Van der Pol Expansions of L-Series

D. Borwein* and J.M. Borwein†

January 25, 2005

Abstract. We provide concise series representations for various L-series integrals. Different techniques are needed below and above the abscissa of absolute convergence of the underlying L-series.

Key words. Dirichlet series integrals, Hurwitz zeta functions, Plancherel theorem, L-series.

Classification numbers. 11M35, 11M41, 30B50

1 Preliminaries

In [7] the following integral evaluation is obtained.

\[ \int_0^\infty \frac{(3 - 2\sqrt{2}\cos(t \log 2))}{t^2 + 1/4} \left| \zeta \left( \frac{1}{2} + it \right) \right|^2 dt = \pi \log 2. \]  

(1)

This identity turns out—formally—to be a case of a rather pretty class of evaluations given in Theorem 1 [1].

We let \( \lambda_n \) represent a formal Dirichlet series, with coefficients \( \lambda_n \) real in the first two sections but complex in the final two sections, and we set \( s := \sigma + i \tau \) with \( \sigma = \Re(s) > 0 \). We refer to [2, 3, 5] for other, largely standard, details. We shall consider the following integral:

\[ \iota_\lambda(\sigma) := \int_0^\infty \left| \frac{\lambda(s)}{s} \right|^2 d\tau = \frac{1}{2} \int_{-\infty}^\infty \left| \frac{\lambda(s)}{s} \right|^2 d\tau, \]

(2)
as a function of \( \lambda \).

It is convenient to recall, [1], that for \( u, a > 0 \)

\[ \int_0^\infty \frac{\cos(at)}{t^2 + u^2} dt = \frac{\pi}{2u} e^{-au}. \]

*Dept. of Mathematics, University of Western Ontario, London Ontario, N6A 5B7. Research supported by NSERC. E-mail: dborwein@uwo.ca

†Faculty of Computer Science, Dalhousie University, Halifax NS, B3H 1W5. Research supported by NSERC, the Canada Foundation for Innovation and the Canada Research Chair Program. E-mail: jborwein@cs.dal.ca
2 Integrals Involving $s$ with Large Real Part

Theorem 1 [1] For $\lambda(s) := \sum_{n=1}^{\infty} \frac{\lambda_n}{n^s}$ and $s = \sigma + i \tau$ with fixed $\sigma = \Re(s) > 0$ such that the Dirichlet series is absolutely convergent we have

$$i_\lambda(\sigma) := \int_0^\infty \left| \frac{\lambda(s)}{s} \right|^2 d\tau = \frac{\pi}{2\sigma} \sum_{n=1}^{\infty} \frac{\Lambda_n^2 - \Lambda_{n-1}^2}{n^{2\sigma}}$$

(3)

where $\Lambda_n := \sum_{k=1}^{n} \lambda_k$, $\Lambda_0 := 0$.

More generally, given absolutely convergent Dirichlet series $\alpha(s) := \sum_{n=1}^{\infty} \frac{\alpha_n}{n^s}$ and $\beta(s) := \sum_{n=1}^{\infty} \frac{\beta_n}{n^s}$ we have

$$\int_0^\infty \frac{\alpha(s) \beta(s)}{\sigma^2 + \tau^2} d\tau = \frac{\pi}{2\sigma} \sum_{n=1}^{\infty} \frac{A_n B_n - A_{n-1} B_{n-1}}{n^{2\sigma}}$$

(4)

where $A_n := \sum_{k=1}^{n} \alpha_k$ and $B_n := \sum_{k=1}^{n} \beta_k$.

Proof. Let $\lambda_N(s) := \sum_{n=1}^{N} \frac{\lambda_n}{n^s}$. Then

$$\int_0^\infty \frac{|\lambda_N(s)|^2}{\tau^2 + \sigma^2} d\tau = \int_0^\infty \sum_{N\geq n,m>0} \frac{\lambda_n \lambda_m}{(nm)^\sigma} \int_0^\infty \frac{(n/m)^{i\tau}}{\tau^2 + \sigma^2} d\tau + \pi \frac{\lambda_n \lambda_m}{(nm)^\sigma} \int_0^\infty \frac{1}{\tau^2 + \sigma^2} d\tau$$

$$= \frac{\pi}{\sigma} \sum_{N\geq n,m>0} \frac{\lambda_n \lambda_m}{(nm)^\sigma (n/m)\sigma} + \frac{\pi}{2\sigma} \sum_{n=1}^{N} \frac{\lambda_n^2}{n^{2\sigma}}$$

$$= \frac{\pi}{\sigma} \sum_{n=1}^{N} \frac{\lambda_n A_n}{n^{2\sigma}} - \frac{\pi}{2\sigma} \sum_{n=1}^{N} \frac{\lambda_n^2}{n^{2\sigma}} = \frac{\pi}{2\sigma} \sum_{n=1}^{N} \frac{\Lambda_n^2 - \Lambda_{n-1}^2}{n^{2\sigma}}.$$}

