Lipschitz functions with maximal Clarke subdifferentials are staunch

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ABSTRACT. In a recent paper we have shown that most non-expansive Lipschitz functions (in the sense of Baire’s category) have a maximal Clarke subdifferential. In the present paper, we show that in a separable Banach space the set of non-expansive Lipschitz functions with a maximal Clarke subdifferential is not only of generic, but also staunch.

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1 Introduction and Definitions

Lipschitz functions with maximal subdifferentials provide counter-examples in nonsmooth analysis and differentiability theory. In a recent paper [1], we showed that the set of Lipschitz functions with maximal subdifferentials is residual in the space of all non-expansive functions. The purpose of this note is to strengthen this by showing that, in a separable-setting the set of all non-expansive Lipschitz functions with maximal subdifferentials is not only of residual but also staunch, by which we mean the complement of the set is $\sigma$-porous. We now recall the appropriate notion of porosity.

Let $(Y, d)$ be a complete metric space. We denote by $B(y, r)$ the closed ball of center $y \in Y$ and radius $r > 0$. A subset $E \subset Y$ is called porous in $(Y, d)$ if there exist $0 < \alpha \leq 1$ and $r_0 > 0$ such that for each $0 < r \leq r_0$ and each $y \in Y$, there exists $z \in Y$ for which

$$B(z, \alpha r) \subset B(y, r) \setminus E.$$  \hfill (1)

A subset of the space $Y$ is called $\sigma$-porous in $(Y, d)$ if it is a countable union of porous subsets in $(Y, d)$. All $\sigma$-porous sets are of the first category. If $Y$ is a finite dimensional Euclidean space, then $\sigma$-porous sets are of Lebesgue measure 0. The class of $\sigma$-porous sets is much smaller than the class of sets which have measure 0 and are of the first category. In fact, every complete metric space without isolated points contains a closed nowhere dense set which is not $\sigma$-porous [6].

Throughout, $X$ is a separable Banach space with norm $\| \cdot \|$, and its topological dual is denoted by $X^*$ with dual unit ball $B^*$. We use $S_X$ to denote the unit sphere of $X$. Let $A \subset X$ be a bounded open convex set. For a real-valued $f : A \to \mathbb{R}$ we say that $f$ is $K$-Lipschitz on $A$ if $K > 0$ and $|f(x) - f(y)| \leq K\|x - y\|$ for all $x, y \in A$. When $K = 1$, $f$ is called nonexpansive. The Clarke derivative of $f$ at point $x$ in the direction $v$ is given by

$$f^c(x; v) := \limsup_{y \to x} \frac{f(y + tv) - f(y)}{t},$$

while the Clarke subdifferential $\partial_c f$ is given by:

$$\partial_c f(x) := \{ x^* \in X^* | \langle x^*, v \rangle \leq f^c(x; v) \text{ for all } v \in X \}.$$

Note that $f^c(x; v)$ is upper semicontinuous as a function of $(x, v)$. Being nonempty and weak$^*$ compact convex valued, the multifunction $\partial_c f : A \to 2^{X^*}$ is norm-to-weak$^*$ upper semicontinuous. Detailed properties about Clarke subdifferentials can be found in [3], which is a sort of bible for nonsmooth analysts.

2 The Main Result

Let $C$ be a weak$^*$–compact convex subset of $X^*$. Recall that the support function of $C$ is the function $\sigma_C : X \to \mathbb{R}$ defined by

$$\sigma_C(v) := \sup \{ \langle x^*, v \rangle | x^* \in C \}.$$
$\sigma_C$ is sublinear, and Lipschitz with Lipschitz rate $K := \sup\{\|x^*\| : x^* \in C\}$. Consider

$$\mathcal{N}_C := \{f | f : A \to R \text{ and } f(x) - f(y) \leq \sigma_C(x - y) \text{ for all } x, y \in A\}.$$ 

Since each $f \in \mathcal{N}_C$ satisfies $f(x) - f(y) \leq K\|x - y\|$ for all $x, y \in A$, $\mathcal{N}_C$ is a special class of $K$–Lipschitz functions defined on $A$.

For $f, g \in \mathcal{N}_C$, set

$$\rho(f, g) := \sup_{x \in A} |f(x) - g(x)|.$$ 

One can easily verify that $(\mathcal{N}_C, \rho)$ is a complete metric space.

Our central result may now be stated.

**Theorem 1** Assume that $X$ is a separable Banach space and let $A \subset X$ be a bounded open convex subset of $X$. In the complete metric space $(\mathcal{N}_C, \rho)$, there exists a subset $G$ such that $\mathcal{N}_C \setminus G$ is $\sigma$-porous in $(\mathcal{N}_C, \rho)$, and such that each $f \in G$ has $\partial_c f \equiv C$ on $A$.

**Proof.** Fix $x \in A$, $v \in S_X$ and a natural number $k$. Consider

$$G(x, v, k) := \left\{ f \in \mathcal{N}_C \mid f(x + tv) - f(x) - \sigma_C(v) \geq -\frac{1}{k} \text{ for some } 0 < t < \frac{1}{k} \right\}.$$ 

We shall show that $\mathcal{N}_C \setminus G(x, v, k)$ is porous in $(\mathcal{N}_C, \rho)$.

