^*_\beta \rangle | + |\langle \tilde{x}_\beta - x^*_\beta, y^*_\beta \rangle |
\leq |\langle \tilde{x}_\beta - x^*_\beta, y^*_\beta \rangle | + K_x \|y^*_\beta - y^*_\beta\| + K_y \|x^*_\beta - x^*_\beta\| + |\langle \tilde{x}_\beta - x^*_\beta, y^*_\beta \rangle | \to 0.
\end{align*}
\]

As epi \( \tilde{f} = \text{epi } f \) (which due to convexity and the Krein–Smulian theorem) we have the existence of \((\tilde{y}_\beta, y^*_\beta) \to \beta_\beta \times s (y^*_\beta, y^*_\beta)\) with \( \|y^*_\beta\| \leq K_y \), some \( K_y > 0 \) and

\[
\tilde{f}(y^*_\beta, y^*_\beta) \to \beta \tilde{f}(y^*_\beta, y^*_\beta).
\]

Suppose \((x^*_\beta, x^*_\beta) \in \partial \{ \tilde{f} \} \) \( (y^*_\beta, y^*_\beta) \) then we have for all \( (v^*_\beta, v^*_\beta) \in X^* \times X^* \) that

\[
\tilde{f}(y^*_\beta, y^*_\beta) + \langle (v^*_\beta, v^*_\beta) - (y^*_\beta, y^*_\beta), (x^*_\beta, x^*_\beta) \rangle \leq \tilde{f}(v^*_\beta, v^*_\beta).
\]

Using (17) with \((v^*_\beta, v^*_\beta) = (\tilde{v}, v^*_\beta) \in \tilde{X} \times X^* \) and (15), (16) and (13) we have \( \forall \varepsilon > 0 \) and any \((\tilde{x}_\beta, x^*_\beta) \to \beta_\beta \times s (x^*_\beta, x^*_\beta)\) with \( \|\tilde{x}_\beta\| \leq K_x \), that eventually

\[
\tilde{f}(y^*_\beta, \tilde{y}_\beta) + \langle (\tilde{v}, v^*_\beta) - (y^*_\beta, \tilde{y}_\beta), (\tilde{x}_\beta, x^*_\beta) \rangle \leq \tilde{f}(v^*_\beta, v^*_\beta), \forall (v^*_\beta, v^*_\beta) \in X \times X^*.
\]

where we have make use of Lemma 9, that is when \((\tilde{y}, y^*_\beta) \in \text{dom } \tilde{f} \subseteq \tilde{X} \times X^* \) then \( \tilde{f}(\tilde{y}, y^*_\beta) = f(y, y^*_\beta) \). Thus for all \( \varepsilon > 0 \) we have eventually

\[
(x^*_\beta, x^*_\beta) \in \partial \varepsilon f (y^*_\beta, y^*_\beta).
\]
Suppose Theorem 11 commutativity of operations in Figure 3.

Now use the fact that
\[ \lim_{\beta} f(y_{\beta}, y_{\beta}) = f(y^{**}, y^{**}) \]
for all \( y^{**} \). As \( \varepsilon > 0 \) is arbitrary we have shown the existence if a nets \( (\hat{v}_{\beta}, v_{\beta}) \rightarrow \hat{w}^{*+s} (x^{**}, x^{*}) \) and \( (\hat{z}_{\beta}, z_{\beta}) \rightarrow \hat{w}^{*+s} (y^{**}, y^{*}) \) with \( \|\hat{v}_{\beta}\| \leq 2K_y, \|\hat{z}_{\beta}\| \leq 2K_x \) and \( (v_{\beta}, v_{\beta}) \in \partial (F_M) (z_{\beta}, z_{\beta}) \) for all \( \beta \). That is,
\[
\text{Graph } \partial (f) \supseteq \text{Graph } \partial \left( \hat{f} \right).
\]

For the reversed inclusion take nets \( (\hat{x}_{\beta}, x_{\beta}) \rightarrow \hat{w}^{*+s} (x^{**}, x^{*}) \) and \( (\hat{y}_{\beta}, y_{\beta}) \rightarrow \hat{y}^{*+s} \) \( (y^{**}, y^{*}) \) with \( \|\hat{y}_{\beta}\| \leq K_y \), \( \|\hat{x}_{\beta}\| \leq K_x \), for some \( K_x, K_y > 0 \) and \( (\hat{x}_{\beta}, x_{\beta}) \in \partial (F_M) (y_{\beta}, y_{\beta}) \) for all \( \beta \). Then we have

\[
f(y_{\beta}, y_{\beta}) + ((v^{*}, v^{*}) - (y_{\beta}, y_{\beta}), (x_{\beta}, x_{\beta})) \leq f(v, v^{*}), \quad \forall (v, v^{*}) \in X \times X^{*},
\]
and so for all \( (\hat{v}, v^{*}) \in \hat{X} \times X^{*} \)

\[
\hat{f}(y^{*}, y^{**}) + ((\hat{v}, v^{*}) - (y^{*}, y^{**}), (x^{*}, x^{**})) \leq \lim_{\beta} \inf \left\{ f(y^{*}, y_{\beta}) + ((v^{*}, v^{*}) - (y_{\beta}, y_{\beta}), (x_{\beta}, x_{\beta})) \right\} \leq \hat{f}(\hat{v}, v^{*}).
\]

Now use the fact that
\[
(v^{**}, v^{*}) \mapsto \hat{f}(y^{*}, y^{**}) + ((v^{**}, v^{*}) - (y^{*}, y^{**}), (x^{*}, x^{**}))
\]
is continuous with respect to \( bw^{*} \times s \) topology on \( X^{**} \times X^{*} \). Consequently

\[
\hat{f}(y^{*}, y^{**}) + ((v^{**}, v^{*}) - (y^{*}, y^{**}), (x^{*}, x^{**})) \leq \hat{f}(v^{**}, v^{*}), \quad \forall (v^{**}, v^{*}) \in X^{**} \times X^{*},
\]
or
\[
\text{Graph } \partial (f) \subseteq \text{Graph } \partial \left( \hat{f} \right).
\]

We may now prove an important closure theorem (this essentially a small extension the results on page 168 of [13]). One can view this result as verifying commutativity of operations in Figure 3.

**Theorem 11** Suppose \( f : X \times X^{*} \rightarrow \mathbb{R} \) is proper, \( s \times bw^{*} \)-closed convex function. Then

\[
\inf_{(\hat{v}_{\beta}, y_{\beta}) \rightarrow \hat{w}^{*+s} (y^{**}, y^{*})} \lim_{\beta} \inf \hat{f}^{*}(\hat{v}_{\beta}, y_{\beta}) = \hat{f}^{*}(y^{**}, y^{*}). \quad (18)
\]
Proof. First note that \( \text{epi} \hat{f}^* \) being \( s \times bw^* \)-closed is also strongly closed and so by the Phelps-Brondsted-Rockafellar Lemma \( \text{epi} \hat{f}^* \) is the intersection of all half spaces formed from (non-vertical) supporting hyper-planes i.e. half spaces formed from elements of \( \text{Graph} \partial \hat{f}^* \). By (14) we have \( \text{Graph} \partial \hat{f}^* = (\text{Graph} \hat{\partial}(f))^\dagger \) (where in this instance the transpose operator \( ^\dagger \) applies to pairs of elements in \( X^{**} \times X^* \)). By the continuity implied in (15) with respect to the \( s \times w^* \) convergent, norm bounded nets we have \( \text{epi} \hat{F}_M^* \) generated as the \( bw^* \times s \)-closure of the convex set generated as the intersection of all half spaces formed from elements in \( \text{Graph} \partial \hat{f} \subseteq (X^* \times \hat{X}) \times (\hat{X} \times X^*) \). Writing this as a closure operation (invoking the Krein–Smulian theorem) we obtain (18).

This allows the following important observation to be made.

Corollary 12 Let \( M \) be a monotone set in \( X \times X^* \). Then \( \hat{F}_M^* \) is a representative function on \( X^* \times X^{**} \).

Proof. The result follows immediately from Theorem 11, Lemma 3, (7) and (9).

4 Conditions for Maximality in a Banach Space

In this section we derive conditions that may be used to deduce maximality of monotone sets defined by certain representative functions. The proofs of sum theorem is based on these results. Thus, we present the technical machinery in this section with the consequences following in later sections. Proofs of some recent results are included for the readers convenience. The following is a consequence of Rockafellar’s version of the duality theorem.

Theorem 13 ([21]) Let \( X \) be a normed space, \( f : X \to \mathbb{R} \) be a proper and convex function, \( g : X \to \mathbb{R} \) be convex and continuous and suppose \( f + g \geq \alpha \) on \( X \). Then there exists \( x^* \in X^* \) such that \( f^*(x^*) + g(-x^*) \leq -\alpha \).

For \((x, x^*) \in X \times X^*\) denote
\[
\Delta(x, x^*) = \frac{1}{2} \|x\|^2 + \langle x, x^* \rangle + \frac{1}{2} \|x^*\|^2
\]
\[
\geq \frac{1}{2} \left( \|x\|^2 - 2 \|x\| \|x^*\| + \|x^*\|^2 \right) \geq (\|x\| - \|x^*\|)^2 \geq 0.
\]

Similarly for \((x^{**}, x^*) \in X \times X^*\) denote \( \Delta(x^{**}, x^*) = \frac{1}{2} \|x^{**}\|^2 + \langle x^{**}, x^* \rangle + \frac{1}{2} \|x^*\|^2 \)

On \( X \times X^* \) we use the norm \( \|(x, x^*)\|^2 := \|x\|^2 + \|x^*\|^2 \) and on \( X \times X^* \) we use \( \|(x^{**}, x^*)\|^2 := \|x^{**}\|^2 + \|x^*\|^2 \), since then
\[
\left( \frac{1}{2} \|\cdot\|^2 \right)^*(x^{**}, x^*) = \frac{1}{2} \left( \|x^*\|^2 + \|x^{**}\|^2 \right) = \frac{1}{2} \|(x^{**}, x^*)\|^2.
\]
Lemma 14 ([25], Lemma 1.3) Let \( h : X \times X^* \to \overline{\mathbb{R}} \) be proper, convex and \((w, w^*) \in X \times X^*\) and

\[
(x, x^*) \in X \times X^* \implies h(x, x^*) - \langle x, x^* \rangle + \Delta(w - x, w^* - x^*) \geq 0.
\]

Then there exists \((x^*, x^{**}) \in X^* \times X^{**}\) such that

\[
\hat{h}^*(x^*, x^{**}) - \langle x, x^* \rangle + \Delta(w - x^*, w^* - x^*) \leq 0.
\]

Proof. Let \( \eta_{(w, w^*)} (x, x^*) := -\langle x, x^* \rangle + \Delta(w - x, w^* - x^*) \). Then since

\[
\eta_{(w, w^*)} (x, x^*) = \langle w, w^* \rangle - \langle (x, x^*), (w, w^*) \rangle + \frac{1}{2} \| (w, w^*) - (x, x^*) \|^2
\]

we have \((x, x^*) \mapsto \eta_{(w, w^*)} (x, x^*) \) is convex and norm-continuous. Then

\[
\eta^*_{(w, w^*)} (-x^*, -x^{**}) = \\
= \sup_{(z, z^*) \in X \times X^*} \left\{ \langle (z, z^*), (-x^{**}, -x^*) \rangle - \left( \langle w, w^* \rangle - \langle (z, z^*), (w, w^*) \rangle + \frac{1}{2} \| (w, w^*) - (z, z^*) \|^2 \right) \right\}
\]

\[
= \langle w, w^* \rangle - \langle (x^{**}, x^*), (w, w^*) \rangle + \sup_{(z, z^*) \in X \times X^*} \left\{ \langle (z, z^*) - (w, w^*), (w, w^*) - (x^{**}, x^*) \rangle - \frac{1}{2} \| (w, w^*) - (z, z^*) \|^2 \right\}
\]

\[
= \langle w, w^* \rangle - \langle (x^{**}, x^*), (w, w^*) \rangle + \frac{1}{2} \| (w, w^*) - (x^{**}, x^*) \|^2 = \eta_{(w, w^*)} (x^{**}, x^*). \]

Note that this only relies on the identity (19). Now invoke Theorem 13 with the property that \( f(x, x^*) \geq \langle x, x^* \rangle \) for all \((x, x^*) \in X \times X^*\). As (20) implies

\[
\inf_{X \times X^*} \{ h(x, x^*) + \eta_{(w, w^*)} (x, x^*) \} \geq 0
\]

we deduce that

\[
\min_{X^* \times X^{**}} \left\{ \hat{h}^*(x^*, x^{**}) + \eta_{(w, w^*)} (-x^{**}, -x^*) \right\} \leq 0.
\]

This clearly implies (21).

When \( M \subseteq X \times X^* \) is a maximal monotone set we can always embed

\[
J_{X \times X^*} (M) \subseteq X^{**} \times X^*
\]

and extend \( J_{X \times X^*} (M) \) to a larger monotone set \( \tilde{M} \) within \( X^{**} \times X^* \) (many such extensions have been proposed in the literature).

This set has the following useful property: When \((\tilde{u}, u^*)\) is monotonically related to \(\tilde{M}\), as it must then be monotonically related to \(M\) also, we have \((\tilde{u}, u^*) \in M\). We will see that this property is more widely held by monotone sets \( M \) which we do not know a-priori are maximal. Note also that when we choose \( \tilde{M} \) a maximal extension of \( J_X (M) \) within \( X^{**} \times X^* \) then its Fitzpatrick function satisfies \( F_{\tilde{M}} (x^{**}, x^*) \geq (x^{**}, x^*) \) for all \((x^{**}, x^*) \in X^{**} \times X^* \) and consequently
also $\mathcal{F}_M(\tilde{x}, x^*) \geq \langle \tilde{x}, x^* \rangle$. By [14, Prop. 4] we know that the following is a monotone set:

$$\{(x, x^*) \mid \mathcal{F}_M(x, x^*) = \langle x, x^* \rangle \}.$$ 

Clearly $\mathcal{F}_M(x, x^*) \leq \langle x, x^* \rangle$ if and only if $(x, x^*)$ is monotonically related to $\tilde{M}$ and by a before mentioned property it follows that $(x, x^*) \in M$. Thus,

$$\{(x, x^*) \mid \mathcal{F}_M(x, x^*) = \langle x, x^* \rangle \} \subseteq M.$$ 

Note that as $\tilde{M}$ is maximal

$$\{(x^*, x^*) \mid \mathcal{F}_M(x^*, x^*) = \langle x^*, x^* \rangle \} = \tilde{M} \supseteq J_{X \times X^*}(M)$$

and so

$$\{(x, x^*) \mid \mathcal{F}_M(x, x^*) = \langle x, x^* \rangle \} = \tilde{M} \cap (J_X(X) \times X^*) \supseteq J_X(M).$$

Thus $\mathcal{F}_M$ restricted to $X \times X^*$ is a representative function of $M$ and $M$ is the restriction to $X \times X^*$ of a maximal monotone set $\tilde{M}$ in $X^* \times X^*$. We will now study this phenomenon in more detail.

**Lemma 15** Suppose $f : X \times X^* \rightarrow \mathbb{R}$ is a representative function. Suppose in addition that $\hat{f}^*(x^*, x^{**}) \geq \langle x^*, x^* \rangle$ (and hence $f^*(x^*, x) \geq \langle x, x^* \rangle$). Then

1. If $(u^*, u) \in X^* \times X$ is monotonically related to $M$, we have $(u^*, u) \in M_{\tilde{f}}$.

2. The restrictions $R_{X^* \times X} (M_{\tilde{f}}) = M_{\tilde{f}}$, $R_{X \times X^*} (M_{\tilde{f}}) = (M_{\tilde{f}})^\dagger$ for any $M_{\tilde{f}}$ a maximal monotone extension of $(M_{\tilde{f}})^\dagger$ (viewed as a monotone set within $X^* \times X^*$).

**Proof.** Let $(u^*, u) \in X^* \times X$ be monotonically related to $M_{\tilde{f}}$, (i.e., $\langle u^* - y^*, u - y^{**} \rangle \geq 0$ for all $(y^*, y^{**}) \in M_{\tilde{f}}$). As $\Delta \geq 0$ we have $f(z, z^*) - \langle z, z^* \rangle + \Delta (u - z, u^* - z^*) \geq 0$ for all $(z, z^*) \in X \times X^*$. Thus by Lemma 14 there exists $(x^*, x^{**}) \in X^* \times X^*$ such that

$$\hat{f}^*(x^*, x^{**}) - \langle x^*, x^{**} \rangle + \Delta (u - x^{**}, u^* - x^*) \leq 0. \quad (22)$$

It follows that $\hat{f}^*(x^*, x^{**}) \leq \langle x^*, x^{**} \rangle$. Hence $\hat{f}^*(x^*, x^{**}) = \langle x^*, x^{**} \rangle$ or $(x^*, x^{**}) \in M_{\tilde{f}}$ and also by (22) we have

$$\Delta (u - x^{**}, u^* - x^*) = \frac{1}{2} \|u - x^{**}\|^2 + (u - x^{**}, u^* - x^*) + \frac{1}{2} \|u^* - x^*\|^2 = 0. \quad (23)$$

Because $\langle u - x^{**}, u^* - x^* \rangle \geq 0$ for all $(x^*, x^{**}) \in M_{\tilde{f}}$, as $(u, u^*)$ is monotonically related to $M_{\tilde{f}}$, from (23) it follows that that $(u, u^*) = (x^{**}, x^*) \in M_{\tilde{f}}$, and that $x^{**}$ is actually in $X$. That is $(u, u^*) \in R_{X^* \times X} (M_{\tilde{f}})$ the restriction of $M_{\tilde{f}}$ into $X^* \times X$. 