Next, we observe that

$$|\lambda_N(s)|^2 \leq \left( \sum_{n=1}^{\infty} \frac{|\lambda_n|}{n^\sigma} \right)^2 = M < \infty,$$

where $M$ is independent of $\tau$. Hence

$$\left| \frac{\lambda_N(s)}{s} \right|^2 \leq \frac{M}{\tau^2 + \sigma^2},$$

and (3) follows by Lebesgue’s theorem on dominated convergence on letting $N \to \infty$. We can establish (4) in a similar fashion. $\square$
Note that, by Dirichlet’s test, the final series in (3) is convergent for all \( \sigma > 0 \)
when \( \Lambda_n \) is bounded, but we cannot automatically guarantee that it is equal
to the integral in (3) in this case, or even that the integral is finite. Simple
continuation arguments will not work. This in part motivates the first example
and the following section.

It is, however, easy now to check that
\[
\langle \alpha, \beta \rangle := \int_0^\infty \frac{\alpha(s) \overline{\beta(s)}}{\sigma^2 + \tau^2} \, d\tau
\]
defines an extended-value inner product on the space of Dirichlet series with
\( \langle \alpha, \alpha \rangle_\sigma = \iota_\alpha(\sigma) \), which is typically finite for \( \sigma \) large enough.

In the sequel, we let \( L_\mu(s) := \sum_{n=1}^{\infty} \left( \frac{\mu}{n} \right) n^{-s} \) denote the
primitive \( L \)-function corresponding to the Kronecker symbol \( \left( \frac{\mu}{n} \right) \), [2]. Below
\( \lfloor x \rfloor \) is the integer part of \( x \) for \( x > 0 \) extended by \( \lfloor -x \rfloor = -\lfloor x \rfloor \), while \( \{x\} := x - \lfloor x \rfloor \) denotes the
fractional part.

**Example 1** For the Riemann zeta function \( (\zeta = L_1) \), and for \( \sigma > 1 \) Theorem
1 applies and yields
\[
\frac{\sigma}{\pi} \iota_\zeta(\sigma) = \zeta(2\sigma - 1) - \frac{1}{2} \zeta(2\sigma),
\]
as \( \lambda_n = 1 \) and \( \Lambda_n = n \). By contrast it is known—see equation (69) of [4]—that
on the critical line
\[
\frac{1}{2\pi} \iota_\zeta \left( \frac{1}{2} \right) = \log(\sqrt{2\pi}) - \frac{1}{2} \gamma.
\]

More broadly, for \( 0 < \sigma < 1 \), Crandall has found that
\[
\iota_\zeta(\sigma) = \pi \int_0^1 \sum_{n \geq 0} \frac{\theta^2}{(n + \theta)^{1+2\sigma}} \, d\theta = \pi \int_0^1 \theta^2 \zeta(1 + 2\sigma, \theta) \, d\theta,
\]
where \( \zeta(s, a) \) is the Hurwitz zeta function—which is easy to compute. This
devolves from van der Pol’s representation
\[
\frac{\zeta(s)}{s} = -\int_{-\infty}^\infty e^{-s\omega} (e^{\omega} - \lfloor e^{\omega} \rfloor) e^{-i\tau \omega} \, d\omega,
\]
valid in the critical strip \( s = \sigma + i\tau \) with \( 0 < \sigma < 1 \). Identity (5) may be
obtained via
\[
\int_{-\infty}^\infty e^{-s\omega} (e^{\omega} - \lfloor e^{\omega} \rfloor) \, d\omega = \int_0^\infty t^{-s-1}(t - \lfloor t \rfloor) \, dt = \sum_{n=0}^{\infty} \int_n^{n+1} t^{-s-1}(t - \lfloor t \rfloor) \, dt
\]
\[
= \sum_{n=0}^{\infty} \int_0^1 \theta \zeta(1 + s, \theta) \, d\theta = \sum_{n=0}^{\infty} \int_0^1 \theta (\theta + n)^{-s-1} \, d\theta
\]
\[
= -\frac{1}{s} \lim_{N \to \infty} \left( \sum_{n=1}^{N} \frac{1}{n^s} - \frac{N^{1-s}}{1-s} \right)
\]
For the Example 2 applying the result in (3) with \( \lambda \) as \( \frac{1}{n^s} \). We have