According to (1), it suffices to find $0 < \alpha \leq 1$ such that for each $r \in (0, 1/k)$ and each $f \in \mathcal{N}_C$ there exists $h_2 \in \mathcal{N}_C$ for which

$$B(h_2, \alpha r) \subset B(f, r) \cap G(x, v, k).$$

Of course, here $h_2$ relies on $r$, but $\alpha$ only relies on $(x, v, k)$.

To meet this goal, we define $h : X \to R$ by

$$h(\tilde{x}) := f(x) - \frac{r}{4} + \sigma_C(\tilde{x} - x),$$

and set

$$h_1 := \min\{f, h\}, \quad h_2 := \max\{f - \frac{r}{2}, h_1\}. \quad (2)$$

Clearly, $h_2 \in \mathcal{N}_C$ and $f - r/2 \leq h_2 \leq f$, so that

$$\rho(h_2, f) \leq \frac{r}{2}.$$ 

Set

$$\alpha := \frac{\min\{d_{X \setminus A}(x), 1\}}{8(\sigma_C(v) + \sigma_C(-v) + 1)} \cdot \frac{1}{k}. \quad (3)$$

If we let

$$t := \frac{\min\{d_{X \setminus A}(x), 1\}}{4(\sigma_C(v) + \sigma_C(-v) + 1)} r, \quad (4)$$

where $d_{X \setminus A}(x) := \inf\{\|x - y\| : y \in X \setminus A\}$, then $0 < t < 1/k$ and $x + tv \in A$. Note that $d_{X \setminus A}(x) > 0$ because $A$ is open and $x \in A$. Now

$$h(x + tv) = f(x) - \frac{r}{4} + t\sigma_C(v).$$
Since
\[ f(x) - f(x + tv) \leq \sigma_C(-tv), \]
we have
\[ f(x + tv) \geq f(x) - \sigma_C(-tv) = f(x) - t\sigma_C(-v). \]
The choice of \( t \) implies
\[ t(\sigma_C(v) + \sigma_C(-v)) \leq \frac{r}{4}, \]
so that
\[ f(x) - \frac{r}{4} + t\sigma_C(v) \leq f(x) - t\sigma_C(-v). \]
It follows that \( h(x + tv) \leq f(x + tv) \), and so \( h_1(x + tv) = h(x + tv) \) by (2). On the other hand,
\[ f(x + tv) - \frac{r}{2} \leq f(x) - \frac{r}{4} + t\sigma_C(v), \]
since \( f(x + tv) - f(x) \leq \sigma_C(tv) \). Therefore, by (2),
\[ h_2(x + tv) = f(x) - \frac{r}{4} + t\sigma_C(v) \quad \text{and} \quad h_2(x) = f(x) - \frac{r}{4}. \]
This means
\[ \frac{h_2(x + tv) - h_2(x)}{t} = \sigma_C(v). \tag{5} \]
Assume that \( g \in B(h_2, \alpha r) \). We will show that \( g \in G(x, v, k) \). Indeed, by (5), (4), (3),
\[
\frac{g(x + tv) - g(x)}{t} - \sigma_C(v)
= \frac{(g - h_2)(x + tv) - (g - h_2)(x)}{t} + \frac{h_2(x + tv) - h_2(x)}{t} - \sigma_C(v)
\geq \frac{-2\alpha r}{t} = -2\alpha r t^{-1} = -2\alpha r \left[ \frac{\min\{d_{X\setminus A}(x), 1\}}{4(\sigma_C(v) + \sigma_C(-v) + 1)} \right]^{-1}
= \alpha \cdot \frac{8(\sigma_C(v) + \sigma_C(-v) + 1)}{\min\{d_{X\setminus A}(x), 1\}} = -\frac{1}{k}.
\]
Therefore,
\[ \{ g \in \mathcal{N}_C : \rho(g, h_2) \leq \alpha r \} \subset G(x, v, k). \tag{6} \]
If \( \rho(g, h_2) \leq \alpha r \), then
\[
\rho(g, f) \leq \rho(g, h_2) + \rho(h_2, f) \leq \alpha r + \frac{r}{2} \leq \frac{r}{2} + \frac{r}{2} = r.
\]
Thus
\[ \{ g \in \mathcal{N}_C : \rho(g, h_2) \leq \alpha r \} \subset \{ g \in \mathcal{N}_C : \rho(g, f) \leq r \}. \]
When combined with (6), this inclusion implies that
\[ \mathcal{N}_C \setminus G(x, v, k) \quad \text{is indeed porous in} \quad (\mathcal{N}_C, \rho). \tag{7} \]
Now let \( \{x_n : n \geq 1\} \) be norm dense in \( A \), \( \{v_m : m \geq 1\} \) be norm dense in \( S_X \). Set

\[
G := \bigcap_{n=1}^{\infty} \bigcap_{m=1}^{\infty} \bigcap_{k=1}^{\infty} G(x_n, v_m, k).
\]