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On the other hand we know that the restriction of $\hat{f}^*$ to $X^* \times X$ is also a representative function (indeed that of $M_f$) and so $R_{X^* \times X}(M_{\hat{f}^*}) = M_f$. Thus, if $(u^*, u) \in X^* \times X$ is monotonically related to $M_{\hat{f}^*}$, then $(u, u^*) \in M_f$. Clearly, when $(u^*, u) \in X^* \times X$ is monotonically related to $M_{\tilde{f}^*}$, then it is monotonically related to $M_f$ and so the same results follow for $M_{\tilde{f}^*}$. □

An immediate corollary is the following.

**Corollary 16** Suppose $f : X \times X^* \to \mathbb{R}$ is a representative function. Suppose in addition that $\hat{f}^*(x^*, x^{**}) \geq \langle x^{**}, x^* \rangle$ (and hence $f^*(x^*, x) \geq \langle x, x^* \rangle$). Let $M_f$ be any maximal monotone extension of $J_{X \times X}(M_f^{\dag})$ (viewed as a subset of $X^{**} \times X^*$).

Then $R_{X \times X} \left( M_f^{\dag} \right) = M_f^{\dag}$, and indeed $M_f = M_{\tilde{f}^*}$ for some maximal extension of $M_{\tilde{f}^*}^{\dag}$.

**Proof.** Note that we have $J_{X \times X}(M_f^{\dag}) \subseteq M_{\tilde{f}^*}$ (for any extension as defined in Lemma 15) and so $J_{X \times X}(M_f^{\dag}) \subseteq M_f \subseteq M_{\tilde{f}^*}$, for some such extension $M_f$.

As $M_f$ is maximal we have $M_f = M_{\tilde{f}^*}$ and hence $R_{X \times X} \left( M_f^{\dag} \right) = R_{X \times X} \left( M_{\tilde{f}^*}^{\dag} \right) = M_f^{\dag}$. □

An elegant result that holds in all Banach spaces is possible by using the composition result of [4] in conjunction with Theorem 17 below. This requires a qualification assumption and so in order to obtain the best result we delay statement of the qualification assumption to the next section.

The following theorem allows us to make the inference that the requisite Fitzpatrick function is indeed a representative function. This is often a crucial step toward proving the maximality of the underlying monotone set.

**Theorem 17** Suppose $f : X \times X^* \to \mathbb{R}$ is a representative function. Suppose in addition that there exists an family of subspaces $Y$ of $X$ such that $\cup_{Y \in \mathcal{Y}} Y = X$ and a maximal monotone extension $\hat{T} := M_f$ of the monotone set $J_{X \times X}(M_f^{\dag}) \subseteq X^{**} \times X^*$ with the property that whenever

$$A : Y \to X^{**}$$

is the embedding of a subspace $Y \in \mathcal{Y}$ into $X^{**}$ we have $T_A := A^* \circ \hat{T} \circ A$ maximal monotone from $Y$ to $Y^*$.

Then when $\hat{f}^*(x^*, x^{**}) \geq \langle x^{**}, x^* \rangle$ for all $(x^*, x^{**}) \in X^* \times X^{**}$ we have $F_{M_f} : X \times X^* \to \mathbb{R}$ a representative function and $M_f = M_f^{\dag}$.

**Proof.** Let $Y \in \mathcal{Y}$ then for each $x \in Y$ we have $A(x) = J_X(x) = \hat{x}$ and when $z^* \in Y^*$ there exists $x^* \in X^*$ with $z^* = x^*|_Y$. Let $(x, z^*) \in T_A$ then there exists $\hat{x} \in J_X(X)$ and $x^* \in \hat{T}(\hat{x})$ such that $z^* = x^*|_Y$. Consequently using Lemma
15 for all \( \hat{x} \in Y \) and \( x^* \in X^* \) we have for \( z^* = X^*|_Y \)

\[
\mathcal{F}_{\tilde{M}_f} (\hat{x}, x^*) = \sup_{(y, y^*) \in (M_f)^\dagger} \{ \langle y, x^* \rangle + \langle y^*, \hat{x} \rangle - \langle y, y^* \rangle \}
\]

\[
= \sup_{(y, y^*) \in \tilde{T} \cap (Y \times X^*)} \{ \langle y, x^* |_Y \rangle + \langle y^* |_Y, \hat{x} \rangle - \langle y, y^* |_Y \rangle \}
\]

\[
\geq \sup_{(y, y^*) \in \tilde{T}} \{ \langle y, x^* \rangle + \langle y^* , \hat{x} \rangle - \langle y, y^* \rangle \}
\]

\[
= \sup_{(y, y^*) \in \tilde{T}} \{ \langle y, z^* \rangle + \langle y^*, x \rangle - \langle y, y^* \rangle \} = \mathcal{F}_{M_f} (x, z^*) \quad (24)
\]

As \( T_A \) is maximal on \( Y \times Y^* \) we have for all \( (\hat{x}, z^*) \in Y \times Y^* \) with \( z^* = x^* |_Y \) that

\[
\mathcal{F}_{T_A} (x, z^*) \geq \langle \hat{x}, z^* \rangle \quad \text{and so}
\]

\[
\mathcal{F}_{T_A} (x, z^*) = \sup_{(y, y^*) \in T_A} \{ \langle y, x^* |_Y \rangle + \langle y^*, x \rangle - \langle y, y^* \rangle \}
\]

\[
= \sup_{(y, y^*) \in T_A} \{ \langle y, x^* \rangle + \langle y^*, \hat{x} \rangle - \langle y, y^* \rangle \}
\]

\[
= \mathcal{F}_{T_A} (\hat{x}, x^*) \geq \langle \hat{x}, z^* \rangle = \langle \hat{x}, x^* \rangle \quad (25)
\]

where \( \mathcal{F}_{T_A} (\hat{x}, x^*) \) is the Fitzpatrick function of \( T_A \) embedded in \( X^{**} \times X^* \). As \( J_{X \times X^*} (M_f^\dagger) \subseteq (M_f^\dagger)^\dagger \subseteq \tilde{T} \) we have on \( X^{**} \times X^* \) pointwise,

\[
\mathcal{F}_{T} \geq \mathcal{F}_{M_f^\dagger} \geq \mathcal{F}_{M_f^\dagger}.
\]

Evaluating at \( (\hat{x}, x^*) \) for \( x \in Y \) we obtain from (24) and (25)

\[
\mathcal{F}_{M_f^\dagger} (x, x^*) = \mathcal{F}_{\tilde{M}_f} (\hat{x}, x^*) \geq \langle \hat{x}, x^* \rangle. \quad (26)
\]

Since this holds for any choice of subspace \( Y \subseteq Y \) we have (26) holding for all \( x \in X \). Thus \( \mathcal{F}_{M_f^\dagger} \) is a representative function on \( X \times X^* \).

As \( f^* (x^*, \hat{x}) \geq \langle \hat{x}, x^* \rangle \) for all \( (x^*, x) \in X^* \times X \) we have \( f^* \) is a representative function for \( M_f^\dagger \). By [5, Prop. 2] we have \( \mathcal{P}_{M_f^\dagger} (\cdot, \cdot) \geq \langle \cdot , \cdot \rangle \geq \mathcal{F}_{M_f^\dagger} (\cdot, \cdot) \geq \langle \cdot , \cdot \rangle \). Using the duality with respect to the pairing of \( X \times X^* \) with \( X^* \times X \), on taking conjugates we have

\[
\mathcal{P}_{M_f^\dagger} = \mathcal{F}_{M_f^\dagger} \geq f \geq \mathcal{P}_{M_f^\dagger} = \mathcal{F}_{M_f^\dagger}. \quad (27)
\]

As \( \mathcal{F}_{M_f^\dagger} (\cdot, \cdot) \geq \langle \cdot , \cdot \rangle \) by (27)

\[
M_f = \{ (x^*, x) \mid \mathcal{P}_{M_f^\dagger} (x, x^*) = \langle x, x^* \rangle \} \subseteq \{ (x^*, x) \mid f (x, x^*) = \langle x, x^* \rangle \} = M_f.
\]

Also (27) implies

\[
M_f \subseteq \{ (x^*, x) \mid \mathcal{F}_{M_f^\dagger} (x, x^*) = \langle x, x^* \rangle \}
\]

and [5, Prop. 2] says that \( \mathcal{F}_{M_f^\dagger} (x, x^*) = \langle x, x^* \rangle \) implies \( \mathcal{P}_{M_f^\dagger} (x, x^*) = \langle x, x^* \rangle \) (or \( (x, x^*) \in M_f^\dagger \)) when \( \mathcal{F}_{M_f^\dagger} \) is representative. So \( M_f \subseteq M_f^\dagger \). 

\[
\]
Remark 18 As $R_{X \times X} \left( \tilde{M} \right) = M_f$, we have $T_A := A^* \circ M_f : Y \to Y^*$. When $f^*$ is a representative function for $T : X \rightrightarrows X^*$ we have $T_A = A^* \circ T \circ A$ where $A$ is the embedding of $Y$ in $X$ (because $R_{X \times X} \left( \tilde{T} \right) = T$).

We cannot deduce from the fact that $F_T$ is a representative function that $T$ is maximal (graph closure may easily fail). In [5] a monotone operator $T$ is said to be almost maximal when $F_T$ is a representative function i.e. $F_T (x, x^*) \geq \langle x, x^* \rangle$. The close relationship between almost maximality and maximality for sums of maximal monotone operators is noted in [5]. The significance of Theorem 17 is that it gives sufficient conditions for almost maximality. We will use this to deduce maximality of a sum of two maximal monotone operators.

5 Some Properties of Operators in Finite Dimensions

We will be using an embedding of operators into a finite dimensional subspace and will impose conditions to obtain a maximal operator within finite dimensional subspaces. Consequently the properties finite dimensional maximal monotone operators are of some interest. In the ensuing analysis we find that we need to reconstruct a finite dimensional maximal monotone operator from information provided only within the relative interior of its domain. This problem is consider in this section along with some related issues as preliminaries to the development of a proof of a sum theorem.

An important piece of our technical machinery is the following composition theorem first proved by Borwein [4]. When we take $A$ to denote the embedding of a finite dimensional subspace $Y$ into a nonreflexive Banach space $X$ we obtain a monotone operator $T_A := A^* \circ T \circ A : Y \to Y^*$ on a finite dimensional space $Y$.

We provide and version of the composition theorem under some alternative assumptions. By $\text{lin} C$ we denote the smallest linear subspace containing the set $C$. We write $0 \in \text{sqri} S$ if $Z := \text{span} S$ is a closed subspace and $0 \in \text{core}_Z S$ (the core relative to $Z$). This has been referred to as the strong quasi-relative interior condition in the literature [12]. The presumption of a strong quasi-relative interior is generally a stronger assumption than the presumption of a relative interior $\text{ri} S$, which corresponds to the interior relative to the affine hull of $S$. Indeed, if $S$ is closed $\text{sqri} S \subseteq \text{ri} S$ (see Theorem 3.6 of [12]).

Theorem 19 Let $T : X \to X^*$ is maximal monotone. Suppose in addition that $A : Y \to X$ is a continuous linear operator between two Banach spaces with $Y$ reflexive. Then $T_A := A^* \circ T \circ A$ is maximal on $Y$ under any one of the following conditions.

1. When we have $0 \in \text{Pr}_X \text{dom} F_T$ and $\text{sqri} (\text{Pr}_X \text{dom} F_T) \cap \text{Range} A \neq \emptyset$. 19
2. When we have $0 \in \text{core} \left( \text{co dom} \, T + \text{range} \, A \right)$.

**Proof.** We only prove 1 as 2 is proved in [4]. Take $x \in \text{sqri} \left( \text{Pr}_X \, \text{dom} \, F_T \right) \cap \text{Range} \, A$. By the definition of the strong quasi-relative interior we have $Z := \text{cone} \left( \text{Pr}_X \, \text{dom} \, F_T - x \right) \subseteq X$ is a closed subspace with $Z + x = \text{affine} \left( \text{Pr}_X \, \text{dom} \, F_T \right)$ (see Proposition 3.4 of [12]). Since $0 \in \text{Pr}_X \, \text{dom} \, F_T$ we have $0 \in \text{affine} \left( \text{Pr}_X \, \text{dom} \, F_T \right)$ and so is this is a subspace. As $x \in \text{affine} \left( \text{Pr}_X \, \text{dom} \, F_T \right)$ we have

$$Z = \text{affine} \left( \text{Pr}_X \, \text{dom} \, F_T \right) - x = \text{affine} \left( \text{Pr}_X \, \text{dom} \, F_T \right) = \text{lin} \left( \text{Pr}_X \, \text{dom} \, F_T \right).$$

Next we form a new linear mapping $\tilde{A}$ with the graph

$$\text{Graph} \, \tilde{A} := \text{Graph} \, A \cap (Y \times Z).$$

We note that when $Ay \not\in Z$ we have $Ay \not\in \text{Pr}_X \, \text{dom} \, F_T$ and so $F_T \left( Ay, x^* \right) = +\infty$ for any $x^*$. Thus

$$\inf \left\{ F_T \left( Ay, x^* \right) \mid A^* x^* = y^* \right\} = \inf \left\{ F_T \left( \tilde{A} y, x^* \right) \mid A^* x^* = y^* \right\}.$$

As $x \in \text{core} \left( \text{Pr}_X \, \text{dom} \, F_T \right)$ (a Banach space). Let

$$F \left( y, y^* \right) = \inf \left\{ F_T \left( \tilde{A} y, x^* \right) \mid A^* x^* = y^* \right\} = \inf \left\{ F_T \left( Ay, x^* \right) \mid A^* x^* = y^* \right\}$$

and note that the condition (28) allows us to apply the duality result in [14, Prop. 13]. That is

$$F^* \left( y^*, y \right) = \min \left\{ F_T^* \left( x^*, Ay \right) \mid \tilde{A}^* x^* = y^* \right\},$$

using the reflexivity of $Y$. We now note that

$$F \left( y, y^* \right) = \inf \left\{ F_T \left( Ay, x^* \right) \mid A^* x^* = y^* \right\} \geq \inf \left\{ \left\langle y, A^* x^* \right\rangle \mid A^* x^* = y^* \right\} = \left\langle y, y^* \right\rangle$$

due to the maximality of $T$. Also note that $Ay = \tilde{A} y$ whenever $Ay \in \text{Pr}_X \, \text{dom} \, F_T \subseteq Z$. Since $\text{Pr}_X \, \text{dom} \, P_T \subseteq \text{Pr}_X \, \text{dom} \, F_T$ we have

$$F^* \left( y^*, y \right) = \min \left\{ F_T^* \left( x^*, Ay \right) \mid \tilde{A}^* x^* = y^* \right\} = \min \left\{ P_T \left( Ay, x^* \right) \mid \tilde{A}^* x^* = y^* \right\} \geq \min \left\{ \left\langle \tilde{A}^* x^*, y \right\rangle \mid \tilde{A}^* x^* = y^* \right\} \geq \min \left\{ \left\langle A^* x^*, y \right\rangle \mid A^* x^* = y^* \right\} = \left\langle y^*, y \right\rangle.$$
As $Y$ is reflexive we may apply [15, Prop. 1] to deduce that $F$ (or $F^*$) is a representative function of the maximal monotone set

$$M_F = \{(y, y^*) \in Y \times Y^* \mid F (y, y^*) = \langle y, y^* \rangle \} = \{(y, y^*) \in Y \times Y^* \mid \mathcal{F}_T (Ay) = \langle y, y^* \rangle \text{ for some } A^* x^* = y^* \} = \{(y, y^*) \in Y \times Y^* \mid (Ay, x^*) \in T \text{ and } A^* x^* = y^* \} = \{(y, y^*) \in Y \times Y^* \mid y^* \in (A^* \circ T \circ A) (y) \}.$$

Remark 20 The same argument as used in Theorem 19 can be used to show that we have

$$F^* (y^*, x^{**}) = \min \{ \mathcal{P}_T (x^{**}, x^*) \mid A^* x^* = y^* \} \text{ for } F (y, x^*) = \mathcal{F}_T (Ay, x^*). \text{ Hence }$$

$$F^* (y^*, Ay) = \min \{ \mathcal{P}_T (Ay, x^*) \mid A^* x^* = y^* \}$$

is a representative function of the maximal monotone set $T_A := A^* \circ T \circ A$, under the same assumptions as Theorem 19. Thus $F^* (y^*, Ay) = +\infty$ if $Ay \notin \text{Pr}_X \text{ dom } \mathcal{F}_T$.

This last observation may be made in more generality as we will shortly show.

Lemma 21 Suppose $T$ is a maximal monotone operator on a Banach space $X$. Then

$$\text{Pr}_X \text{ co } T = \text{co } \text{Pr}_X T = \text{co } \text{dom } T. \quad \text{(29)}$$

Proof. First note that $x \in \text{Pr}_X \text{ co } T$ implies the existence of $x^*$ such that $(Ax, x^*) \in T$ and hence the existence of $(x_i, x^*_i) \in T$ with $(Ax, x^*) = \sum_i \lambda_i (x_i, x^*_i) = (\sum_i \lambda_i x_i, \sum_i \lambda_i x^*_i)$. Thus $Ax = \sum_i \lambda_i x_i \in \text{co Pr}_X T$ and so $\text{Pr}_X \text{ co } T \subseteq \text{co Pr}_X T$. Also $\text{Pr}_X \text{ co } T \supseteq \text{co Pr}_X T$ since $\text{co Pr}_X T$ is the smallest convex set containing $\text{Pr}_X T$. That is, (29) holds.

We will need to use the following partial conjugate formula (first noted by [14] in the case when $X$ is reflexive).