\[
\sigma_{2N} = 2^{1-s} \sigma_N \to \sum_{n=1}^{\infty} \frac{(-1)^n}{n^s} = -(1 - 2^{1-s}) \zeta(s),
\]

which implies that \( \sigma_\infty = \zeta(s) \), as required.

We now write (5) as a Fourier transform: \( \zeta(s)/s = -\mathcal{F}_e(e^{-\sigma \omega} \{e^{\omega s}\}) \) and so obtain

\[
2t_\zeta(\sigma) = \int_{-\infty}^{\infty} |\zeta(s)/s|^2 \, d\tau = 2\pi \int_{-\infty}^{\infty} e^{-2\sigma \omega}|e^\omega - |e^\omega||^2 \, d\omega
\]

from the \( \mathcal{L}_2 \) Plancherel theorem [8]. Now

\[
\int_{-\infty}^{\infty} e^{-2\sigma \omega}|e^\omega - |e^\omega||^2 \, d\omega = \int_{0}^{\infty} t^{-2\sigma-1}[t - [t]]^2 \, dt = \sum_{n=0}^{\infty} \int_{n}^{n+1} t^{-2\sigma-1}[t - [t]]^2 \, dt
\]

\[
= \sum_{n=0}^{\infty} \int_{0}^{1} \theta^2 (\theta + n)^{-2\sigma-1} \, d\theta = \int_{0}^{1} \theta^2 \sum_{n=0}^{\infty} (\theta + n)^{-2\sigma-1} \, d\theta,
\]

as required. Note that all terms are absolutely convergent which legitimates the operations. We have also established, inter alia, that

\[
\int_{-\infty}^{\infty} t^2 \zeta(1+2\sigma, \theta) \, d\theta = \int_{0}^{1} \theta^2 \zeta(1+2\sigma, \theta) \, d\theta.
\]

Moreover, reversing the order of integration and summation above leads to

\[
\zeta(\sigma) = \pi \sum_{n=0}^{\infty} \int_{0}^{1} \frac{\theta^2}{(n+\theta)^{1+2\sigma}} \, d\theta = -\frac{\pi}{2\sigma} \left( \frac{2\zeta(2\sigma - 1)}{2\sigma - 1} + \zeta(2\sigma) \right)
\]

which in the limit as \( \sigma \to 1/2 \) recaptures the evaluation quoted above. Recapitulating, we have

\[
\frac{\sigma}{\pi} t_\zeta(\sigma) = \begin{cases} \frac{1}{2} [\zeta(2\sigma) + 2\zeta(2\sigma - 1)/(2\sigma - 1)], & 0 < \sigma < 1; \\ \frac{1}{2} [\zeta(2\sigma) - 2\zeta(2\sigma - 1)], & 1 < \sigma < \infty. \end{cases}
\]

There are similar formulae for \( s \mapsto \zeta(s-k) \) with \( k \) integral. For instance, applying the result in (3) with \( \zeta_1 := t \mapsto \zeta(t+1) \) yields

\[
\frac{1}{\pi} \int_{0}^{\infty} \frac{[\zeta(3/2 + i\tau)]^2}{1/4 + \tau^2} \, d\tau = \frac{1}{\pi} t_{\zeta_1} \left( \frac{1}{2} \right) = 2\zeta(2,1) + \zeta(3) = 3\zeta(3),
\]

on using Euler’s result that \( \zeta(2,1) := \sum_{n=2}^{\infty} 1/n^2 \sum_{k=1}^{n-1} 1/k = \zeta(3) \).