In view of (7) and that

\[
\mathcal{N}_C \setminus G = \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcup_{k=1}^{\infty} (\mathcal{N}_C \setminus G(x_n, v_m, k)),
\]

the set \( \mathcal{N}_C \setminus G \) must be \( \sigma \)-porous in \( (\mathcal{N}_C, \rho) \). If \( f \in G \), then for each \( x_n, v_m, k \), we have \( f \in G(x_n, v_m, k) \); that is,

\[
\frac{f(x_n + t_{n,m,k}v_m) - f(x_n)}{t_{n,m,k}} - \sigma_C(v_m) \geq -\frac{1}{k},
\]

for some \( 0 < t_{n,m,k} < \frac{1}{k} \). When \( k \to \infty \), from the definition of \( f^o \) it follows that

\[
f^o(x_n; v_m) \geq \limsup_{t \downarrow 0} \frac{f(x_n + tv_m) - f(x_n)}{t} \geq \sigma_C(v_m),
\]

and consequently,

\[
f^o(x_n; v_m) \geq \sigma_C(v_m) \quad \text{for all } n, m \geq 1. \quad (8)
\]

Since \( \{x_n : n \geq 1\} \) is dense in \( A \) and \( \{v_m : m \geq 1\} \) is dense in \( S_X \), for every \( x \in A \) and \( v \in S_X \), we may find sequences \( (x_n) \) and \( (v_m) \) such that \( x_n \to x \) and \( v_m \to v \). By the upper semicontinuity of \( f^o \) and continuity of \( \sigma_C \), from (8) we get

\[
f^o(x; v) \geq \sigma_C(v). \quad (9)
\]

Since \( f \in \mathcal{N}_C \), for every \( y \in A, t > 0 \),

\[
f(y + tv) - f(y) \leq \sigma_C(tv).
\]

Dividing both sides by \( t \), and taking the lim sup as \( y \to x \) and \( t \downarrow 0 \) produces

\[
f^o(x; v) \leq \sigma_C(v).
\]

Together with (9), we obtain

\[
f^o(x; v) = \sigma_C(v) \quad \text{for } x \in A, v \in S_X.
\]

Dually, \( \partial_c f(x) = C \) for every \( x \in A \), and the proof of the theorem is complete. \hfill \Box

Observe that

\[
\mathcal{N}_{B^*} := \{f | f : A \to R \text{ is nonexpansive with respect to } \| \cdot \| \}.
\]

Theorem 1 gives:

\[
\]
Corollary 1  In the space of nonexpansive functions, \((\mathcal{N}_B^*, \rho)\), the set
\[
\{ f \in \mathcal{N}_B^* | \partial_c f \equiv B^* \text{ on } A \},
\]
has a \(\sigma\)-porous complement in \((\mathcal{N}_B^*, \rho)\).

It is well-known that every locally Lipschitz function \(f\) on an open subset \(A\) of a separable Banach space \(X\) is Gâteaux differentiable everywhere on \(A\) except for possibly a Haar-null subset. We need a result due to Giles and Sciffer [4].

Lemma 1  Let \(f : A \to R\) be a locally Lipschitz function on an open subset \(A\) of a separable Banach space \(X\). Then the set
\[
\{ x \in A | f^+(x; v) = f^0(x; v) \text{ for all } v \in X \},
\]
is residual in \(A\). Here
\[
f^+(x; v) := \limsup_{t \downarrow 0} \frac{f(x + tv) - f(x)}{t}.
\]
Combining Corollary 1 with Lemma 1 gives the following result.

Corollary 2  In the space of nonexpansive functions, \((\mathcal{N}_B^*, \rho)\), the set
\[
\{ f \in \mathcal{N}_B^* | f \text{ is Gâteaux differentiable at most on a first category subset of } A \},
\]
has a \(\sigma\)-porous complement in \((\mathcal{N}_B^*, \rho)\).

Proof.  Let \(f \in \mathcal{N}_B^*\) such that \(\partial_c f \equiv B^* \) on \(A\). Consider the set
\[
S_f := \{ x \in A | f^+(x; v) = f^0(x; v) \text{ for all } v \in X \}.
\]
By Lemma 1, \(S_f\) is a residual set in \(A\). If \(f\) is Gâteaux differentiable at \(x\), then \(f^+(x; v) = \langle \nabla f(x), v \rangle\) for every \(v \in X\), and so \(x \not\in S_f\) since \(\partial_c f(x) = B^*\). Therefore, such \(f\) is at most Gâteaux differentiable on \(A \setminus S_f\), which is a first category subset in \(A\). Since the set
\[
\{ f \in \mathcal{N}_B^* | \partial_c f \equiv B^* \text{ on } A \},
\]
has a \(\sigma\)-porous complement in \((\mathcal{N}_B^*, \rho)\) by Corollary 1, the result is proved. \(\square\)

Finally, for various generic aspects of Lipschitz functions with maximal Clarke subdifferentials on general Banach spaces, we refer readers to [2].

References


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