Lemma 22 Suppose $p : X \times X^* \to \mathbb{R}$ is a proper lower semi–continuous function with $p^* = f$ where the conjugate is taken with respect to the paired spaces $X \times X^*$ and $X^* \times X$ with product topologies formed by endowing $X$ with the strong topology and $X^*$ with the bounded weak* topology. Let $h(u, v) := -(p(u, \cdot))^*(v)$ then for all $x, z \in X$ and $x^* \in X^*$ we have

$$f (x^*, x) = (h (\cdot, x))^* (x^*) \text{ and }$$

$$(f (\cdot, x))^* (z) = (h (\cdot, x))^* (z). \quad \text{(30)}$$

If we place $F (y, x^*) := f (Ay, x^*)$ where $A : Y \to X$ is a continuous linear mapping form the Banach space $Y$ into the the Banach space $X$ we have

$$F^* (y^*, x) = \sup_{z \in Y} \{ (z, y^*) + (h (\cdot, Az))^* (x) \} \quad \text{(31)}$$

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Let $\text{dom} \implies \emptyset$ implying Lemma 23

When $x \not\in X$ note that the vex function where the later function corresponds the lower semi–continuous hull of the con–
we have

Proof. First we note that $x \not\in F$ spaces. Let $P \in \{z, x^*\} = \sup y \in Y \{\langle z, y^*\rangle + \langle x, x^*\rangle - f (A z, x^*)\}$

and then

$$f (x, x^*) = p^* (x^*, x) = \sup u \left(\langle u, x^*\rangle - \sup z^* \{\langle x, z^*\rangle - p (u, z^*)\}\right)$$

$$= \sup u \{\langle u, x^*\rangle - h (u, x)\} = (h (\cdot, x))^* (x^*).$$

Consequently $(f (x, \cdot))^* (z) = (h (\cdot, x))^* (z)$. Now

$$F^* (y, x) = \sup \{\langle z, y^*\rangle + \langle x, x^*\rangle - f (A z, x^*)\}$$

$$= \sup z \in Y \sup x^* \{\langle z, y^*\rangle + \langle x, x^*\rangle - f (A z, x^*)\}$$

$$= \sup z \in Y \{\langle z, y^*\rangle + (f (A z, \cdot))^* (x)\} = \sup z \in Y \{\langle z, y^*\rangle + (h (\cdot, A z))^* (x)\}.$$  

The promised characterisation of the domain of $F^*$ may now be shown.

Lemma 23 Let $A : Y \to X$ be a continuous linear operator between two Banach spaces. Let $F_T$ be the Fitzpatrick function of a monotone operator $T : X \to X^*$ and denote $F (y, x^*) = F_T (A y, x^*) : Y \times X^* \to \mathbb{R}$. Then $F^* (y^*, x) = +\infty$ if $x \not\in \text{Pr}_X \text{dom} \overline{\text{T}_T}. In particular F^* (y^*, A y) = +\infty when A y \not\in \text{co} \text{dom} T.$

Proof. First we note that $\text{P}_T^* = F_T$. Using Lemma 22 with $f = F_T$ and $p = \text{P}_T$ we have

$$(F_T (x, \cdot))^* (z) = (h (\cdot, x))^* (z)$$

where the later function corresponds the lower semi–continuous hull of the convex function $u \mapsto h (u, v) = -(\text{P}_T (u, \cdot))^* (v)$. Now using (31) we have

$$F^* (y^*, x) = \sup z \in Y \{\langle z, y^*\rangle + (h (\cdot, A z))^* (x)\} \geq \langle y, y^*\rangle + (h (\cdot, A y))^* (x).$$

Next note that

$$u \mapsto h (u, A y) = -(\text{P}_T (u, \cdot))^* (A y) = -\sup w^* \{\langle w^*, A y\rangle - \text{P}_T (u, w^*)\}$$

$$= \inf w^* \{\text{P}_T (u, w^*) - \langle w^*, A y\rangle\}.$$  

When $x \not\in \text{Pr}_X (\text{dom} \overline{\text{T}_T})$ then there exists $\delta > 0$ such that $B_\delta (x) \cap \text{Pr}_X (\text{dom} \overline{T}) = \emptyset$ implying

$$(h (\cdot, A y))^* (x) \geq \inf u \in B_\delta (x) \inf w^* \{\text{P}_T (u, w^*) - \langle w^*, A y\rangle\} = +\infty.$$  

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When $T$ is maximal we now need to investigate the nature of the conjugate function of $F_T(Ay, x^*)$, as this will be the choice we make in many results. It is closely related to a representative function for $T_A$.

**Lemma 24** Let $A : Y \to X$ be a continuous linear operator between two Banach spaces. Let $F_T$ be the Fitzpatrick function of a maximal monotone operator $T : X \to X^*$ and denote $F(y, x^*) = F_T(Ay, x^*)$. Then for all $(y^*, y) \in Y^* \times Y$ we have

$$F^*(y^*, Ay) \geq F_{A^* \circ T \circ A}(y^*, y) = \sup_{(z, z^*) \in T \circ A} \{\langle z^*, Ay \rangle + \langle z, y^* \rangle - \langle z^*, Az \rangle\}. \quad (33)$$

**Proof.** By direct calculation

$$F^*(y^*, Ay) = \sup_{(z, z^*) \in Y \times X^*} \{\langle z^*, Ay \rangle + \langle z, y^* \rangle - F(z, z^*)\} \geq \sup_{(z, z^*) \in T \circ A} \{\langle z^*, Ay \rangle + \langle z, y^* \rangle - F_T(Az, z^*)\} = \sup_{(z, z^*) \in T \circ A} \{\langle z^*, Ay \rangle + \langle z, y^* \rangle - \langle Az, z^* \rangle\}.$$  

Also by direct computation

$$F_{A^* \circ T \circ A}(y^*, y) = \sup_{(w^*, w) \in A^* \circ T \circ A} \{\langle w^*, y \rangle + \langle w, y^* \rangle - \langle w, w^* \rangle\} \geq \sup_{(z^*, w) \in T \circ A} \{\langle A^* x^*, y \rangle + \langle w, y^* \rangle - \langle w, A^* x^* \rangle\} = \sup_{(z^*, w) \in T \circ A} \{\langle x^*, Ay \rangle + \langle w, y^* \rangle - \langle Aw, x^* \rangle\}$$

and so (33) holds.

**Corollary 25** Let $A : Y \to X$ be a continuous linear operator between two Banach spaces and let $A^* : X^* \to Y^*$. Let $F_T$ be the Fitzpatrick function of a maximal monotone operator $T : X \to X^*$ and denote $F(y, x^*) = F_T(Ay, x^*)$. Suppose in addition that $A^* \circ T \circ A$ is maximal on $Y \times Y^*$. Then for all $(y^*, y) \in Y^* \times Y$ we have

$$F_{A^* \circ T \circ A}(y^*, y) \geq \langle y, y^* \rangle$$

and

$$F_{A^* \circ T \circ A}(y^*, y) = \langle y, y^* \rangle \iff y^* \in (A^* \circ T \circ A)(y).$$

**Proof.** This follows from the usual properties of the Fitzpatrick function of a maximal monotone set.

Let $0^+ A := \{z \mid x + \lambda z \in A, \forall x \in A, \lambda \geq 0\}$ denote the recession directions of a convex set $C$. We denote the normal cone to a close convex set $C$ at a point $x \in C$ by

$$N_C(x) := \{x^* \in X^* \mid \langle y - x, x^* \rangle \leq 0 \text{ for all } y \in C\}.$$  

The following results is probably well known and easily proved.
Lemma 26 Suppose $M$ is a maximal monotone operator on a Banach space $X$. Let $C = \text{co dom } M$ and $x \in \text{dom } M$. Then

$$0^+ M(x) = N_C(x).$$ (34)

In the next section in Proposition 34 we will define a maximal monotone operator $Q_A$ on a finite-dimensional space $Y$ using two maximal monotone operators $T$ and $M$ on a nonreflexive space $X$. It is shown that $Q_A$ can only possibly differ from $(T + M)_A$ at boundary points of its domain. The next two results resolve the issue as to equality on the whole domain and finally allow the demonstration of the main sum theorem of the paper. In essence, we show that the maximal monotone set $Q_A$ can be reconstructed from information within the relative interior of its domain along with the addition of certain normal cones.

Proposition 27 Suppose $Q : Y \rightrightarrows Y^*$ is a maximal monotone operator with $\text{int dom } Q \neq \emptyset$ and suppose $Y$ is finite-dimensional. Let $K(y) := Q(y)$ for all $y \in \text{int dom } Q$ and for each $y \in \overline{\text{dom } Q} \cap (\text{int dom } Q)^c$ define

$$K(y) := \text{co} \left( \limsup_{y' \in \text{int dom } Q} Q(y') \right) + N_{\text{dom } Q}(y),$$

with the usual interpretation of $A + B = \emptyset$ if either $A = \emptyset$ or $B = \emptyset$. Then $K$ is maximal monotone and coincides with $Q$.

Proof. As $Q$ is maximal we know that $\text{int dom } Q$ is a convex set [4]. Suppose $(y_0, y_0')$ is monotonically related to $K$. First note by [5, Cor. 16] that $Q$ is maximal monotone locally. Thus when $y_0 \in \text{int dom } Q$ we may choose a neighbourhood $B_{\varepsilon}(y_0) \subseteq \text{int dom } Q$ and hence, as $(y_0, y_0')$ is monotonically related to $Q|_{B_{\varepsilon}(y_0)}$ we must have $y_0' \in Q(y_0) = K(y_0)$.

This leaves the situation where $y_0 \in \overline{\text{dom } Q} \cap (\text{int dom } Q)^c$. Consider two cases, the first when

$$\limsup_{y \in \text{int dom } Q} Q(y) \neq \emptyset.$$

Then, there exists $\bar{r} > 0$ such that $Q(y) \cap B_{\bar{r}}(0) \neq \emptyset$ for all $r \geq \bar{r}$, and the following (standard) argument suffices. As $Y$ is finite-dimensional and $K(y_0)$ is closed, by the Separation theorem and $y_0' \notin K(y_0)$ there exists $u \in X$ and $\alpha$ such that

$$\langle y^*, u \rangle < \alpha < \langle y_0^*, u \rangle,$$

for all $y^* \in K(y_0)$.

As $N_{\text{dom } Q}(y_0) \subseteq K(y_0)$ it follows that $\langle u^*, u \rangle \leq 0$ for all $u^* \in N_{\text{dom } Q}(y_0)$ and so $u \in T_{\text{dom } Q}(y_0)$ (the tangent cone at $y_0$). Let $\alpha < \beta < \langle y_0^*, u \rangle$.

Let $W := \{x^* \mid \langle x^*, u \rangle < \alpha \}$ be an open set containing $K(y_0)$. By definition of $K$ (and the coincidence of Kuratowski-Painleve and epi-distance convergence within finite dimensions), for each $r > \bar{r}$ there exists a neighbourhood $B_{\varepsilon}(y_0)$
such that $K(y) \cap B_r(0) \subseteq W$ for all $y \in B_c(y_0)$. Since $u \in T_{\dom Q}(y_0)$ there exists a $t > 0$ and $v \in V := \{x \mid (y_0, x) > 0\}$ such that $\|u - v\| < (\beta - \alpha)/r$ and $y + tv \in \dom Q \cap B_r(y_0)$. Then $K(y + tv) \cap B_r(0) \subseteq W$ and for all $u^* \in K(y + tv) \cap B_r(0) \subseteq W$ we have

$$0 \leq \langle y^*_0 - u^*, y - y - tv \rangle = -t\langle y^*_0 - u^*, v \rangle \quad \text{implying} \quad \langle y^*_0, v \rangle \leq \langle u^*, v \rangle.$$  

Thus

$$\langle u^*, w \rangle = \langle u^*, v \rangle + \langle u^*, u - v \rangle \geq \langle y^*_0, v \rangle - \|u^*\| \|u - v\| \geq \beta - r (\beta - \alpha)/r = \alpha$$

implying $u^* \notin W$, a contradiction. Thus, $y^*_0 \in K(y_0)$.

This leaves only the case when $K(y_0) = \emptyset$. We shall show that $y_0 \not\in \dom Q$. Once this is shown we have $K|_{\dom Q}$ maximal relative to $\dom Q$ and as we clearly have $K \subseteq Q$ it follows that $K(x) = Q(x)$ for all $x \in \dom Q$. The maximality of $Q$ relative to $Y$ implies is $K$ maximal relative to $Y$ as well. To this end suppose $y_0 \in \dom Q$ (and hence $y_0 \in \dom Q \cap (\int \dom Q)^\circ$), take $y^*_0 \in Q(y^*_0)$ and a sequence $y_n \in \int_Y \dom Q$ such that $\frac{y_n - y_0}{\|y_n - y_0\|} \rightarrow z \in \int T_{\dom Q}(y_0)$ (the latter being nonempty due to the assumption that $\int \dom Q \neq \emptyset$). Then by the monotonicity of $Q$ we have for all $n$,

$$\langle y_n - y_0, y^*_0 - y^*_0 \rangle \geq 0 \quad \text{implying} \quad \langle \frac{y_n - y_0}{\|y_n - y_0\|}, \frac{y^*_n - y^*_0}{\|y^*_n - y^*_0\|} \rangle \geq 0.$$  

On taking subsequences we may assume $\frac{y^n - y_0}{\|y^n - y_0\|} \rightarrow z^*$ and hence

$$\langle z, z^* \rangle \geq 0 \quad \text{for some} \quad z \in \int T_{\dom Q}(y_0). \quad \text{(35)}$$

On the other hand by maximality of $Q$

$$\langle y - y_n, y^*_n - y^*_n \rangle \geq 0 \quad \text{for all} \quad (y, y^*_n) \in Q$$

implying

$$\lim_{n} \langle y - y_n, \frac{y^*_n - y^*_n}{\|y^*_n - y^*_n\|} \rangle \geq 0 \quad \text{or} \quad \langle y - y_0, -z^* \rangle \geq 0 \quad \text{for all} \quad y \in \dom Q. \quad \text{(36)}$$

Consequently, $z^* \in N_{\dom Q}(y_0)$ contradicting (35).

As we are in finite dimensions we may recast the last result.

**Corollary 28** Suppose $Q : Y \rightrightarrows Y^*$ is a maximal monotone operator and suppose $Y$ is finite-dimensional. Let $K(y) := Q(y)$ for all $y \in \ri \dom Q$ and for each $y \in (\dom Q) \cap (\ri \dom Q)^\circ$ define

$$K(y) := \overline{\text{co}} \left( \limsup_{y^* \in \ri \dom Q, y^* \rightharpoonup y} Q(y^*) \right) + N_{\dom Q}(y). \quad \text{(37)}$$

Then $K = Q$. 

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Suppose that \( A \) span \( \text{dom} \) (larger) monotone operator. Consequently, that is finite-dimensional it is complemented in \( X \). As \( Y \) is finite-dimensional it is complemented in \( X \). Without loss of generality we may assume \( 0 \in \text{dom} \; Q \). Let \( Z := \text{span} \; \text{dom} \; Q \). Then relative to \( Z \) we have \( \text{int}_Z \; \text{dom} \; Q \neq \emptyset \) (and \( \text{int}_Z \; \text{dom} \; Q \) is convex due to maximality of \( Q \)).

Let \( A : Z \to Y \) be the embedding of \( Z \) into \( Y \) and note that \( \text{ri} \; (\text{Pr}_X \; \text{dom} \; F_Q) \cap \) Range \( A \neq \emptyset \) so we may apply Theorem 19 to obtain maximality of \( Q_A := A^* \circ Q \circ A \). Apply Proposition 27 to \( Q_A \) to obtain maximality of \( K \) (as defined on \( Z \)). We have \( K = Q \cap (Z \times Z^*) \), and we note, since \( \text{dom} \; Q \subseteq Z \), that \( K \) defined in (37) only differs from \( K \) by the addition of the annihilator \( Z^\perp \). Hence, \( K \) extends \( K \) from \( Z \) to \( Y \) as a maximal monotone operator. Consequently, \( Q \subseteq K \) and the maximality of \( Q \) implies \( Q = K \).

We finish this section by studying the relationship between the normal cones to \( \text{dom} \; M \) and those to \( (\text{dom} \; M) \cap Y \) as the domain of \( M_A \) is equal to \( (\text{dom} \; M) \cap Y \) for any maximal monotone operator \( M : X \to X^* \).

**Lemma 29** Suppose \( M \) is a maximal monotone operator on a Banach space \( X \).

Suppose that \( Y \) is a finite dimensional subspace of \( X \) and let \( \tilde{y} \in \text{bd} \; (\text{dom} \; M \cap Y) \).

Then

\[
M_A (\tilde{y}) = M_A (\tilde{y}) + (N_{\text{dom} \; M \cap Y})_A (\tilde{y})
\]

and

\[
N_{\text{dom} \; M \cap Y} (\tilde{y}) = (N_{\text{dom} \; M})_A (\tilde{y}),
\]

where \( (N_{\text{dom} \; M})_A (\tilde{y}) := (A^* \circ N_{\text{dom} \; M} \circ A) (\tilde{y}) \).

