Implications of Experimental Mathematics for the Philosophy of Mathematics

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Christopher Koch accurately captures a great scientific distaste for philosophizing:

_Whether we scientists are inspired, bored, or infuriated by philosophy, all our theorizing and experimentation depends on particular philosophical background assumptions. This hidden influence is an acute embarrassment to many researchers, and it is therefore not often acknowledged._ (Christof Koch, 2004)

That acknowledged, mathematical philosophy matters more now than it has in nearly a century.

1 Mathematical Knowledge as I View It

Somewhat unusually, I can exactly place the day at registration that I became a mathematician and I recall the reason why. I was about to deposit my punch cards in the ‘honours history bin’. I remember thinking

_If I do study history, in ten years I shall have forgotten how to use the calculus properly. If I take mathematics, I shall still be able to read competently about the War of 1812 or the Papal schism._ (Jonathan Borwein, 1968)

The inescapable reality of objective mathematical knowledge is still with me. Suffice it to say that, nonetheless, my view then of the edifice I was entering is not that close to my view of the one I inhabit nearly 40 years later.

I also know when I became a computer assisted fallibilist. Reading Imre Lakatos’ _Proofs and Refutations_ while a post doctoral student a few years later I was suddenly absolved from the grave sin of error, as I began to understand that missteps, mistakes and errors are the grist of all creative work. The book, his doctorate posthumously published in (1976), is a student conversation of the Euler characteristic. The students are of various philosophical stripes and the discourse benefits from his early work on Hegel with the Stalinist Lukács in Hungary and of later study with Karl Popper at the London School of Economics. I had been prepared for this dispensation by the opportunity to learn a variety of subjects from Michael Dummett. Dummett was at that time completing his rehabilitating study of

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Frege\textsuperscript{4}. A decade later the appearance of the first ‘portable’ computers happily coincided with my desire to decode Srinivasa Ramanujan’s (1887–1920) cryptic assertions about theta functions and elliptic integrals. I realized that by coding his formulae and my own in the \textit{APL} programming language\textsuperscript{5}, I was able to rapidly confirm and refute identities and conjectures and to travel much more rapidly and fearlessly down potential blind alleys. I had become a computer assisted fallibilist, at first somewhat falteringly but twenty years have certainly honed my abilities.

I first discuss my views, and those of others, on the nature of mathematics, and then illustrate these views in a variety of mathematical contexts.\textsuperscript{6}

Kurt G\ödel may well have overturned the mathematical apple cart entirely deductively, but nonetheless he could hold quite different ideas about legitimate forms of mathematical reasoning:

\begin{quote}
\textit{If mathematics describes an objective world just like physics, there is no reason why inductive methods should not be applied in mathematics just the same as in physics.} (Kurt G\ödel\textsuperscript{7}, 1951)
\end{quote}

While we mathematicians have often separated ourselves from the sciences, they have tended to be more ecumenical. For example, a recent review of \textit{Models. The Third Dimension of Science}\textsuperscript{8} chose a mathematical plaster model of a Clebsch diagonal surface as its only illustration. Similarly, authors seeking examples of the aesthetic in science often choose iconic mathematics formulae such as $E = MC^2$.

Let us start by fixing concepts and then turn to the implications for mathematics and for mathematical philosophy. Above all, I hope to persuade you of the power of mathematical experimentation—it is also fun—and that the traditional accounting of mathematical learning and research is largely an ahistorical caricature. Let us recall three terms.

\textbf{mathematics, n.} a group of related subjects, including algebra, geometry, trigonometry and calculus, concerned with the study of number, quantity, shape, and space, and their inter-relationships, applications, generalizations and abstractions.

This definition—taken from the Collins Dictionary—makes no immediate mention of proof, nor of the means of reasoning to be allowed. The Webster’s Dictionary contrasts:

\textbf{induction, n.} any form of reasoning in which the conclusion, though supported by the premises, does not follow from them necessarily; and

\textbf{deduction, n.} a process of reasoning in which a conclusion follows necessarily from the premises presented, so that the conclusion cannot be false if the premises are true.

\textit{b. a conclusion reached by this process.}

\textsuperscript{5}A ‘read only’ very high level language, APL was a fine tool albeit with a steep learning curve.
\textsuperscript{6}A considerably more detailed treatment of many of these topics is to be found in Dave Bailey and my book \textit{Mathematics by Experiment: Plausible Reasoning in the 21st Century.}
\textsuperscript{7}Taken from a previously unpublished manuscript in his \textit{Collected Works}, Volume III.
I, like Gödel, suggest that both should be entertained in mathematics. In the sequel, I shall talk broadly about experimental and heuristic mathematics, giving accessible, primarily visual and symbolic, examples.