Example 2 For the alternating zeta function, \( \alpha := s \mapsto (1 - 2^{1-s})\zeta(s) \), we recover, [7]:

\[
\frac{\sigma}{\pi} t_\alpha(\sigma) = \frac{1}{2} \alpha(2\sigma),
\]

as \( \lambda_n = (-1)^{n+1}, \Lambda_n = (1 - (-1)^n)/2 \) and \( \Lambda_n^2 - \Lambda_{n-1}^2 = (-1)^{n+1}/2 \).
Set $\sigma := 1/2$. Then
\[
\frac{|(1/2 + it)|^2}{1/4 + t^2} = \frac{|1 - 2^{1/2 - it}|^2 |\zeta(1/2 + it)|^2}{1/4 + t^2} = \left(3 - 2\sqrt{2}\cos(t\ln(2))\right) \frac{|\zeta(1/2 + it)|^2}{1/4 + t^2}
\]
is precisely the integrand in (1). Thus, since $\alpha(2\sigma) = \log 2$ we see that $\iota_\alpha(1/2) = \pi \log 2$, is the evaluation in [7]. Note that to justify the exchange of sum and integral implicit in (3) we have to more carefully analyse the integrand, since $1/2$ is below the abscissa of absolute convergence of the series. This leads to the approach in the section below on the Hurwitz zeta function.

**Example 3** For the Catalan zeta function ($\beta := L_{-4}$):
\[
\sigma \frac{\pi \iota_\beta(\sigma) = \frac{1}{2} \beta(2\sigma)}{\iota_\beta(2\sigma)}
\]
as $\lambda_{2n} = 0, \lambda_{2n+1} = (-1)^n$ and again $\Lambda_n^2 - \Lambda_{n-1}^2 = \lambda_n$.

For $L_{-3}$, the same pattern holds, in that $\frac{\sigma}{\pi} \iota_{L_{-3}}(\sigma) = \frac{1}{2} L_{-3}(2\sigma)$, but not for $L_{-5}, L_{-7}$, and so on. In general $L_{d+1}$ does not lead to output which is again a primitive L-series mod $d$. For example,
\[
\frac{\sigma}{\pi} \iota_{L_5}(\sigma) = -\sum_{\delta \in \mathbb{Z}} \frac{(-1)^n \mod 5}{n^{2\sigma}},
\]
and
\[
\frac{\sigma}{\pi} \iota_{L_{-5}}(\sigma) = L_{-5}(2\sigma) - \frac{1}{2} L_{-4}(2\sigma).
\]
These are not character sums, though always the coefficients repeat modulo $d$.

Finally, let $\vartheta := s \mapsto (1 + 2^{-s})^{-1}$. Then again $\frac{\sigma}{\pi} \iota_{\vartheta}(\sigma) = \frac{1}{2} \vartheta(2\sigma)$.

For each of these examples, the defect of Theorem 1 is that, as we have seen, it only directly applies when $\sigma$ is large enough. The van der Pol approach offers a nice alternative, especially in the critical strip.

### 3 Van der Pol’s Approach to Hurwitz Zeta

Recall that
\[
\zeta(s, a) := \sum_{n=0}^{\infty} \frac{1}{(n + a)^s},
\]
appropriately continued for $\sigma < 1$. Thus, $\zeta(s) = \zeta(s, 1)$. We first work out a Fourier transform representation for $\zeta(a, s)$. For $a, \sigma > 0$, we have
\[
\int_{\log a}^{\infty} e^{-s\omega} \{e^\omega - a\} \, d\omega = \int_a^{\infty} t^{-s-1} (t - a - \lfloor t - a \rfloor) \, dt 
\]

\[
= \sum_{n=0}^{\infty} \int_{n+a}^{n+1+a} t^{-s-1} (t - a - \lfloor t - a \rfloor) \, dt 
\]

\[
= \int_0^1 \theta \zeta(1+s,\theta+a) \, d\theta = \sum_{n=0}^{\infty} \int_0^1 \theta (\theta + n + a)^{-s-1} \, d\theta 
\]

\[
= -\frac{1}{s} \lim_{N \to \infty} \left( \sum_{n=1}^N \frac{1}{(n+a)^s} - \frac{(N + a)^{1-s} - a^{1-s}}{1-s} \right) 
\]

\[
= -\frac{1}{s} \zeta(s, a + 1) - \frac{a^{1-s}}{s(1-s)},
\]

this evidently being true for \( \sigma > 1 \) and so for \( 0 < \sigma < 1 \) by analytic continuation—both sides of (7) clearly being analytic for \( \sigma > 0 \). Hence

\[
\int_{-\infty}^{\infty} e^{-s\omega} \max(e^\omega - a, 0) \, d\omega = -\frac{1}{s} \zeta(s, a + 1) - \frac{a^{1-s}}{s(1-s)}. \quad (8)
\]