**Proof.** Maximality of \( M \) implies that \( M = M + N_{\text{dom} \; M} \), since the latter is a (larger) monotone operator. Consequently,

\[
M_A (\tilde{y}) = (A^* \circ M \circ A) (\tilde{y}) = (A^* \circ M \circ A) (\tilde{y}) + (A^* \circ N_{\text{dom} \; M} \circ A) (\tilde{y}) = M_A (y) + (N_{\text{dom} \; M})_A (\tilde{y}).
\]

Observe that both cones in (38) reside in \( Y^* \) and that \( \tilde{y} \in Y \), giving \( A\tilde{y} = \tilde{y} \). When \( y^* \in N_{\text{dom} \; M \cap Y} (\tilde{y}) \) we have

\[
\langle y^*, x - \tilde{y} \rangle \leq 0 \quad \text{for all} \quad x \in (\text{dom} \; M) \cap Y.
\]

As \( Y \) is finite-dimensional it is complemented in \( X \). We may thus write \( X = Y \oplus Z \) with \( Z \) closed. Extend \( y^* \) to \( x^* \in X^* \) via linearity by placing \( \langle x^*, x \rangle = 0 \) for \( x \in Z \) and \( \langle x^*, y \rangle = \langle y^*, y \rangle \) for \( y \in Y \). Then we have

\[
\langle x^* , x - \tilde{y} \rangle \leq 0 \quad \text{for all} \quad x \in \text{dom} \; M.
\]

That is \( x^* \in N_{\text{dom} \; M} (\tilde{y}) \) with \( x^*|_Y = y^* \) and we have

\[
y^* = A^* x^* \in (A^* \circ (N_{\text{dom} \; M}) \circ A) (\tilde{y}) = (N_{\text{dom} \; M})_A (\tilde{y}),
\]

establishing \( N_{\text{dom} \; M \cap Y} (\tilde{y}) \subseteq (N_{\text{dom} \; M})_A (\tilde{y}) \).

Now take \( y^* \in (N_{\text{dom} \; M})_A (\tilde{y}) \) then there exists \( x^* \in N_{\text{dom} \; M} (A\tilde{y}) \) with \( y^* = A^* x^* \) and so \( \langle x - A\tilde{y}, x^* \rangle \leq 0 \) for all \( x \in \text{dom} \; M \). Thus we also have
\[ \langle x - Ay, x^* \rangle \leq 0 \text{ for all } x \in \text{dom} \, M \text{ and consequently for all } x = Ay' \in \text{dom} \, M \cap \hat{Y} \]
we have
\[ \langle Ay' - Ay, x^* \rangle = \langle y' - y, A^*x^* \rangle = \langle y' - y, y^* \rangle \leq 0 \text{ for all } y' \in \text{dom} \, M \cap \hat{Y}. \]
It follows that \( \langle y' - y, y^* \rangle \leq 0 \) for all \( y' \in \text{dom} \, M \cap \hat{Y} \) or \( y^* \in N_{\text{dom} \, M \cap \hat{Y}}(y) \).

6 Conditions for Maximality of a Sum of Maximal Monotone Operators

As promised, we will now study when the sum of two maximal monotone operators is maximal. There are important outstanding problems in nonreflexive spaces regarding the type of qualification that ensures maximality of a sum of maximal monotone operators. The following analysis is motivated by the desire to prove a generalization of the classic result of Rockafellar [18]. Consequently, we deal with this problem in some detail.

In [15] the following was observed. Let \( f, g : X \times X^* \to \mathbb{R} \) be proper, closed convex functions and

\[
(f \square_2 g)(x, x^*) := \inf \{ g(x, y_1^*) + h(x, y_2^*) \mid x^* = y_1^* + y_2^*, y_1^*, y_2^* \in X^* \}.
\]

Then
\[
(f \square_2 g)^*(x^*, x^{**}) \leq (f^* \square_1 g^*)(x^*, x^{**}) := \inf \{ f^*(x_1^*, x^{**}) + g^*(x_2^*, x^{**}) \mid x_1^* + x_2^* = x^*, x_1^*, x_2^* \in X^* \}.
\]

(39)

In [15] the problem of proving \( T + M \) is maximal in a reflexive space is reduced to questions regarding equality in (39) and the nature of the conjugates \( f^* \) and \( g^* \). When we use \( f = F_M \) for a maximal monotone operator then by Corollary 12 we have \( f^*(x^*, x^{**}) = \hat{F}_M(x^*, x^{**}) \geq \langle x^{**}, x^* \rangle \) on \( X^{**} \times X^* \). Let \( \hat{x} = J_X(x) \in X^{**} \) and recall that such inequalities imply a corresponding one in the embedded space \( X \times X^* \) in that \( f^*(x^*, x) = F_M(x, x^*) = \hat{F}_M(x^*, \hat{x}) \geq \langle x, x^* \rangle \) on \( X \times X^* \). The following is a slight generalization of a recent result in [27].

Theorem 30 Suppose \( T \) and \( M \) are maximal monotone operators and \( f \) and \( g \) are their respective representative functions. Suppose that \( f \) and \( g \) may be chosen so that equality holds in (39) for all \( (x^*, x^{**}) \in X^* \times X^{**} \) with the infimum being attained in the definition of \( (f^* \square_1 g^*) \)(\( x^*, x^{**} \)) for some \( x_1^*, x_2^* \in X^* \).

Then both \( h := (f \square_2 g) \) and \( h^* = (f^* \square_1 g^*) \) are representative functions with \( (h^*)^\dagger \) a representative function for \( T + M \).

If \( f \) and \( g \) are chosen so that \( f^*(\cdot, \cdot) \geq \langle \cdot, \cdot \rangle \) and \( g^*(\cdot, \cdot) \geq \langle \cdot, \cdot \rangle \) then \( h^*(\cdot, \cdot) \geq \langle \cdot, \cdot \rangle \) on \( X^* \times X^{**} \).
Proof. Let \( h := (f \boxempty g) \). First note that as \( f (x, y_1^*) \geq \langle x, y_1^* \rangle \) and \( g (x, y_2^*) \geq \langle x, y_2^* \rangle \) we have
\[
h(x, x^*) = \inf \{ f (x, y_1^*) + g (x, y_2^*) | x^* = y_1^* + y_2^*, y_1^*, y_2^* \in X^* \}
\[
\geq \inf \{ \langle x, y_1^* \rangle + \langle x, y_2^* \rangle | x^* = y_1^* + y_2^*, y_1^*, y_2^* \in X^* \} = \langle x, x^* \rangle
\]
and so \( f \) is a representative function. Since \( T \) and \( M \) are monotone \( \hat{f}^* (x_1^*, x^{**}) \geq \langle x_1^*, x^{**} \rangle \) and similarly \( \hat{g}^* (x_2^*, x^{**}) \geq \langle x_2^*, x^{**} \rangle \) (as noted in Corollary 12). A similar calculation shows
\[
\hat{h}^* (x^*, x^{**}) = \left( \hat{f}^* \boxempty_1 \hat{g}^* \right) (x^*, x^{**}) = \langle x^*, x^{**} \rangle.
\]
Because \( g \) and \( h \) are representative for their respective maximal monotone sets we have
\[
M_f := \{ (x, x^*) \in X \times X^* | f (x, x^*) = \langle x, x^* \rangle \}
\[
= \{ (x, x^*) \in X \times X^* | f^* (x^*, x) = \langle x^*, x \rangle \} = M_{f^*} = T.
\]
Similarly, \( M_g = M_{g^*} = M \). Let \( (x, x^*) \in M_{h^*} \). By assumption, there exists \( x_1^*, x_2^* \in X^* \) with \( x_1^* + x_2^* = x^* \) such that
\[
h^* (x^*, x) = (f^* \boxempty_1 g^*) (x^*, x) := f^* (x_1^*, x) + g^* (x_2^*, x) = \langle x^*, x \rangle = \langle x_1^*, x \rangle + \langle x_2^*, x \rangle.
\]
As \( f^* (x_1^*, x) \geq \langle x_1^*, x \rangle \) similarly \( g^* (x_2^*, x) \geq \langle x_2^*, x \rangle \) and
\[
f^* (x_1^*, x) − \langle x_1^*, x \rangle + g^* (x_2^*, x) − \langle x_2^*, x \rangle = 0
\]
we have \( f^* (x_1^*, x) = \langle x_1^*, x \rangle \) and \( g^* (x_2^*, x) = \langle x_2^*, x \rangle \). Hence \( (x, x^*) \in M_{f^*} = T \) and \( (x, x^*) \in M_{g^*} = M \) with \( x^* = x_1^* + x_2^* \in T(x) + M(x) \). Thus \( M_{h^*} \) is contained in the graph of \( T + M \) which is itself a monotone set. Suppose now that \( (x, x_1^*) \in M_{f^*} = T \) and \( (x, x_2^*) \in M_{g^*} = M \) then as \( f^* (x_1^*, x) = \langle x_1^*, x \rangle \) and \( g^* (x_2^*, x) = \langle x_2^*, x \rangle \) we have
\[
h^* (x^*, x) = (f^* \boxempty_1 g^*) (x^*, x) \leq f^* (x_1^*, x) + g^* (x_2^*, x) = \langle x_1^*, x \rangle + \langle x_2^*, x \rangle = \langle x^*, x \rangle
\]
implying \( h^* (x^*, x) = \langle x^*, x \rangle \). Thus, \( (h^*)^\dagger \) is a representative function for the sum \( T + M \).

It is shown in a Banach space in [25] that for \( g, h \) proper, lower semi-continuous convex functions we have
\[
(f \boxempty_2 g)^* (x^*, y^*) = (f^* \boxempty_1 g^*) (x^*, y^*)
\]
under the assumption that
\[
0 \in \text{sqri} (\text{Pr}_X (\text{dom } f) − \text{Pr}_X (\text{dom } g))
\]
where \( \text{Pr}_X \) is the projection onto \( X \) and we write \( 0 \in \text{sqri } S \) iff \( Z := \text{span } S \) is a closed subspace and \( 0 \in \text{core}_Z S \) (the core relative to \( Z \)).

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As to the nature of the function \( (f \square g) \), we next note that it does produce a representative function for \( T + M \) under another weak qualification assumption. Recall, that in order to show a convex set is bounded weak\(^*\) closed it is sufficient to show it contains the limits of all bounded and weak\(^*\) convergent nets taken from the set [13].

**Theorem 31** Suppose \( T \) and \( M \) are maximal monotone operators from a Banach space \( X \) to \( X^* \). Suppose in addition there exists representative functions \( g \) and \( h \) for \( T \) and \( M \) respectively such that

\[
0 \in \text{core}(\text{Pr}_X (\text{dom} \ f^*) - \text{Pr}_X (\text{dom} \ g^*))
\]

Then \( h := (f \square g) (x, x^*) \) is proper, \( s \times bw^*\)-closed. Moreover, \( h \) is exact and also is a representative function for \( T + M \).

**Proof.** For each \( x \in X \) and \( K > \inf_{x^* \in X^*} (f \square g) (x, x^*) \) let

\[
H (K, M, r) := \{ (x_1^*, x_2^*) \in X^* \times X^* \mid f (x, x_1^*) + g (x, x_2^*) \leq K \\
\text{with} \quad \| x_1^* + x_2^* \| \leq r \text{ and } \| x \| \leq M \}.
\]

We claim that there exists \( C (K, M, r, u, v) \) such that for all \((x^*, y^*) \in H (K, M, r)\) we have \((x^*, u) + (y^*, v) \leq C (K, M, r, u, v)\).

Indeed, since (41) holds \((u - v) = \lambda (a^* - b^*)\) for some \( \lambda > 0 \) and some \((a^*, a) \in \text{dom} \ g^* \) and \((b^*, b) \in \text{dom} \ h^*\). Thus \( u - v = \lambda (a^* - b)\). Then using the Fenchel inequality

\[
(x^*, u) + (y^*, v) = \lambda (x^*, a) + \lambda (y^*, b) + \langle x^* + y^*, v - \lambda b \rangle
\]

\[
= \lambda ((x^*, a) + \langle x, a^* \rangle) + \lambda ((y^*, b) + \langle x, b^* \rangle)
\]

\[
+ \langle x^* + y^*, v - \lambda b \rangle - \lambda (x, a^* + b^*)
\]

\[
\leq \lambda (f^*(a^*, a) + f(x, x^*) + g^*(b^*, b) + g(x, y^*))
\]

\[
+ \| x^* + y^* \| \| v - \lambda b \| + \lambda M \| a^* + b^* \|
\]

\[
\leq \lambda (K + f^*(a^*, a) + g^*(b^*, b) + M \| a^* + b^* \|)
\]

\[
+ r \| v - \lambda b \| := C (K, M, r, u, v).
\]

Note that this bound only depends on \( x \) in so far as we require that the inequalities \( \| x \| \leq M \) and \( K \geq (f \square g) (x, x^*) \) are satisfied.

Thus, by the uniform boundedness principle we have \( H (K, M, r) \) contained in a ball whose radius depends only on the choice of \( K \), \( M \) and \( r \). As \( H (K, M, r) \) is clearly closed it is weak\(^*\) compact. We now show that the level sets of \((f \square g)\) are \( s \times bw^*\)-closed. Take \((x_\beta, x_\beta^*) \in \{ (x, x^*) \mid (f \square g) (x, x^*) \leq K \} \) such that \( \{ x_\beta^* \} \) is a bounded weak\(^*\) convergent net converging to \( x^* \) and \( x_\beta \) strongly converging to \( x \). Let \( \varepsilon_\beta > 0 \) with \( \varepsilon_\beta \downarrow 0 \) and \( r > 0 \) such that \( \max \{ \varepsilon_\beta, \| x_\beta^* \| \} \leq r \) along with \( \| x_\beta \| \leq 2 \| x \| := M \)

\[
(u_\beta^*, v_\beta^*) \in H (K + \varepsilon_\beta, M, r) \quad \text{with} \quad u_\beta^* + v_\beta^* = x_\beta^*
\]
and by the previous observation we have $\| (u^\star_{\beta}, v^\star_{\beta}) \|$ uniformly bounded for all $\beta$. Thus on passing to a subnet, we may assume $\left( u^\star_{\beta}, v^\star_{\beta} \right) \xrightarrow{w^\star} (u^*, v^*)$ for some $u^*, v^* \in X^*$ with $u^* + v^* = x^*$. Then, since $x^\beta \rightarrow x^*$, we conclude

$$(g \Box_2 h)(x, x^*) \leq f(x, x^* - v^*) + g(x, v^*)$$

$$\leq \liminf_{\beta} g\left( f(x_\beta, x^\beta_\beta - v^\beta_{\beta}) + \liminf_{\beta} g(x_\beta, v^\beta_{\beta}) \right)$$

$$\leq \liminf_{\beta} \left( f(x_\beta, x^\beta_\beta - v^\beta_{\beta}) + g(x_\beta, v^\beta_{\beta}) \right) \leq \liminf (K + \varepsilon_{\beta}) = K.$$ 

Thus the level-set of $f \Box_2 g$ is $s^\star$-bw$^\star$-closed and so $(x, x^*) \mapsto (f \Box_2 g)(x, x^*)$ is $s \times bw^\star$-lower semi–continuous.

To verify properness, suppose to the contrary that $(f \Box_2 g)(x^*) = -\infty$ for some $(x, x^*)$. Then $\lim_{\beta} (f(x_\beta, x^* - v^\beta_{\beta}) + g(x_\beta, v^\beta_{\beta})) = -\infty$ for some net $v^\beta_{\beta}$. Without loss of generality we may assume $\lim_{\beta} (f(x_\beta, x^* - v^\beta_{\beta}) + g(x_\beta, v^\beta_{\beta})) \leq 0$ for all $\beta$ and since $(x^* - v^\beta_{\beta}, v^\beta_{\beta}) \in H(0, \|x^*\|, x^*_\beta)$ with this set being uniformly bounded, we have $v^\beta_{\beta} \xrightarrow{w^\star} v^*$ on taking a subnet. Then

$$-\infty < f(x, x^* - v^*) + g(x, v^*) \leq \liminf_{\beta} f(x_\beta, x^* - v^\beta_{\beta}) + \liminf_{\beta} g(x_\beta, v^\beta_{\beta})$$

$$\leq \liminf_{\beta} \left( f(x_\beta, x^* - v^\beta_{\beta}) + g(x_\beta, v^\beta_{\beta}) \right) = -\infty,$$

a contradiction, and so $(f \Box_2 g)$ is proper.

Next, to show $(f \Box_2 g)(x, x^*)$ is exact, we take $K = (f \Box_2 g)(x, x^*)$ and $\varepsilon_{\beta} > 0$ with $\varepsilon_{\beta} \downarrow 0$. Then there exists

$$(u^\star_{\beta}, v^\star_{\beta}) \in H(K + \varepsilon_{\beta}, \|x\|, x) \quad \text{with} \quad u^\star_{\beta} + v^\star_{\beta} = x^*.$$

As $\left\| (u^\star_{\beta}, v^\star_{\beta}) \right\|$ is bounded, on taking $w^\star$-convergent subnet we may assume

$$\left( u^\star_{\beta}, v^\star_{\beta} \right) \rightarrow (u^*, v^*)$$

with $u^* + v^* = x^*$ and so

$$(f \Box_2 g)(x, x^*) = f(x, x^* - v^*) + g(x, v^*),$$

giving the exactness of $(f \Box_2 g)$.

Finally

$$\{(x, x^*) \mid (f \Box_2 g)(x, x^*) = \langle x, x^* \rangle \}$$

$$= \{(x, x^*) \mid \exists v^* \in X^* \text{ such that } f(x, x^* - v^*) + g(x, v^*) = \langle x, x^* \rangle \}$$

$$= \{(x, x^*) \mid \exists v^* \in X^* \text{ such that } f(x, x^* - v^*) \rangle x, x^* - v^* \langle + g(x, v^*) \rangle x, v^* \rangle 0 \}$$

$$= \{(x, x^*) \mid \exists v^* \in X^* \text{ such that } f(x, x^* - v^*) = \langle x, x^* - v^* \rangle \text{ and } g(x, v^*) = \langle x, v^* \rangle \}$$

$$= \{(x, x^*) \mid \exists v^* \in X^* \text{ such that } x^* - v^* \in T(x) \text{ and } v^* \in M(x) \}$$

$$= \{(x, x^*) \mid x^* \in (T + M)(x) \}.$$
The previous result gives us a way of deducing that \((g \Box h)\) is a representative function for \(T + M\) which is not provided directly by Theorem 30 under the assumption of (41). In order to obtain a sum theorem for two maximal monotone operators we need the following chain of results that investigates the nature of a particular representative function \(H (y, y^*) : = \inf \{ h (Ay, x^*) \mid A^* x^* = y^* \}\) (with \(h : = (f \Box g)\)) and its conjugate. Note that under the qualification assumption (41) we have \(h\) a representative function for \(T + M\) and so for all \((y, y^*)\) we have

\[
H (y, y^*) = \inf \{ h (Ay, x^*) \mid A^* x^* = y^* \} \\
\geq (Ay, x^*) = \langle y, A^* x^* \rangle = \langle y, y^* \rangle.
\]

Consequently \(H\) may be used to define a monotone set \(Q_A\). Note also that in order that \(y \in \Pr_X \dom H \) there must exist \(x^*\) such that \(+\infty > h (Ay, x^*)\) implying

\[
Ay \in \Pr_X \dom h \quad \text{or} \quad y \in \Pr_X \dom h \cap \hat{Y}.
\]

Hence under the assumption of (41) and the exactness of the infimal convolution

\[
\Pr_X \dom H = \Pr_X \dom h \cap \hat{Y} = \Pr_X \dom f \cap \Pr_X \dom g \cap \hat{Y}.
\]

Finally note that when the qualification assumption (41) holds, as \(h\) is a representative function for \(T + M\), for \(x^* \in (T + M)(Ay)\) with \(y^* = A^* x^*\) we have

\[
H (y, y^*) = \inf \{ h (Ay, x^*) \mid A^* x^* = y^* \} \\
= (Ay, x^*) = \langle y, A^* x^* \rangle = \langle y, y^* \rangle
\]

and so \((y, y^*) \in Q_A\). Consequently we always have

\[
(T + M)_A \subseteq Q_A.
\]

**Proposition 32** Let \(A : Y \to X\) be a continuous linear operator between two Banach spaces. Let \(f, g : X \times X^* \to \overline{\mathbb{R}}\) be proper, lower semi–continuous function with

\[
\emptyset \neq \text{Range } A \cap \text{[core } \Pr_X (\dom f)\] \cap \Pr_X (\dom g).
\]

Then for \(h : = (f \Box g)\) and \(H (y, y^*) : = \inf \{ h (Ay, x^*) \mid A^* x^* = y^* \}\) we have

\[
H^* (y^*, y) = (F^* \Box G^*) (y^*, Ay)
\]

for all \((y^*, y) \in Y^* \times Y\), where \(F (y, x^*) = f (Ay, x^*)\) and \(G (y, x^*) = g (Ay, x^*)\). Moreover, the infimal convolution is exact.

**Proof.** We have

\[
H^* (v^*, v) = \sup_{(y, y^*)} \left\{ \langle y, v^* \rangle + \langle y^*, v \rangle - \inf_{x^*} \{ h (Ay, x^*) \mid A^* x^* = y^* \} \right\}
\]

\[
= \sup_{(x^*, y^*)} \{ \langle y, v^* \rangle + \langle y^*, v \rangle - h (Ay, x^*) \mid A^* x^* = y^* \}
\]

\[
= \sup_{(x^*, y^*)} \{ \langle y, v^* \rangle + \langle y^*, v \rangle \\
- \inf \{ f (Ay, x_1^*) + g (Ay, x_2^*) \mid x_1^* + x_2^* = x^* \} \mid A^* x^* = y^* \}
\]

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where we have used the Lagrange multiplier rule [7, Cor. 4.4.4] to absorb the constraint $A^*x_1^* + A^*x_2^* - y^* = 0$ into the minimization. This is valid if

$$V(x_1^*, x_2^*, y, y^*) := f(Ay, x_1^*) + g(Ay, x_2^*) - (y, v^*) - \langle y^*, v \rangle$$

is lower semicontinuous and under the condition that $0 \in \text{core}(A(\text{dom } V))$, where $A(x_1^*, x_2^*, y, y^*) := A^*x_1^* + A^*x_2^* - y^* \in Y^*$. This condition is verified when $\text{dom } V \neq \emptyset$, which is implied by (45) because

$$A^{-1}(\text{Pr}_X(\text{dom } f)) \cap A^{-1}(\text{Pr}_X(\text{dom } g)) = \text{Pr}_Y(\text{dom } F) \cap \text{Pr}_Y(\text{dom } G) \neq \emptyset.$$

Continuing

$$H^*(v^*, v) = \max_{y \in Y} \inf_{(x_1^*, x_2^*, y, y^*)} \{ f(Ay, x_1^*) + \langle Ay, x_1^* \rangle + g(Ay, x_2^*) + \langle Ay, x_2^* \rangle - \langle y, v^* \rangle - \langle y^*, v + y \rangle \}$$

$$= \max_{\tilde{y} \in Y} \inf_{(x_1^*, y, y^*)} \{ \inf_{x_2^* = x^*} \{ f(Ay, x_1^*) + g(Ay, x_2^*) \} - \langle -Ay, x^* \rangle - \langle y, v^* \rangle - \langle y^*, v + y \rangle \}$$

$$= \min_{y^* \in Y^*} \sup_{(x^*, y)} \left\{ \sup_{(x^*, y)} \{ -\langle Ay, x^* \rangle + \langle y, v^* \rangle - \inf_{x_1^* + x_2^* = x^*} \{ f(Ay, x_1^*) + g(Ay, x_2^*) \} \} + \langle y^*, v + y \rangle \right\}.$$

Let $F(y, x^*) := f(Ay, x^*)$ and $G(y, x^*) := g(Ay, x^*)$. Now use [25, Thm 4.1] to evaluate

$$\sup_{(x^*, y)} \left\{ -\langle Ay, x^* \rangle + \langle y, v^* \rangle - \inf_{x_1^* + x_2^* = x^*} \{ F(y, x_1^*) + G(y, x_2^*) \} \right\}$$

$$= \min_{y_1^* + y_2^* = y^*} \{ F^*(y_1^*, -Ay) + G^*(y_2^*, -Ay) \} = (F^* \square I_G^*) (v^*, -Ay)$$

under the assumption that

$$0 \in \text{sqri} (\text{Pr}_Y(\text{dom } F) - \text{Pr}_Y(\text{dom } G)). \quad (46)$$

Now $y \in \text{Pr}_Y(\text{dom } F)$ iff there exists $x^{**}$ such that $(Ay, x^{**}) \in \text{dom } f$ iff $y \in A^{-1}(\text{Pr}_X(\text{dom } f)$ and so (46) is implied by

$$0 \in \text{sqri} \left[ A^{-1}(\text{Pr}_X(\text{dom } f) - A^{-1}(\text{Pr}_X(\text{dom } g) \right].$$
Suppose \( \dom dom \) Theorem 33 theorem outlines conditions that imply a sum theorem. Where \( F \) defining \( \hat \) respectively. In the following we will extending each Proof. By Theorem 19 we know that

\[
Z := A^{-1}(X) = \text{cone } A^{-1}(\text{Pr}_X(\dom f) - z) = \text{cone } [A^{-1}(\text{Pr}_X(\dom f)) - x]
\]

\[
\subseteq \text{cone } [A^{-1} \text{Pr}_X(\dom f) - A^{-1} \text{Pr}_X(\dom g)] \subseteq \text{cone } A^{-1}(X) = Z,
\]

due the attainment of the minimum at \( \hat y = -v \).

To invoke the theorem [15, Prop. 1] or of [3] that ensures maximality of \( Q_A \) (on the reflexive space \( Y \)) we need to investigate the conjugate of \( H \). The next theorem outlines conditions that imply a sum theorem.

**Theorem 33** Suppose \( T \) and \( M \) are maximal monotone operators on a Banach space \( X \) and that \( A : Y \to X \) is an embedding of a finite-dimensional subspace \( \hat Y \subseteq X \) into \( X \). In addition suppose the following condition hold:

\[
\emptyset \neq \hat Y \cap \text{int dom } T \cap \text{co dom } M. \tag{47}
\]

Let \( h := (f \square 2g) \) and \( H(y, y^*) := \inf_{x^*} \left\{ h \left( Ay, x^* \right) \mid A^* x^* = y^* \right\} \) with \( f = \mathcal{F}_T \), \( g = \mathcal{F}_M \). Then

\[
H^*(y, y^*) \geq \langle y, y^* \rangle \quad \text{for all } (y, y^*) \in Y \times Y^*. \tag{48}
\]

Consequently \( Q_A \) is maximal on \( Y \) and \( \text{dom } Q_A \) is convex.

**Proof.** By Theorem 19 we know that \( T_A := A^* \circ T \circ A \) is maximal since \( \text{core } (\text{Pr}_X \text{ dom } \mathcal{F}_T) \cap \text{Range } A \neq \emptyset \). Appealing to Corollary 25 we have for all \( (y^*, y) \in Y^* \times Y \) that \( F^*(y^*, Ay) \geq \mathcal{F}_{A \circ T \circ A} (y^*, y) \geq \langle y, y^* \rangle \). Note that we do not know \textit{a priori} that \( M_A := A^* \circ M \circ A \) is maximal, when \( A \) is the embedding of the finite-dimensional subspace \( Y \) into \( X \). As \( Y \) is finite dimensional we know that \( Y \) is complemented by \( Z \), a closed subspace, and so there exist continuous projections onto \( Y \) and \( Z \) which we denote by \( P_X : X \to Y \) and \( P_Z : X \to Z \), respectively. In the following we will extending each \( y^* \in Y^* \) to \( \tilde y^* \in X^* \) by defining \( \tilde y^* (x) = y^* (x) \) if \( x \in Y \) and \( \tilde y^* (x) = 0 \) if \( x \in Z \) and then extending via linearity to the whole space. Now use Proposition 32 and so we have for some \( y_1^*, y_2^* \in Y \) such that \( y^* = y_1^* + y_2^* \)

\[
\begin{align*}
H^* (y, y^*) &= (F^* \square_1 G^*) (y^*, Ay) \\
&= F^* (y_1^*, Ay) + G^* (y_2^*, Ay)
\end{align*}
\]

where \( F(y, x^*) = \mathcal{F}_T \left( Ay, x^* \right) \) and \( G(y, x^*) = \mathcal{F}_M \left( Ay, x^* \right) \).
Now consider the function $H(x^*, y) := \left(\hat{F} \square_2 \hat{G}\right)(y, x^*)$ where $\hat{F}(y, x^*) = F_T(P_Y(y), x^*)$ and $\hat{G}(y, x^*) = F_M(P_Y(y), x^*)$. By the results of [25] that under the qualification assumption

$$0 \in \text{core} \left[\left(P_{TY} \text{ dom } \hat{F} - P_{TY} \text{ dom } \hat{G}\right) + Z\right]$$

we have

$$H^* (y^*, x^{**}) := \left(\hat{F}^{**} \square_1 \hat{G}^{**}\right)(y^*, x^{**}) : X^* \times X^{**} \to \mathbb{R},$$

where the latter infimal convolution is exact and so $H^*$ is proper convex. Now $P_{TY} \text{ dom } F = P_{TY} \text{ dom } F_T \cap \hat{Y}$ along with $\text{int}_Y P_{TY} \text{ dom } F_T \neq \emptyset$ due to (47). Thus within $Y$ we have

$$Y = \text{cone}_Y \left(P_{TY} \text{ dom } \hat{F} - P_{TY} \text{ dom } \hat{G}\right)$$

verifying (50). Restricting to $\hat{X} \subseteq X^{**}$ we now consider the mapping

$$(y^*, x) \mapsto H^* (y^*, x) := H^* (y^*, \hat{x}).$$

Next we note that

$$(\hat{y}^*, z) = (\hat{y}^*, P_Y(z)) = (y^*, P_Y(z))$$

because $z = P_Y(z) + P_Z(z)$ and hence

$$\hat{F}^*(\hat{y}^*, x) = \sup_{(z, z^*) \in X \times X^*} \left\{\langle \hat{y}^*, z \rangle + \langle x, z^* \rangle - \hat{F}(z, z^*)\right\}$$

$$= \sup_{(z, z^*) \in X \times X^*} \left\{\langle \hat{y}^*, z \rangle + \langle x, z^* \rangle - F_T(P_Y(z), z^*)\right\}$$

$$= \sup_{(P_Y(z), z^*) \in Y \times X^*} \left\{\langle \hat{y}^*, P_Y(z) \rangle + \langle x, z^* \rangle - F_T(P_Y(z), z^*)\right\}$$

$$= \sup_{(z, z^*) \in Y \times X^*} \left\{\langle y^*, z \rangle + \langle x, z^* \rangle - F_T(Ay, z^*)\right\} = F^*(y^*, x).$$

From (49) we have $H^*(y, y^*) = H^* (\hat{y}^*, Ay)$ for all $(y, y^*) \in Y \times Y^*$. Using the exactness of the convolution

$$H^* (y^*, x) = F^*(y^*_1, x) + G^*(y^*_2, x)$$

$$= (F^*(y^*_1, x) - (y^*_1, x)) + (y^*, x) - (y^*_2, x) + G^*(y^*_2, x)$$

$$\geq (y^*, x) + F^*(y^*_1, x) - (y^*_1, x) - \sup_{y^*_2 \in X^*} \left\{\langle y^*_2, x \rangle - G^*(y^*_2, x)\right\}$$

$$= (y^*, x) + (F^*(y^*_1, x) - (y^*_1, x)) - \sup_{y^*_2 \in X^*} \left\{\langle y^*_2, P_Y(x) \rangle - G^*(y^*_2, x)\right\}$$

$$= (y^*, x) + (F^*(y^*_1, x) - (y^*_1, x)) - (G^*(\cdot, x))^* (P_Y(x))$$

(51)

where we denote

$$(G^*(\cdot, x))^* (P_Y(x)) = \sup_{y^* \in X^*} \left\{\langle y^*, P_Y(x) \rangle - G^*(y^*, x)\right\}.$$
Now use Lemma 22 with \( f = \mathcal{F}_M \) and \( p = \mathcal{P}_M \) with \( h(u, v) := -(\mathcal{P}_M (u, \cdot))^* (v) \) (and a bounded linear operator \( P_Y : X \rightarrow X \)) to obtain via (31) that
\[
G^* (y^*, x) = \sup_{z \in X} \left\{ (z, y^*) + (h(\cdot, P_Y(z)))^* (x) \right\}
= \sup_{z \in X} \{ (z, y^*) - k(z, x) \}
\]
where \( k(z, x) := - (h(\cdot, P_Y(z)))^* (x) \). Consequently
\[
(G^* (\cdot, x))^* (P_Y(x)) = (k(\cdot, x))^* (P_Y(x)) \leq k(P_Y(x), x) = -(h(\cdot, P_Y(x)))^* (x), \tag{52}
\]
where \((h(\cdot, P_Y(x)))^* (x)\) is the lower semi–continuous hull of the function
\[
u \mapsto h(u, P_Y(x)) = - (\mathcal{P}_M (u, \cdot))^* (P_Y(x))
= \inf_{w^*} \{ \mathcal{P}_M (u, w^*) - \langle w^*, P_Y(x) \rangle \}. \tag{53}
\]
Consequently
\[
\mathcal{H}^* (y^*, x) \geq \langle \hat{y}^*, x \rangle + (F^* (g_1^*, x) - \langle \hat{y}_1^*, x \rangle) - (k(\cdot, x))^* (P_Y(x)) \tag{54}
\geq \langle \hat{y}^*, x \rangle + (F^* (g_1^*, x) - \langle \hat{y}_1^*, x \rangle) + (h(\cdot, P_Y(x)))^* (x) \tag{55}
\]
By Lemma 23 if \( u \notin \Pr_X \text{dom} \mathcal{P}_M \) then \( \mathcal{P}_M (u, w^*) = +\infty \) for all \( w^* \) and hence \( h(u, P_Y(x)) = +\infty \). In particular \((h(\cdot, P_Y(x)))^* (x) = +\infty \) for all \( x \notin \text{ccdom} M \). Now consider \( x, w \in \Pr_X \text{dom} \mathcal{P}_M \) and on writing out the closure operations in detail we have (with limits taken with respect to the strong topology)
\[
(k(\cdot, x))^* (w) = \lim_{z \rightarrow w} \left( \inf_{i \rightarrow x} (k(z, x)) \right)
= \lim_{z \rightarrow w} \left( \inf_{i \rightarrow x} \left( \lim_{u \rightarrow z} (\mathcal{P}_M (u, \cdot))^* (z) \right) \right)
= \lim_{z \rightarrow w} \left( \lim_{u \rightarrow x} (\mathcal{P}_M (u, \cdot))^* (z) \right)
= \inf_{\{z \rightarrow w\}} \left( \lim_{u \rightarrow x} (\mathcal{P}_M (u, \cdot))^* (z) \right)
\leq \inf_{\{z \rightarrow w\}} \lim_{u \rightarrow x} (\mathcal{P}_M (u, \cdot))^* (z) = \varepsilon \lim_{u \rightarrow x} (\mathcal{P}_M (u, \cdot))^* (w). \tag{56}
\]
As \( w \mapsto (k(\cdot, x))^* (w) \) is proper and lower semi–continuous convex we may take the conjugate twice and still preserve the inequality between the left and right hand sides i.e.
\[
(k(\cdot, x))^* (w) \leq \left[ e \lim_{u \rightarrow x} (\mathcal{P}_M (u, \cdot))^* \right]^* (w). \tag{57}
\]
35
Now we argue that \( e\text{-lim sup}_{u \to x} (P_M(u, \cdot))^* \) never attains \(-\infty\) (it is certainly lower semi-continuous and convex). If \( e\text{-lim sup}_{u \to x} (P_M(u, \cdot))^* \) attains \(-\infty\) then the right hand side of (57) would identically equal to \(-\infty\) forcing \((k(\cdot, x))^*(\cdot) \equiv -\infty\). But then using (54) we have \( H^* (y^*, \cdot) \equiv +\infty\), a contradiction to the properness of \( H^* \). Note also that \( P_M(u, w^*) = \epsilon_M^* (u, w^*) \) with the conjugate taken with the respect to the paired spaces \( X \times X^* \) and \( X^* \times X \) with \( X \) endowed with the strong topology and \( X^* \) endowed with the bounded weak* topology. Thus \((u, w^*) \mapsto P_M(u, w^*)\) is jointly \( s \times bw^* \) lower semi-continuous and convex. Thus

\[
\left( bw^*-e\text{-lim inf}_{u \to x} P_M(u, \cdot)(x^*) = \liminf_{u \to x, w^* \to u^*} P_M(u, w^*) = P_M(x, x^*) \right.
\]

with \( x \mapsto P_M(x, x^*) \) a closed, convex function. Thus we may invoke Theorem 48 (see the Appendix) using the parametrised families \( f_u(\cdot) := (P_M(u, \cdot))^*(\cdot) \) and \( f_u^*(\cdot) := P_M(u, \cdot) \) to deduce that

\[
\left[ e\text{-lim sup}_{u \to x} (P_M(u, \cdot))^* \right]^* (w) \leq \left[ bw^*-e\text{-lim inf}_{u \to x} P_M(u, \cdot) \right]^* (w), \tag{58}
\]

where the bounded weak* epi-limit infimum may be written as

\[
 bw^*-e\text{-lim inf}_{u \to x} P_M(u, \cdot)(x^*) = \liminf_{u \to x, v^* \to u^*} P_M(u, v^*). \tag{59}
\]