Equivalently, for \( 0 < a < 1, 0 < \sigma < 1 \)

\[
\int_{-\infty}^{\infty} e^{-s\omega}(e^\omega - [e^\omega - a]) \, d\omega = \int_0^{\infty} e^{-s\omega}(\{e^\omega - a\} + a) \, d\omega + \frac{1}{1-s} = -\frac{1}{s} \zeta(s, a + 1). \quad (9)
\]

Now consider the limit, \( \sigma(s, a) \), of

\[
\sigma_N(s, a) := \sum_{n=1}^N \frac{1}{(n+a)^s} - \frac{(N + a)^{1-s} - a^{1-s}}{1-s}
\]

for \( 0 < s, a < 1 \). For \( a = 0 \) we saw earlier that the limit is \( \zeta(s) \). Inter alia, we have proven:

**Proposition 1** For \( \sigma > 0 \) and \( 0 \leq a < 1 \) the limit, \( \sigma(s, a) \), below exists

\[
\sigma(s, a) := \lim_{N \to \infty} \left( \sum_{n=1}^N \frac{1}{(n+a)^s} - \frac{(N + a)^{1-s} - a^{1-s}}{1-s} \right) = \zeta(s, a + 1).
\]

Another form of the limit is

\[
\sigma(s, a) = -s \int_0^1 t \zeta(1+s, a+t) \, dt,
\]

and indeed \( \sigma(s, 0) = \zeta(s) \).

Also for any L-series whose coefficients repeat modulo \( N \) we may write

\[
\lambda(s) := \sum_{n=1}^{\infty} \frac{\lambda_n}{n^s} = \sum_{m=0}^{\infty} \sum_{k=1}^{N} \frac{\lambda_k}{(mN+k)^s} = \frac{1}{N^s} \sum_{k=1}^{N} \lambda_k \zeta \left( s, \frac{k}{N} \right); \quad (10)
\]
strictly this is true for \( \sigma > 1 \) in the first place and the equating of the extreme terms for \( \sigma > 0 \) follows by analytic continuation.

We next wish to combine this with (9) in the following form

\[
-\frac{1}{N^s} \int_{-\infty}^{\infty} e^{-\sigma \omega} \left( e^\omega - \left\lfloor \frac{e^\omega - k}{N} \right\rfloor \right) e^{-i\tau \omega} d\omega = \frac{1}{s N^s} \zeta \left( \frac{k}{N} \right) - \frac{1}{s k^s},
\]

for \( k = 1, 2, \ldots, N, \ 0 < \sigma < 1 \).

We first change variables—\( w \mapsto \omega - \log N \)—to obtain

\[
-\int_{-\infty}^{\infty} e^{-\sigma \omega} \left( e^\omega - \left\lfloor \frac{e^\omega - k}{N} \right\rfloor \right) e^{-i\tau \omega} d\omega = \frac{1}{s N^s} \zeta \left( \frac{k}{N} \right) - \frac{1}{s k^s},
\]

so that

\[
-\int_{-\infty}^{\infty} e^{-\sigma \omega} \sum_{k=1}^{N} \lambda_k \left( e^\omega - \left\lfloor \frac{e^\omega + N - k}{N} \right\rfloor \right) e^{-i\tau \omega} d\omega = \frac{\lambda(s)}{s} - \frac{1}{s} \sum_{k=1}^{N} \frac{\lambda_k}{k^s}.
\]

We have proven:

**Theorem 2** For \( 0 < \sigma < 1 \)

\[
-\int_{-\infty}^{\infty} e^{-\sigma \omega} \sum_{k=1}^{N} \lambda_k \left( e^\omega - \left\lfloor \frac{e^\omega + N - k}{N} \right\rfloor \right) e^{-i\tau \omega} d\omega = \frac{\lambda(s)}{s}.
\] (11)