Combining this with (54), (56), (58) and (59) we obtain

\[
H^* (y^*, x) \geq \langle \hat{y}^*, \hat{x} \rangle + (F^* (y_1^*, x) - \langle \hat{y}_1^*, x \rangle) - (P_M(x, \cdot))^* (P_Y(x)). \tag{60}
\]

When \( x \in \hat{Y} \cap (\text{Pr}_X \text{ dom } P_M) \) we have \( x = P_Y(x) \) and so using \( \langle w^*, x \rangle - P_M(x, w^*) \leq 0 \) for all \( w^* \) we have

\[
(P_M(x, \cdot))^* (x) = \sup_{w^*} \{ \langle w^*, x \rangle - P_M(x, w^*) \} \leq 0. \tag{61}
\]

Combining (61) this with (51) and (52) we have for \( x \in \hat{Y} \cap (\text{Pr}_X \text{ dom } P_M) \)

\[
H^* (y^*, x) \geq \langle \hat{y}^*, \hat{x} \rangle + (F^* (y_1^*, x) - \langle \hat{y}_1^*, x \rangle).
\]

Now we use Proposition 32 to obtain \( H^* (y^*, y^*) = H^* (y^*, Ay) \) for all \( (y, y^*) \in Y \times Y^* \). Thus using Lemma 24, Corollary 25, Theorem 19 part 2 and int dom \( T \neq \emptyset \) we have

\[
H^* (y, y^*) = H^* (\hat{y}^*, Ay) \geq \langle \hat{y}^*, Ay \rangle + (F^* (y_1^*, Ay) - \langle \hat{y}_1^*, Ay \rangle)
\]

\[
\geq \langle A^* \hat{y}^*, y \rangle + (F_{A^*} \cap A^* (y_1^*, y) - \langle A^* \hat{y}_1^*, y \rangle)
\]

\[
= \langle y^*, y \rangle + (F_{A^*} (y_1^*, y) - \langle y_1^*, y \rangle) \geq \langle y^*, y \rangle
\]

whenever \( \hat{y} \in (\text{Pr}_X \text{ dom } P_M) \cap \hat{Y} \). When \( x \notin \text{Pr}_X \text{ dom } P_M \) then \( (P_M(x, \cdot))^* (P_Y(x)) = -\infty \) and via (60) we have \( H^* (y^*, x) = +\infty \). Consequently \( H^* (y, y^*) = H^* (y^*, Ay) \geq \langle y^*, y \rangle \) for all \( (y, y^*) \in Y \times Y^* \).
The final assertion follows from the observation that all maximal monotone operators on reflexive spaces have semiconvex domains and that maximality of $Q_A$ follows from the converse of (10) proved in [15].

The representative function $H$ will now be used to define a maximal monotone set $Q_A$ which we must compare with $(T + M)_A$.

**Proposition 34** Suppose $T$ and $M$ are maximal monotone operators on a Banach space $X$ and that $A : Y \to X$ is an embedding of a finite-dimensional subspace $\hat{Y} \subseteq X$ into $X$. Define a monotone mapping $Q_A$ by placing $h := (f \square 2g)$, $H (y, y^*) := \inf_{x^*} \{ \langle Ay, x^* \rangle | A^* x^* = y^* \}$ and defining the graph of $Q_A$ as

$$Q_A := \{(y, y^*) | H (y, y^*) = \langle y, y^* \rangle \}.$$  

Suppose

$$\emptyset \neq (Pr_X \text{ dom } P_M) \cap \text{ int } (Pr_X \text{ dom } P_T) \cap \hat{Y}.  \tag{62}$$

Then for all $y \in \text{ int } (\text{ dom } T) \cap \hat{Y}$ and any $y^* \in Y^*$ we have

$$Q_A (y) \subseteq (T + M)_A (y).  \tag{63}$$

**Proof.** Taking $x_0 \in (Pr_X \text{ dom } P_M) \cap \text{ int } (Pr_X \text{ dom } P_T)$ and translating the origin to $x_0$ we may assume with out loss of generality that $0 \in (Pr_X \text{ dom } P_M) \cap \text{ int } (Pr_X \text{ dom } P_T)$. Note that since the qualification assumption implies

$$0 \in \text{ core } [(Pr_X \text{ dom } P_T) - (Pr_X \text{ dom } P_M)]$$

we have $h := f \square 2g$ is exact. Then $H (y, y^*) = \langle y, y^* \rangle$ exactly when

$$0 = \inf_{(v_1^*, v_2^*)} \{ (f (Ay, v_1^*) - \langle Ay, v_1^* \rangle) + (g (Ay, v_1^*) - \langle Ay, v_2^* \rangle) | A^* v_1^* = y^* \}$$

$$= \inf_{(v_1^*, v_2^*)} \{ (f (Ay, v_1^*) - \langle Ay, v_1^* \rangle) + (g (Ay, v_1^*) - \langle Ay, v_2^* \rangle) | A^* (v_1^* + v_2^*) = y^* \},  \tag{64}$$

where both terms inside the infimum are positive. Now $Ay \in Pr_X (\text{ core } T) \cap Pr_X (\text{ core } M)$, otherwise $(Ay, v_1^*) + g (Ay, v_2^*) > (Ay, v_1^* + v_2^*)$ irrespective of the choice of $v_1^*, v_2^*$. Consequently the domain of $Q_A$ is contained in $Pr_X (\text{ core } T) \cap Pr_X (\text{ core } M) \cap \hat{Y}$ (a convex set in a finite-dimensional subspace). Denote the standard $\varepsilon$-enlargement, for $\varepsilon > 0$, by

$$M_\varepsilon := \{(x, x^*) | (g(x, x^*) - \langle x, x^* \rangle \leq \varepsilon \}$$

with $T_\varepsilon$ defined similarly. We now observe that that by Lemma 2.2 part c) of [23] and that fact that $f = (f^*)^*$ (the conjugates taken with respect to the paired spaces $\sigma^{w^* \times s} (X^* \times X)$ and $\sigma^{s \times w^*} (X \times X^*)$) that for all $s_0 \in \text{ int } Pr_X \text{ dom } P_T \subseteq \text{ int } Pr_X \text{ dom } f^*$ we have the existence of $K > 0, \eta \in (0, 1]$ such that $\| s - s_0 \| \leq \eta$ and $(y, y^*) \in X \times X^*$ we have $f (y, y^*) + K \| y - s \| - \langle s, y^* \rangle \geq \eta (\| y^* \| - K)$. Consequently for all $s^* \in T_\varepsilon (s)$ we have (on letting $y = s$ and $y^* = s^*$)

$$\varepsilon \geq \eta (\| s^* \| - K) \implies \text{ diam } T_\varepsilon (s) \leq K + \frac{\varepsilon}{\eta} \quad \text{ for all } \| s - s_0 \| \leq \eta.$$
Thus, by the uniform boundedness principle we have $\text{cone } (P_{\varepsilon} v_{\varepsilon})$ and obtain $v_{\varepsilon}$ bounded, and so $(\varepsilon, \beta)$ tend to zero. Equation (64) implies for all $v_{\varepsilon}^\beta \in M_\varepsilon \circ A (y)$ such that $A^* (v_{\varepsilon}^\beta + v_{\varepsilon}^\beta) = y^*$ we have

$$f(Ay, v_{\varepsilon}^\beta) - \langle Ay, v_{\varepsilon}^\beta \rangle \to 0. \quad (65)$$

Consequently there is a $\beta_\varepsilon$ such that for $\beta \geq \beta_\varepsilon$ we have $v_{\varepsilon}^\beta \in T_{\varepsilon} \circ A (y)$ and $v_{\varepsilon}^\beta \in T_{\varepsilon} \circ A (y)$. Let $A^* v_{\varepsilon}^\beta = y_{\varepsilon}^\beta$ then for all $y_{\varepsilon}^\beta \in (A^* \circ M_\varepsilon \circ A) (y) = (M_\varepsilon) A (y)$ such that $A^* v_{\varepsilon}^\beta = y^* - y_{\varepsilon}^\beta \in (A^* \circ T_{\varepsilon} \circ A) (y) =: (T_{\varepsilon}) A (y)$ we have (65). Now (65) implies

$$\sup_{(x,x^*) \in T} (Ay - x, x^* - v_{\varepsilon}^\beta) \to 0.$$

Hence for all $(x, x^*) \in T$ and for $\beta \geq \beta_\varepsilon \lor \beta_\varepsilon$ such that $\sup_{(x,x^*) \in T} (Ay - x, x^* - v_{\varepsilon}^\beta) \leq \varepsilon$ it follows that

$$\langle v_{\varepsilon}^\beta, x \rangle \leq \langle x, x^* \rangle - \langle Ay, x^* \rangle + \langle Ay, v_{\varepsilon}^\beta \rangle + \varepsilon$$

$$= \langle x, x^* \rangle - \langle Ay, x^* \rangle + \langle y, A^* v_{\varepsilon}^\beta \rangle + \varepsilon$$

$$= \langle x, x^* \rangle - \langle Ay, x^* \rangle + \langle y, y^* - y_{\varepsilon}^\beta \rangle + \varepsilon$$

Taking convex combinations and limits we have for all $x \in P_{\varepsilon} (\text{dom } T_{\varepsilon})$, $\beta \geq \beta_\varepsilon \lor \beta_\varepsilon$ and $(x, x^*) \in \text{dom } T_{\varepsilon}$ that

$$\langle v_{\varepsilon}^\beta, x \rangle \leq T_{\varepsilon} (x, x^*) - \langle Ay, x^* \rangle + \langle y, y^* - y_{\varepsilon}^\beta \rangle + \varepsilon$$

$$\leq T_{\varepsilon} (x, x^*) - \langle Ay, x^* \rangle + S(T_{\varepsilon}) A (y) + \varepsilon := K(x, x^*, A, y). \quad (66)$$

As $0 \in \text{int } (P_{\varepsilon} \text{dom } T_{\varepsilon}) \neq 0$ we have (66) implying a bound for all $x \in X = \text{cone } (P_{\varepsilon} \text{dom } T_{\varepsilon})$ where the right hand side bound is independent $\{v_{\varepsilon}^\beta\}_\beta$.

Thus, by the uniform boundedness principle we have $\{v_{\varepsilon}^\beta\}_\beta$ bounded whenever $(T_{\varepsilon}) A (y)$ is bounded. Using [23, Thm 3.6] ensures that

$$\text{int } \text{dom } T = \text{int } (P_{\varepsilon} \text{dom } T_{\varepsilon}) \supseteq \text{int } (P_{\varepsilon} \text{dom } T_{\varepsilon}) \supseteq \text{int } \text{dom } T. \quad (67)$$

It follows that $\text{int } (P_{\varepsilon} \text{dom } T_{\varepsilon}) = \text{int } \text{dom } T$ and within this set $T$ is locally bounded, and so $(T_{\varepsilon}) A (y)$ is bounded for $Ay \in \text{int } (\text{dom } T)$.

Consequently, for these $y$ we have $\{v_{\varepsilon}^\beta\}_\beta$ bounded and we may take a sub-net and obtain $v_{\varepsilon}^\beta \to y_{\varepsilon}^\beta$, $v_{\varepsilon}^\beta \in T_{\varepsilon} \circ A (y)$. It follows that $A^* v_{\varepsilon}^\beta \to y_{\varepsilon}^\beta \in (A^* \circ T_{\varepsilon} \circ A) (y) := (T_{\varepsilon}) A (y)$ and as $A^* v_{\varepsilon}^\beta = y^* - y_{\varepsilon}^\beta$ it also follows that $\{y^* - y_{\varepsilon}^\beta\}_\beta$ converges. Let $\text{w-}\lim_{\beta} y_{\varepsilon}^\beta = y_2^\beta$ and so $y^* - y_2^\beta = y_1^\beta$. 
Thus, there exists $y^*_2 \in (M_\varepsilon)_A (y)$ with $\text{w}^*\text{-lim}_\varepsilon y^*_2 = y^*_2$ and by the weak* closedness of $(M_\varepsilon)_A (y)$ we have $y^*_2 \in (M_\varepsilon)_A (y)$ where $\varepsilon > 0$ is arbitrary. Consequently,

$$y^* - y^*_1 \in \cap_{\varepsilon > 0} (M_\varepsilon)_A (y) = A^* \circ (\cap_{\varepsilon > 0} M_\varepsilon) \circ A (y)$$

$$= (A^* \circ M \circ A) (y) = M_A (y).$$

Similarly $v^*_1 \to \text{w}^* v^*_1 \in T \circ A (y)$ and $A^* v^*_1 = y^* - y^*_2$ we have $A^* v^*_1 = y^* - y^*_2$ with $y^*_1 \in T_A (y)$. Thus

$$y^* \in T_A (y) + M_A (y) = (T + M)_A (y).$$

We have shown that for all

$$\hat{y} = Ay \in [\text{int} (Pr_X \text{dom } P_T)] \cap \hat{Y} = [\text{int} (\text{dom } T)] \cap \hat{Y}$$

it follows that

$$H (y, y^*) = \langle y, y^* \rangle \Rightarrow y^* \in (T + M)_A (y).$$

Thus, (63) holds and we are done. \hfill \blacksquare

The containment of the monotone relations $Q_A$ and $(T + M)_A$ may only fail on the boundary of $\text{dom } T$. By definition it is clear that

$$\text{dom } (T + M)_A = \text{dom } M \cap \hat{Y} \cap \text{dom } T.$$ We need to extend this inclusion to all of $\text{dom } Q_A$ and to do so we need to characterise its domain. This will occupy our attention for next few results.

**Corollary 35** Suppose $T$ and $M$ are maximal monotone operators on a Banach space $X$ and that $A : Y \to X$ is an embedding of a finite-dimensional subspace $\hat{Y} \subseteq X$ into $X$. Define a monotone mapping $Q_A$ as in Proposition 34. Suppose the qualification assumption (62) holds then

$$\text{dom } M \cap \hat{Y} \cap \text{dom } T \subseteq \text{dom } Q_A,$$

$$\text{int } \text{dom } T \cap \text{dom } Q_A \subseteq \text{dom } (T + M)_A = \text{dom } M \cap \hat{Y} \cap \text{dom } T \quad \text{and}$$

$$\text{int } \text{dom } T \cap \text{dom } Q_A \subseteq \text{dom } M \cap \hat{Y} \cap \text{int } \text{dom } T.$$

**Proof.** From Proposition 34 we have

$$\text{int } \text{dom } T \cap \text{dom } Q_A \subseteq \text{dom } (T + M)_A = \text{dom } M \cap \hat{Y} \cap \text{dom } T.$$

Let $Ay \in \text{dom } T \cap \text{dom } M \cap \hat{Y}$. First observe that even when the support of some $x^* \in (T + M) \circ A (y)$ does not intersect $Y$ we still have $A^* x^* = x^* |_Y = 0$ and so for $y^* = 0$ (using that fact that $H$ is a representative function (42)),

$$0 = \langle y, y^* \rangle \leq H (y, y^*) = \inf \left\{ h \left( Ay, \hat{x}^* \right) \mid A^* \hat{x}^* = 0 \right\}$$

$$\leq h (Ay, x^*) = \langle Ay, x^* \rangle = \langle y, A^* x^* \rangle = 0 = \langle y, y^* \rangle$$

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giving \( y \in \text{dom} \, Q_A \) with \( Q_A(y) = \{0\} \).

Otherwise, for any \( x^* \in (T + M) \circ A(y) \), with \( y^* = x^*|_Y \) and then

\[
\langle y, y^* \rangle \leq H(y, y^*) = \inf_{z^*} \{ h(Ay, z^*) \mid A^*z^* = y^* \} \\
\leq h(Ay, x^*) = \langle Ay, x^* \rangle = \langle y, A^*x^* \rangle = \langle y, y^* \rangle
\]

implying \( y^* \in Q_A(y) \) i.e. \( y \in \text{dom} \, Q_A \) and

\[
Q_A(y) \supseteq \{ y^* = x^*|_Y \mid x^* \in (T + M) \circ A(y) \} = (T + M)_A(y).
\]

Consequently, \( \text{dom} \, Q_A \supseteq \text{dom} \, M \cap \hat{Y} \cap \text{dom} \, T \). Combining this observation with (69) we have the first two inclusions (68). For the last inclusion we take \( x \in (\text{int dom} \, T) \cap \text{dom} \, Q_A \) and so there exists \( x_n \to x \) with \( x_n \in \text{dom} \, Q_A \). Thus eventually \( x_n \in \text{int dom} \, T \) and so \( x_n \in (\text{int dom} \, T) \cap \text{dom} \, Q_A \subseteq \text{dom} \, M \cap \hat{Y} \cap \text{int dom} \, T \) by (69). Taking limits gives the last inclusion of (68).

We need the exact form of \( \text{dom} \, Q_A \) and \( \text{dom} \, H \).

**Proposition 36** Suppose \( T \) and \( M \) are maximal monotone operators on a Banach space \( X \) and that \( A : Y \to X \) is an embedding of a finite-dimensional subspace \( \hat{Y} \subseteq X \) into \( X \). Define a monotone mapping \( Q_A \) via the representative function \( H \) as in Proposition 34 with \( f(x, x^*) = \mathcal{F}_T(x, x^*) \) and \( g(x, x^*) = \mathcal{F}_M(x, x^*) \). Suppose \( \text{dom} \, M \cap \text{int dom} \, T \neq \emptyset \) holds and \( \text{dom} \, M \cap \hat{Y} \cap \text{int dom} \, T \neq \emptyset \) then

\[
\text{Pr}_X \text{dom} \, H \supseteq (\text{co dom} \, M) \cap \hat{Y} \cap \text{dom} \, T = (\text{Pr}_X \text{co} \, M) \cap \hat{Y} \cap \text{dom} \, T,
\]

and so when \( Q_A \) is maximal

\[
\text{Pr}_X \text{dom} \, H = \text{dom} \, Q_A \supseteq (\text{co dom} \, M) \cap \hat{Y} \cap \text{dom} \, T.
\]

**Proof.** First observe that \( \text{dom} \, M \cap \hat{Y} \subseteq \text{Pr}_X \text{dom} \, \mathcal{F}_M \) and so \( \text{co} \left( \text{dom} \, M \cap \hat{Y} \right) \subseteq \text{Pr}_X \text{dom} \, \mathcal{F}_M \) (because \( \mathcal{F}_M \) is a convex function) implying by (43)

\[
\text{Pr}_X \text{dom} \, H \supseteq \text{co} \left( \text{dom} \, M \cap \hat{Y} \right) \cap \text{dom} \, T = \text{co} \left( \text{dom} \, M \cap \hat{Y} \right) \cap \text{dom} \, T
\]

where the last equality of (71) follows from \( \text{co} \left( \text{dom} \, M \cap \hat{Y} \right) \cap \text{int dom} \, T \neq \emptyset \) and standard convex analysis results.

When \( Q_A \) is maximal on \( Y \) (a finite dimensional (reflexive) space) by the semi–convexity of \( \text{dom} \, Q_A \) we have \( \text{dom} \, Q_A \) convex and

\[
\text{dom} \, Q_A = \text{Pr}_X \text{dom} \, H \supseteq \text{co} \left( \text{dom} \, M \cap \hat{Y} \right) \cap \text{dom} \, T.
\]
The domain of $Q_A$ could differ from that of $(T + M)_A$, an issue that needs to be resolved at least for the case when $f(x, x^*) = F_T(x, x^*)$ and $g(x, x^*) = F_M(x, x^*)$.

**Corollary 37** Suppose $T$ and $M$ are maximal monotone operators on a Banach space $X$ and that $A : Y \to X$ is an embedding of a finite-dimensional subspace $\hat{Y} \subseteq X$ into $X$. Define a monotone mapping $Q_A$ via the representative function $H$ as in Proposition 34 with $f(x, x^*) = F_T(x, x^*)$ and $g(x, x^*) = F_M(x, x^*)$.

Suppose that the assumptions of Theorem 33 hold then

$$\text{dom} \ Q_A \subseteq \text{Pr}_X \text{ dom} \ H \subseteq \text{dom} \ T \cap \hat{Y} = \text{co} \left( \text{dom} \ T \cap \hat{Y} \right).$$

Consequently

$$\begin{align*}
\text{dom} \ Q_A &= \text{dom} \ (T + M)_A \\
&= \text{dom} \ M \cap \hat{Y} \cap \text{int dom} \ T = \text{co} \left( \text{dom} \ M \cap \hat{Y} \cap \text{int dom} \ T \right) \\
&= \left( \text{co dom} \ M \right) \cap \hat{Y} \cap \text{dom} \ T. 
\end{align*}$$

(73) (74)

**Proof.** Using the semi–convexity property for $T$ (since $T$ is maximal monotone and $\text{int dom} \ T \neq \emptyset$, see [23, Thm 3.6] Theorem 3.8) we have $\text{co} \text{ Pr}_X T = \text{Pr}_X T$ and so

$$A^{-1} \left( \text{Pr}_X T \right) \subseteq \text{co} \ A^{-1} \left( \text{Pr}_X T \right) \subseteq A^{-1} \left( \text{co} \text{ Pr}_X T \right) = A^{-1} \left( \text{co} \text{ Pr}_X T \right).$$

where the second inclusion follows from the observation that the set $\text{co} A^{-1} \left( \text{Pr}_X T \right)$ is the smallest convex set containing $A^{-1} \left( \text{Pr}_X T \right)$ and $A^{-1} \left( \text{Pr}_X \text{ co} T \right)$ is a convex set containing $A^{-1} \left( \text{Pr}_X T \right)$. Thus we have the equalities

$$A^{-1} \left( \text{Pr}_X \text{ co} T \right) = \text{co} A^{-1} \left( \text{Pr}_X T \right) = \text{co} \left( \text{Pr}_X T \cap \hat{Y} \right) = \text{co} \left( \text{dom} T \cap \hat{Y} \right)$$

and

$$A^{-1} \left( \text{Pr}_X T \right) = A^{-1} \left( \text{dom} T \right) = \text{dom} T \cap \hat{Y}.$$

Using Theorem 3.8 of [23, Thm 3.6] states that

$$\text{Pr}_X \text{ dom} \ F_T = \text{int} \left( \text{Pr}_X \text{ dom} \ F_T \right) = \text{int dom} T = \text{dom} T$$

and combining this with (43) we have $\text{Pr}_X \text{ dom} \ H \subseteq \text{dom} T \cap \hat{Y}$. In particular

$$\text{dom} Q_A = \text{dom} Q_A \cap \text{dom} T.$$

Via maximality $\text{dom} Q_A$ is convex, using the conclusion of Theorem 33 it follows that

$$\begin{align*}
\text{dom} Q_A &= \text{co} \left( \text{dom} Q_A \right) \supseteq \text{co} \left( \text{dom} M \cap \hat{Y} \cap \text{dom} T \right) \\
&= \text{co} \left( \text{dom} \ (T + M)_A \right) \supseteq \text{co} \left( \text{dom} M \cap \hat{Y} \cap \text{int dom} T \right). 
\end{align*}$$

(76)
By Theorem 2.13 of [6], the fact that $\text{qri} A = \text{ri} A$ if $A$ is finite dimensional, that fact that $\text{dom} Q_A \cap \text{int} \text{dom} T \neq \emptyset$ and $\text{ri} \text{dom} Q_A \neq \emptyset$ (because $\text{dom} Q_A$ is a convex set in finite dimensions) we have

$$\text{ri} \text{dom} Q_A = \text{ri}(\text{dom} Q_A \cap \text{dom} T) = \text{ri} \text{dom} Q_A \cap \text{int} \text{dom} T$$

$$\subseteq \text{dom} Q_A \cap \text{int} \text{dom} T \subseteq \text{dom} M \cap \hat{Y} \cap \text{int} \text{dom} T \subseteq \text{dom} (T + M)_A,$$

where we have used the last inclusion of Corollary 35. Thus

$$\text{dom} Q_A = \text{ri} \text{dom} Q_A \subseteq (\text{dom} M \cap \hat{Y} \cap \text{int} \text{dom} T).$$

Combining this with (68), (76) gives (73). As

$$\left(\text{co} \text{dom} M\right) \cap \hat{Y} \cap \text{dom} T \supseteq \text{co} \left(\text{dom} M \cap \hat{Y} \cap \text{int} \text{dom} T\right)$$

then (72) gives the equality (74).

The following give a first step toward semi-convexity of $\text{dom} M$ for an arbitrary maximal monotone mapping on a nonreflexive space.

**Corollary 38** Suppose $M$ is a maximal monotone operator on a Banach space $X$ and $\hat{Y}$ a finite dimensional subspace. Suppose that the assumption (47) of Theorem 33 holds. Then

$$\text{co} \left(\text{dom} M \cap \hat{Y}\right) = \text{dom} M \cap \hat{Y} = \left(\text{co} \text{dom} M\right) \cap \hat{Y}. \quad (77)$$

Consequently we have

$$\text{dom} Q_A = \text{dom} M \cap \hat{Y} \cap \text{dom} T \quad (78)$$

$$= \text{dom} M \cap \hat{Y} \cap \text{dom} T \cap \hat{Y} = \left(\text{co} \text{dom} M\right) \cap \hat{Y} \cap \text{dom} T.$$

**Proof.** In any space within which there exist monotone operators there exists a monotone operator $T$ with $\text{dom} T = X$. This follows from the results of [4] in which it is shown that any monotone operator with bounded range has a maximal extension with domain the whole space. Using a maximal monotone operator $T$ with $\text{dom} T = X$ we apply Corollary 37 to obtain (77). In particular $\text{dom} M \cap \hat{Y}$ is convex. Now take some other monotone operator $T$ which only has $\text{int} \text{dom} T \neq \emptyset$. When $\text{dom} M \cap \hat{Y} \cap \text{int} \text{dom} T \neq \emptyset$ we have

$$\text{dom} M \cap \hat{Y} \cap \text{int} \text{dom} T = \text{dom} M \cap \hat{Y} \cap \text{dom} T$$

As $\text{int} \text{dom} T \cap \hat{Y} \neq \emptyset$ we have $\text{dom} T \cap \hat{Y} = \text{dom} T \cap \hat{Y}$ which will give the second equality in (78) after the following is observed. Take $x \in \text{dom} M \cap \hat{Y} \cap \text{int} \text{dom} T$ then there exists $x_n \to x$ with $x_n \in \text{dom} M \cap \hat{Y} \cap \text{int} \text{dom} T$. For $n$ large $x_n \in \text{int} \text{dom} T$ and $x_n \in \text{dom} M \cap \hat{Y}$ implies the existence of $z_m \to x_n$ with
A suitable translation. Take to this end, we take that suffices to show that there exists a representative function. By Proposition 1 of \[15\] and the reflexivity of every finite-dimensional subspace it Hence, on identifying the translations with the original operators we may assume 0 \in \text{core } \Pr_X \cap \Pr_Y \cap \text{int } \text{dom } T and hence

\text{dom } M \cap \text{int } \text{dom } T = \text{dom } M \cap Y \cap \text{int } \text{dom } T = \text{dom } Q_A.

These results culminate in the next generalization of Rockafellar’s sum theorem [18].

**Theorem 39 (Sum Theorem)** Suppose \( T \) and \( M \) are maximal monotone operators on a Banach space \( X \). In addition suppose

\[ \emptyset \neq \text{dom } M \cap \text{int } \text{dom } T. \] (79)

Then \( T + M \) is a maximal monotone operator.

**Proof.** Our assumption imply 0 \in \text{core } \Pr_X \cap \Pr_Y \cap \text{int } \text{dom } T and \( T \) and \( M \) are both representative functions and since \( f^* = \widehat{F}_T \) and \( g^* = \widehat{F}_M \) we have \( f^* \geq \langle \cdot, \cdot \rangle \) (resp. \( g^* \geq \langle \cdot, \cdot \rangle \)) on \( X^* \times X^{**} \) (as observed in Corollary 12). Thus, by Theorem 30 with \( h := (f \square g) \), we deduce that \( \widehat{h}^* = \left( \widehat{f}^* \square g^* \right) \) is a representative functions with \( (h^*)^\dagger \) being a representative function for \( T + M \) and with \( h^* \geq \langle \cdot, \cdot \rangle \) on \( X^* \times X^{**} \).

In order to apply Theorem 17 (or rather Remark 18) we first show that for every finite-dimensional subspace \( Y \) of \( X \) such that \( Y \cap \text{core } \Pr_X \cap \Pr_Y \neq \emptyset \) we have

\( (T + M)_A := A^* \circ (T + M) \circ A \)

maximal where \( A : Y \to X \) is the embedding of \( Y \) into \( X \). As \( Y \) is reflexive it suffices to show that there exists a representative function \( H \) of \( (T + M)_A \) such that \( H \geq \langle \cdot, \cdot \rangle \) and \( H^* \geq \langle \cdot, \cdot \rangle \) on \( X \times Y^* \) and as proved in Proposition 1 of [15]. To this end, we take \( h := (f \square g) \) and \( H (y, y^*) := \inf_{x^*} \{ h (Ay, x^*) | A^*x^* = y^* \} \). By Theorem 30 since \( h \) is representative, \( H \) is representative, that is,

\[ H (y, y^*) \geq \inf_{x^*} \{ (Ay, x^*) | A^*x^* = y^* \} \geq \langle y, y^* \rangle. \]

By Proposition 33 we have \( H^* (y, y^*) \geq \langle y, y^* \rangle \) and hence \( H^* \) is a representative function. By [15, Prop. 1] and the reflexivity of \( Y \) we know that \( H \) is a

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representative function of a maximal monotone operator given by $QA$ as defined in Proposition 34.

Clearly we can cover $X$ with such finite-dimensional subspaces $Y$—a condition like $\hat{Y} \cap \text{core} (\Pr_X \text{dom} P) \neq \emptyset$ is easily satisfied by taking any $x \in (\text{int dom} T) \cap \text{dom} M$ and forming $Y := \text{span} (Y' \cup \{x\})$ where $Y'$ can be any finite-dimensional subspace. Now we have

$$(\text{dom } M \cap \text{int } (\text{dom } T) \cap \hat{Y}) \neq \emptyset.$$  

We now restrict attention to the subspace $Y$. By Theorem 31 the function $h$ is representative for $T + M$. By Corollary 38 we have

$$\text{dom } QA = (\text{co dom } M) \cap \hat{Y} \cap \text{dom } T.$$  

As $(\text{co dom } M \cap Y) \cap \text{int } (\text{dom } T) \neq \emptyset$ we may apply Theorem 2.13 of [6]—using $qri C = ri C$ when $C$ is finite-dimensional—to deduce that

$$\text{ri dom } QA = \text{ri} \left( (\text{co dom } M) \cap \hat{Y} \cap \text{dom } T \right) \subseteq \text{int } (\text{dom } T) \cap \hat{Y}.$$  

By Proposition 34 we know $QA (y) \subseteq (T + M)_A (y)$ for all $y \in \text{int } (\text{dom } T) \cap \hat{Y}$ and any $y^* \in Y^*$. Thus the inclusion $QA \subseteq (T + M)_A$ holds on $\text{ri dom } Q$. Now apply Corollary 28 to deduce

$$QA (y) = \overline{\text{co lim sup}_{y' \to y} Q_A (y')} + N_{\text{dom } QA} (y).$$  

We now use the fact that $(T + M)_A$ has closed graph [5]. This follows from the observation that $T + M = T + M$ since its representative function $f := (g \boxdot 2 h)$ is $s \times bw^*$ closed and

$$T + M = \{(x, x^*) | (g \boxdot 2 h) (x, x^*) - (x, x^*) \leq 0\}.$$  

Consequently,

$$(T + M)_A := (T + M) \cap (Y \times (X^*/Y^\perp))$$

is a closed set in the finite-dimensional subspace $Y \times (X^*/Y^\perp) \simeq Y \times Y^*$. That is, it is isomorphic to the finite-dimensional space $Y \times Y^*$, see [13, pp. 123]. Thus, $(T + M)_A$ has closed convex images. Using this observation and (63)

$$\overline{\text{co lim sup}_{y' \to y} Q_A (y')} \subseteq \text{lim sup}_{y' \to y} (T + M)_A (y')$$

$$y' \in \text{ri dom } Q_A \quad y' \in [\text{int } (\text{dom } T) \cap \hat{Y}]$$

$$\subseteq (T + M)_A (y).$$  

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Let \( \hat{y} \in \text{bd} \, \text{dom} \, Q_A = \text{bd} \left( \text{dom} \, T \cap \hat{Y} \cap \text{dom} \, M \cap \hat{Y} \right) \). As \( \text{dom} \, \text{M} \cap \text{int} \, (\text{dom} \, T) \cap \hat{Y} \neq \emptyset \) we have 0 \in \text{core}_Y (\text{dom} \, T \cap \hat{Y} - \text{dom} \, M \cap \hat{Y}) and so

\[
N_{\text{dom} \, Q_A} \left( y \right) = N_{\text{dom} \, M \cap \hat{Y} \cap \text{dom} \, T \cap \hat{Y}} \left( y \right) = N_{\left( \text{dom} \, T \cap \hat{Y} \right)^c \cap \hat{Y}} \left( y \right) + N_{\left( \text{dom} \, M \cap \hat{Y} \right)^c \cap \hat{Y}} \left( y \right).
\]

(84)

By Corollary 28, Lemma 29, (84) and (83)

\[
Q_A \left( y \right) = \omega \limsup_{y' \to y \atop y' \in \text{dom} \, Q_A} N \left( y' \right) + N_{\text{dom} \, Q_A^{\text{c}}} \left( y \right)
\]

\[
\subseteq T_A \left( y \right) + N_{\left( \text{dom} \, T \right)^c \cap \hat{Y}} \left( y \right) + M_A \left( y \right) + N_{\left( \text{dom} \, M \cap \hat{Y} \right)^c \cap \hat{Y}} \left( y \right)
\]

\[
\subseteq T_A \left( y \right) + \left( N_{\text{dom} \, T} \right)_A \left( y \right) + M_A \left( y \right) + \left( N_{\text{dom} \, M} \right)_A \left( y \right)
\]

\[
= T_A \left( y \right) + M_A \left( y \right) = \left( T + M \right)_A \left( y \right).
\]

Since \( Q_A \) is maximal, equality ensues and proves the maximality of \( \left( T + M \right)_A \).