In particular, when \( \Lambda_N = \sum_{k=1}^{N} \lambda_k = 0, \)

\[
\frac{\lambda(s)}{s} = 2\pi \int_{0}^{\infty} e^{-2\sigma \omega} \left| \sum_{k=1}^{N} \lambda_k \left( \frac{e^\omega + N - k}{N} \right) \right|^2 d\omega
\]

\[
= \int_{-\infty}^{\infty} e^{-\sigma \omega} \sum_{k=1}^{N} \lambda_k \left( \frac{e^\omega + N - k}{N} \right) e^{-i\tau \omega} d\omega
\]

\[
= \int_{0}^{\infty} e^{-\sigma \omega} \sum_{k=1}^{N} \lambda_k \left( \frac{e^\omega + N - k}{N} \right) e^{-i\tau \omega} d\omega,
\] (12)

and

\[
\int_{-\infty}^{\infty} \left| \frac{\lambda(s)}{s} \right|^2 d\tau = 2\pi \sum_{n=0}^{\infty} \int_{0}^{N} \frac{1}{(Nn + t)^{2\sigma + 1}} \left| \sum_{k=1}^{N} \lambda_k \left( \frac{t}{N} + \frac{N - k}{N} \right) \right|^2 dt
\]

\[
= \frac{2\pi}{N^{2\sigma}} \int_{0}^{\infty} \zeta(2\sigma + 1, \theta) \left( \sum_{k=1}^{N} \lambda_k \left( \frac{\theta}{N} + \frac{N - k}{N} \right) \right)^2 d\theta.
\] (13)
In the case that $\Lambda_N = 0$, using summation by parts we may reexpress the kernel
\[
W_N(\lambda, \theta) := \sum_{k=1}^{N} \lambda_k \left[ \theta + \frac{N - k}{N} \right] = \sum_{k=1}^{N-1} \Lambda_k \chi_k(\theta),
\]
on denoting the characteristic function of the interval $(k/N, (k+1)/N)$ by $\chi_k$. Thus,
\[
\left| \sum_{k=1}^{N} \lambda_k \left[ \theta + \frac{N - k}{N} \right] \right|^2 = \sum_{k=1}^{N-1} |\Lambda_k|^2 \chi_k(\theta).
\]
It follows that
\[
\int_{-\infty}^{\infty} \left| \frac{\lambda(s)}{s} \right|^2 d\tau = 2\pi \sum_{n=0}^{\infty} \sum_{k=1}^{N-1} |\Lambda_k|^2 \int_{k}^{k+1} \frac{1}{(N+n+t)^{2\sigma+1}} dt = \frac{\pi}{\sigma} \sum_{n=0}^{\infty} \sum_{k=1}^{N} \left| \frac{|\Lambda_k|^2 - |\Lambda_{k-1}|^2}{(N+n+k)^{2\sigma}} \right| = \frac{\pi}{\sigma} \sum_{m=1}^{\infty} \frac{|\Lambda_m|^2 - |\Lambda_{m-1}|^2}{m^{2\sigma}},
\]
whenever $\Lambda_N = \sum_{k=1}^{N} \lambda_k = 0$.

**Corollary 1** For $0 < \sigma < 1$ and complex coefficients which repeat modulo $N$:
\[
\int_{-\infty}^{\infty} \left| \frac{\lambda(s)}{s} \right|^2 d\tau = \frac{\pi}{\sigma} \sum_{m=1}^{\infty} \frac{|\Lambda_m|^2 - |\Lambda_{m-1}|^2}{m^{2\sigma}},
\]
whenever $\Lambda_N = \sum_{k=1}^{N} \lambda_k = 0$. 

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*Figure 1: The kernel function $W$ for $L_{+5}$*
This is precisely consistent with the result of Theorem 1, indeed in the case that the coefficients are periodic, the abscissa of absolute convergence is no greater than 1. Note also that when the coefficients are real \( |\Lambda_m|^2 - |\Lambda_{m-1}|^2 = (2\Lambda_{m-1} + \lambda_m)\lambda_m \). When \( N = 1, \lambda_1 = 1 \) we recover from (11) the case of the zeta function. For \( N = 2, \lambda_1 = 1 \equiv -\lambda_2 \) we reobtain from (13) or Corollary 1, a rigorous form of the original evaluation in the MAA Monthly.

In summary we have:

**Theorem 3** (Plancherel) Suppose that \( \alpha, \beta \) have a common period \( N \) such that \( A_N = \sum_{k=1}^{N} \alpha_k = 0, \ B_N = \sum_{k=1}^{N} \beta_k = 0 \). Then

\[
\int_{-\infty}^{\infty} \frac{\alpha(s) \overline{\beta}(s)}{|s|^2} \, ds = \frac{\pi}{\sigma} \sum_{n=1}^{\infty} \frac{A_n \overline{B}_n - A_{n-1} \overline{B}_{n-1}}{n^{2\sigma}},
\]

whenever \( 0 < \sigma < 1 \).