By Theorem 17 we have \( F_{M_A} : X \times X^* \to \mathbb{R} \) a representative function and \( M_A = M_A^e \). Using Theorem 31 we have \( h \) is a representative function of \( T + M \). Thus, we have \( M_A^e = T + M \) and so by Theorem 17, \( F_{T + M} \) is a representative function. We will now argue as in Proposition 5 of [5]. Suppose \( (x, x^*) \) is monotonically related to \( T + M \) then \( F_{T + M} (x, x^*) \leq \langle x, x^* \rangle \) and as \( F_{T + M} \) is a representative function \( F_{T + M} (x, x^*) = \langle x, x^* \rangle \). But by Proposition 2 of [5] we must have \( P_{T + M} \left( x, x^* \right) = \langle x, x^* \rangle \) and so using Theorem 30

\[
\langle x, x^* \rangle = P_{T + M} \left( x, x^* \right) \geq h^* \left( x^*, x \right) = (f^* \sqcup g^*) \left( x^*, x \right) \geq \langle x, x^* \rangle.
\]

As \( h^* \) is a representative of \( T + M \) we have \( x^* \in \left( T + M \right) \left( x \right) \), completing the proof.

By the same methods we may also prove the following composition result.

**Theorem 40 (Composition)** Suppose \( X \) and \( Y \) are Banach spaces, that \( T \) is a maximal monotone operator on \( Y \), and that \( A : X \mapsto Y \), is a bounded linear mapping. Then \( T_A := A^* \circ T \circ A \) is maximal monotone on \( X \) whenever \( 0 \in \text{range} \left( A \right) \cap \text{int} \, \text{dom} \, T \).

We refer to [31, Thm 6] and [5] for results when \( Y \) is assumed reflexive.

A special case is worth recording.

**Corollary 41 (Normal Cones.)** Suppose in an arbitrary Banach space that \( T \) is maximal monotone and \( C \) is closed and convex while either

\[
C \cap \text{int} \, D(T) \neq \emptyset \quad \text{or} \quad D(T) \cap C \neq \emptyset.
\]

Then \( T + N_C \) is maximal monotone.

We finish this section with a corollary extending one in [5] and which answers a quite long-standing open question. Recall that a maximal monotone mapping \( T \) is maximal monotone locally [26] or of type (FPV), if for every open set \( V \) in \( X \) with \( \text{dom} \, T \cap V \neq \emptyset \) the following holds: if \( x \in V \) has the property that \( \langle y^* - x^*, y - x \rangle \geq 0 \) for all \( y^* \in T \left( y \right) \), and all \( y \in V \) then \( x^* \in T \left( x \right) \).
Corollary 42 (Convex Closure.) Suppose \( T \) is maximal monotone on a Banach space, then \( T \) is maximal monotone locally. In particular, \( \text{dom}(T) \) has a convex closure.

**Proof.** We argue as follows. Fix \( x, V \) and \( x^* \) as in the definition of maximal monotonicity locally. We may select a closed convex set \( C \) such that \( x \in \text{int} C \subset V \) and \( \text{dom} T \cap \text{int} C \neq \emptyset \). It follows from Theorem 39 that \( T + N_C \) is maximal. Let \( y^* \in T(y), n^* \in N_C(y), y \in Y \) be given. Then \( \langle y^* + n^* - x^*, y - x \rangle = \langle y^* - x^*, y - x \rangle + \langle n^*, y - x \rangle \geq 0 \) since \( x \in C \). By maximality \( x^* \in T(x) + N_C(x) = T(x) \) since \( x \in \text{int} C \).

The final conclusion follows by results in [4] and earlier. \( \blacksquare \)

7 Appendix

The give an outline of the bounded–weak* epi-limit–infimum and a proof of Theorem 48. This is a minor modification of the proof of Theorem 3.4 of [29]. Similar results may be found in [30] but are framed in a way that makes difficult the direct deduction of the result we require.

**Proposition 43** 1. Suppose \( F : X \to Y \) is a multifunction between normed spaces. Then

\[
\liminf_{v \to w} F(v) = \bigcap \{ C(w) \mid \exists \text{ a net } v_\beta \to w \text{ such that } \lim_{\beta} F(v_\beta) = C(w) \text{ exists} \}.
\]

(85)

2. For a family of lower semicontinuous functions \( f_v : X \to \mathbb{R} \) and a function \( f : X \to \mathbb{R} \). Then we have that \( \varepsilon \)-lim sup \( f_v \leq f \) (86) if and only if for all subnets \( v_\beta \to w \) such that \( \varepsilon \)-lim sup \( f_{v_\beta} = \varepsilon \)-lim inf \( f_{v_\beta} \) we have

\[
\varepsilon \text{-lim sup } f_{v_\beta} \leq f. \tag{87}
\]

3. Consequently

\[
\varepsilon \text{-lim sup } f_v = \sup \left\{ \varepsilon \text{-lim sup } f_{v_\beta} \mid \exists \text{ sub-nets } v_\beta \to w \text{ s.t. } \varepsilon \text{-lim } f_{v_\beta} \text{ exists} \right\}.
\]

(88)

**Proof.** For part 1 we first invoke Mrowka’s compactness theorem (see [2, Theorem 5.2.11]) to deduce that the set on the right–hand–side of (85) is non-empty. Indeed by this result any net admits a Kuratowski–Painlevé convergent subnet.
Clearly the set on the right-hand-side of (85) is larger than the that of the left-hand-side. Now if \( x \notin \liminf_{v \to x} F(v) \) then by Remark 2.2 of [29] or Lemma 1 of [30] we have a \( \varepsilon > 0 \) such that
\[
\limsup_{v \to w} d(x, F(v)) \geq \varepsilon > 0.
\]
Take a net \( \{v_\beta\} \) such that \( v_\beta \to w \) and
\[
\lim_{\beta} d(x, F(v_\beta)) \geq \varepsilon/2
\]
and invoke the Mrowka’s compactness theorem to obtain yet another subnet \( \{v_{\beta_\gamma}\} \) such that \( \lim_{\gamma} d(x, F(v_{\beta_\gamma})) = \lim_{\beta} d(x, F(v_\beta)) \geq \varepsilon/2 > 0 \)
implies
\[
x \notin \lim_{\gamma} F(v_{\beta_\gamma}) \quad \text{and so}
\]
\[
x \notin \bigcap \{C(w) \mid \exists \text{ a net } v_\beta \to w \text{ such that } \lim_{\beta} F(v_\beta) = C(w) \text{ exists} \},
\]
with equality in (87) ensuing.

The results for extended-real-valued functions follow on using \( F(v) := \text{epi } f_v \). Then \( e\text{-lim sup}_{v \to w} f_v \leq f \) if and only if
\[
\text{epi } f \subseteq \liminf_{v \to w} \text{epi } f_v
\]
and similarly \( e\text{-lim sup}_{v \to w} f_v \leq f \) if and only if \( \text{epi } f \subseteq \liminf_{v \to w} \text{epi } f_v \). As
\[
\liminf_{v \to w} f = \bigcap \{\lim \text{epi } f_{v_\beta} \mid \exists \text{ a net } v_\beta \to x \text{ such that } \lim \text{epi } f_{v_\beta} \text{ exists} \}
\]
we have (89) holding if and only if whenever there exists a net \( v_\beta \to w \) such that \( \lim_{\beta} \text{epi } f_{v_\beta} \) exists and we have \( \text{epi } f \subseteq \lim_{\beta} \text{epi } f_{v_\beta} \).

For (88) it is clear that \( e\text{-lim sup}_{v \to w} f_v \) is pointwise greater than the quantity on the right-hand-side of (88). Define
\[
f := \sup \left\{ e\text{-lim sup}_{v} f_{v_\beta} \mid \exists \text{ subnets } v_\beta \to w \text{ s.t. } e\text{-lim sup}_{v} f_{v_\beta} = e\text{-lim inf}_{v} f_{v_\beta} \right\}
\]
then clearly \( f \geq e\text{-lim sup}_{v} f_{v_\beta} \) for all subnets \( v_\beta \to w \) with \( \{\text{epi } f_{v_\beta}\}_{\beta} \) Kuratowski–Painlevé convergent. Thus by part 2 we have \( f \geq \liminf_{v \to w} \text{epi } f_v \), forcing equality.

**Corollary 44** Let \( f_v : X \to \bar{\mathbb{R}} \) be a family of lower semicontinuous convex functions, \( \{f_v\}_{v \in W} \) with \( W \) a neighbourhood of \( w \) and \( f : X \to \bar{\mathbb{R}} \) a lower semicontinuous, convex function. Then
\[
\left( e\text{-lim sup}_{v \to w} f_v \right)^{**} \leq f
\]
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if and only if for all subnets $v_\beta \to w$ such that $e\limsup_\beta f_{v_\beta} = e\liminf_\beta f_{v_\beta}$ we have
\[
e\limsup_\beta f_{v_\beta}^{**} \leq f.
\]

**Proof.** When $f \equiv +\infty$ there is nothing to prove so we proceed with the assumption that $f$ is not identically infinite. Thus when $e\limsup_{v \to w} f_v$ is improper we must have $e\limsup_{v \to w} f_v (x) = -\infty$ at some point and hence $e\limsup_\beta f_{v_\beta} (x) \leq e\limsup_{v \to w} f_v (x) = -\infty$ implying $(e\limsup_\beta f_{v_\beta})^{**} \equiv -\infty$. On the other hand when $e\limsup_{v \to w} f_v$ is proper we have
\[
e\limsup_{v \to w} f_v^{**} = e\limsup_{v \to w} f_v
\]
and Proposition 43 implies $(e\limsup_\beta f_{v_\beta})^{**} \leq f$ for all subnets $v_\beta \to w$ such that $e\limsup_\beta f_{v_\beta} = e\liminf_\beta f_{v_\beta}$.

Now suppose $(e\limsup_{v \to w} f_v)^{**} (x) > f (x)$ for some $x$ and so
\[
e\limsup_{v \to w} f_v^{**} (x) > -\infty.
\]
Then either $e\limsup_{v \to w} f_v$ is proper or identically equal to $+\infty$. In the former case properness implies $(e\limsup_{v \to w} f_v)^{**} = (e\limsup_{v \to w} f_v)$. Applying Proposition 43 part 3 we have the existence of subnets $v_\beta \to w$ such that \{epi $f_{v_\beta}$\} Kuratowski–Painlevé converges and for which $e\limsup_\beta f_{v_\beta} (x) > f (x)$. Observing that by (88), at least one of these nets $f_{v_\beta}$ must have $e\limsup_\beta f_{v_\beta} > -\infty$ we conclude that
\[
e\limsup_\beta f_{v_\beta}^{**} (x) = e\limsup_\beta f_{v_\beta} (x) > f (x).
\]

When $e\limsup_{v \to w} f_v \equiv +\infty$ then invoking (88) also leads to the same conclusion. \hfill \blacksquare

We introduce the terminology of [29] for a family of lower semi–continuous proper function. Recall, that in order to show a convex set is bounded weak∗ closed it is sufficient to show it contains the limits of all bounded and weak∗ convergent nets taken from the set [13]. This observation motivates the following.

**Definition 45** Let $\{f_v\}_{v \in W}$ be a family of functions on $X$ and $\{f^*_v\}_{v \in W}$ the family of conjugate functions on $X^*$ (for a normed space $X$). We denote the bounded–weak∗ upper epi-limit (as $v \to w$) of $\{f^*_v\}_{v \in W}$ by
\[
bw^{*}\text{–lim sup } \text{epi } f^*_v := \{ (x^*, \alpha) \in X^* \times \mathbb{R} \mid \exists \text{ nets } v_\beta \to w; \ (y^*_\beta, \alpha_\beta) \in \text{epi } f^*_{v_\beta} \text{ such that } \alpha_\beta \to \alpha; \ y^*_\beta \text{ norm bounded; } y^*_\beta \rightharpoonup w x^* \}.
\]
The above closely resembles the limit–superior of epigraphs, relative to the bounded–weak* topology on $X^*$ (hence the terminology). The bounded–weak* topology is described in, for example, [13]. Clearly this set recedes to $+\infty$ in the vertical direction and so resembles the epigraph of some function. This prompts us to define

**Definition 46** For $x^* \in X^*$,

$$
(bw^*\text{-}e\text{-lim inf}_{v \to w} f^*_v)(x^*) := \inf \{ \alpha \in \mathbb{R} \mid (x^*, \alpha) \in bw^*\text{-}lim sup_{v \to w} f^*_v \}. \quad (91)
$$

It then follows that

$$
epi_s (bw^*\text{-}lim inf_{v \to w} f^*_v) \subseteq bw^*\text{-}lim sup_{v \to w} epi f^*_v \subseteq epi (bw^*\text{-}lim inf_{v \to w} f^*_v).
$$

(92)

Thus $bw^*\text{-}lim inf_{v \to w} f^*_v$ is essentially a variational limit in the sense of use by Aubin, Rockafellar and Wets. Analogous definitions can be made for nets $\{f_\gamma\}_{\gamma \in I}$ of functions i.e.

$$
bw^*\text{-}lim sup_{\gamma} f^*_\gamma := \{(x^*, \alpha) \in X^* \times \mathbb{R} \mid \exists \text{ subnet } \gamma_\beta : (y^*_\beta, \alpha_\beta) \in epi f^*_\gamma \text{ such that } \alpha_\beta \to \alpha; \ y^*_\beta \text{ norm bounded; } y^*_\beta \overset{w^*}{\to} x^* \},
$$

with $(bw^*\text{-}lim inf_{\gamma} f^*_\gamma)(x^*)$ defined as in (91).

We now state Lemma 3.3 of [29].

**Proposition 47** Let $X$ be a normed space, and $\{f_\beta\}_{\beta \in I}$ a net of proper closed convex extended–real–valued functions on $X$. Suppose also that the strong epi–limit of $\{f_\beta\}_{\beta \in I}$ exists. Also, assume that either:

1. this epi–limit takes a finite value somewhere,

2. or $bw^*\text{-}e\text{-lim inf}_\beta f^*_\beta$ is not identically $+\infty$.

Then

$$
e\text{-}lim sup_\beta f \leq (bw^*\text{-}e\text{-lim inf}_\beta f^*_\beta)^*.
$$

(93)

The proof in [29] shows that the inequality (93) holds at each $x \in \text{dom} [e\text{-lim sup}_\beta f_\beta (\cdot)]$ without any of the additional assumptions stated in Theorem 47. We now weaken the assumption for which (93) holds outside this domain. This next result is proved using an adoption of the proof of Lemma 3.4 of [29].

**Theorem 48** Let $X$ be a normed space, $W$ a topological space; let $\{f_v\}_{v \in W}$ be a family of proper closed convex extended–real–valued functions on $X$. Suppose in addition that $e\text{-lim sup}_{v \to w} f_v > -\infty$. Then

$$
e\text{-lim sup}_{v \to w} f_v)^{**} \leq (bw^*\text{-}e\text{-lim inf}_{v \to w} f^*_v)^*.
$$

(94)
Proof. Recall that the epi–limit–supremum satisfies
\[ \text{epi} \left( \text{e}-\limsup_{\gamma} f_{v_{\beta}} \right) = \liminf_{\gamma} \text{epi} f_{v_{\beta}}, \]
and hence is a closed convex function. We only have to consider nets \( v_{\beta} \to w \) so that there is a strongly epi–convergent subnet \( f_{v_{\beta}} \) with \( \text{e}-\limsup_{\gamma} f_{v_{\beta}} \) finite at some point. Indeed take any \( v_{\beta} \to w \) so that the there is a strongly epi–convergent subnet \( \{ f_{v_{\beta}} \} \) then if we suppose that for all \( x \in \text{dom} \left( \text{e}-\limsup_{\gamma} f_{v_{\beta}} \right) \) we have \( \text{e}-\limsup_{\gamma} f_{v_{\beta}} (x) = -\infty \) (otherwise a finite value is attained) then we find that for all \( x \in \text{dom} \left( \text{e}-\limsup_{\gamma} f_{v_{\beta}} \right) \)
\[ \text{e}-\limsup_{\gamma} f_{v_{\beta}} (x) = \left( \text{e}-\limsup_{\gamma} f_{v_{\beta}} \right)^* (x) = -\infty. \]
For \( x \notin \text{dom} \left( \text{e}-\limsup_{\gamma} f_{v_{\beta}} \right) \) we have \( \left( \text{e}-\limsup_{\gamma} f_{v_{\beta}} \right)^* (x) = -\infty \) while \( \text{e}-\limsup_{\gamma} f_{v_{\beta}} (x) = +\infty. \) Consequently we have
\[ \left( \text{e}-\limsup_{\gamma} f_{v_{\beta}} \right)^* \leq (bw^* - \text{e}-\liminf_{v \to w} f_v^*)^* \]
holding on all of \( X. \) Then we need only to appeal to Proposition 43 and Lemma 47 which considering only subnets \( v_{\beta} \to w \) for which the there is a strongly epi–convergent subnet \( f_{v_{\beta}} \) with \( \text{e}-\limsup_{\gamma} f_{v_{\beta}} \) finite at some point. Then by Proposition 47 (applied to \( f_{v_{\beta}} \)),
\[ \left( \text{e}-\limsup_{\gamma} f_{v_{\beta}} \right)^* \leq \text{e}-\limsup_{\gamma} f_{v_{\beta}} \leq (bw^* - \text{e}-\liminf_{v \to w} f_v^*)^* \]
\[ \leq (bw^* - \text{e}-\liminf_{v \to w} f_v^*)^*. \]
Since this inequality holds for all convergent subnets \( v_{\beta} \to w \) with \( f_{v_{\beta}} \) strongly epi–convergent, by invoking Corollary 44 we conclude that (94) holds.

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