**Proof.** As in Theorem 2 and Corollary 1 we have

\[
\frac{\pi}{\sigma} \sum_{n=1}^{\infty} \frac{A_n \overline{B}_n - A_{n-1} \overline{B}_{n-1}}{n^{2\sigma}} = 2\pi \sum_{n=0}^{\infty} \int_{0}^{N} \frac{1}{(Nn + t)^{2\sigma + 1}} \, W_N(\alpha, t) W_N(\overline{\beta}, t) \, dt
\]

\[
= \frac{2\pi}{N^{2\sigma}} \int_{0}^{1} \zeta(2\sigma + 1, \theta) W_N(\alpha, \theta) W_N(\overline{\beta}, \theta) \, d\theta
\]

\[
= 2\pi \int_{0}^{\infty} e^{-2\sigma \omega} \, W_N(\alpha, e^{i\omega}) W_N(\overline{\beta}, e^{i\omega}) \, d\omega = \int_{-\infty}^{\infty} \frac{\alpha(s) \overline{\beta}(s)}{|s|^2} \, ds.
\]

\[\Box\]

**Example 4** Recall that \( \zeta(p, b) := \sum_{n=2}^{\infty} (-1)^n/n^a \sum_{k=1}^{n-1} 1/k^b \), while \( \zeta(a, \overline{b}) := \sum_{n=2}^{\infty} 1/n^a \sum_{k=1}^{n-1} (-1)^k/k^b \), and \( \zeta(p, \overline{b}) := \sum_{n=2}^{\infty} (-1)^n/n^a \sum_{k=1}^{n-1} (-1)^k/k^b \).

The same approach as in Example 1 applied to (4), (16) produces

\[
\frac{1}{\pi} \int_{0}^{\infty} \frac{\alpha(3/2 + i\tau) \overline{\alpha}(3/2 + i\tau)}{1/4 + \tau^2} \, d\tau = 2 \zeta(\overline{2}, 1) + \zeta(3) = 3 \zeta(2) \log 2 - \frac{9}{4} \zeta(3),
\]

and

\[
\frac{1}{\pi} \int_{0}^{\infty} \frac{\alpha(3/2 + i\tau) \zeta(3/2 + i\tau)}{1/4 + \tau^2} \, d\tau = \zeta(\overline{2}, 1) + \zeta(2, 1) + \alpha(3) = \frac{9}{8} \zeta(2) \log 2 - \frac{3}{4} \zeta(3),
\]

as companions to

\[
\frac{1}{\pi} \int_{0}^{\infty} \frac{\zeta(3/2 + i\tau) \overline{\zeta}(3/2 + i\tau)}{1/4 + \tau^2} \, d\tau = 3 \zeta(3),
\]

since

\[
\zeta(\overline{2}, 1) = \frac{1}{8} \zeta(3), \quad \zeta(2, 1) = \zeta(3) - \frac{3}{2} \zeta(2) \log 2, \quad \zeta(\overline{2}, 1) = \frac{3}{2} \zeta(2) \log 2 - \frac{13}{8} \zeta(3).
\]

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Figure 2: The integrands for $\alpha$ and $L_{-3}$ respectively with $\sigma = 1/2, 1, 2$

4 Final Remarks

Many other similar results obtain. For example:

$$\int_{-\infty}^{\infty} e^{-2\sigma \omega} \{\max(e^\omega - a, 0)\}^2 d\omega = \sum_{n=0}^{\infty} \int_{n+a}^{n+1+a} t^{-2\sigma-1} \{t - a\}^2 dt$$

$$= \sum_{n=0}^{\infty} \int_{0}^{1} \theta^2 (\theta + n + a)^{-2\sigma-1} d\theta$$

$$= \int_{0}^{1} \theta^2 \zeta(1 + 2\sigma, \theta + a) d\theta.$$

While (3) and (16) give an effective way of evaluating the integral, directly evaluating the integral numerically to high precision presents a greater challenge. This is largely because of the severe oscillations of the integrand as illustrated in Figure 2 with $\sigma = 1/2, 1, 2$ for $\alpha$ and $L_{-3}$ respectively. The issue appears to lie in estimating the integrand well and so is intrinsically non-trivial.
References