2 Philosophy of Experimental Mathematics

The computer has in turn changed the very nature of mathematical experience, suggesting for the first time that mathematics, like physics, may yet become an empirical discipline, a place where things are discovered because they are seen.9 (David Berlinski)

The shift from typographic to digital culture shift is vexing for mathematicians. For example, there is still no truly satisfactory way of displaying mathematics on the web—demand certainly not of asking mathematical questions. Also, we respect authority10 but value authorship deeply—however much the two values are in conflict. And we care more about the reliability of our literature than does any other science, indeed we reify this notion.

The traditional central role of proof in mathematics is arguably and perhaps appropriately under siege. Via examples, I intend to pose and answer various questions. I shall conclude with a variety of quotations from our progenitors and even contemporaries:


My Answers. To misquote D’Arcy Thompson (“Form follows Function”), rigor (proof) follows reason (discovery); indeed, excessive focus on rigor has driven us away from our wellsprings. Many good ideas are wrong. Not all truths are provable, and not all provable truths are worth proving . . . Moreover, Near certainty is often as good as it gets—intellectual context (community) matters. Complex human proofs are often very long, extraordinarily subtle and fraught with error—consider, Fermat’s last theorem, the Poincaré conjecture, the classification of finite simple groups, presumably any proof of the Riemann hypothesis. In all these settings, Modern computational tools dramatically change the nature and scale of available evidence.

A more inductive approach can have significant benefits. For example, as there is still some doubt about the proof of the classification of finite simple groups it is important to ask whether the result true but the proof flawed, or if rather there is still perhaps an ‘ogre’ sporadic group even larger than the ‘monster’? What heuristic, probabilistic or computational tools can increase our confidence that the ogre does or does not exist? Likewise, there are experts who still believe the Riemann hypothesis11 (RH) is false and that the billions of

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11All non-trivial zeroes—not negative even integers—of the zeta function lie on the line with real part 1/2.
zeroes found so far are much too small to be representative. In any event, our understanding of the complexity of various crypto-systems relies on (RH) and we should like secure knowledge that any counter-example is enormous.

**Peter Medawar** (1915–87) writing in *Advice to a Young Scientist*(1979) identifies four forms of scientific experiment:

1. The *Kantian experiment*: generating “the classical non-Euclidean geometries (hyperbolic, elliptic) by replacing Euclid’s axiom of parallels (or something equivalent to it) with alternative forms.” All mathematicians perform such experiments while the majority of computer explorations are of the following *Baconian* form.

2. *The Baconian experiment* is a contrived as opposed to a natural happening, it “is the consequence of ‘trying things out’ or even of merely messing about.” Baconian experiments are the explorations of a happy if disorganized beachcomber and carry little predictive power.

3. *Aristotelian demonstrations*: “apply electrodes to a frog’s sciatic nerve, and lo, the leg kicks; always precede the presentation of the dog’s dinner with the ringing of a bell, and lo, the bell alone will soon make the dog dribble.” Arguably our ‘Corollaries’ and ’Examples’ are Aristotelian, they reinforce but do not predict. Medawar goes on to say the most important is:

4. *The Galilean*: is “a critical experiment – one that discriminates between possibilities and, in doing so, either gives us confidence in the view we are taking or makes us think it in need of correction.” The Galilean the only form of experiment which stands to make Experimental Mathematics a serious enterprise. Performing careful, replicable Galilean experiments requires work and care.

**Reuben Hersh**’s arguments for a humanist philosophy of mathematics\(^\text{12}\), as paraphrased below, become even more convincing in our computational setting.

1. Mathematics is human. *It is part of and fits into human culture. It does not match Frege’s concept of an abstract, timeless, tenseless, objective reality.*

2. Mathematical knowledge is fallible. *As in science, mathematics can advance by making mistakes and then correcting or even re-correcting them. The “fallibilism” of mathematics is brilliantly argued in Lakatos’ Proofs and Refutations.*

3. There are different versions of proof or rigor. *Standards of rigor can vary depending on time, place, and other things. The use of computers in formal proofs, exemplified by the computer-assisted proof of the four color theorem in 1977\(^\text{13}\), is just one example of an emerging nontraditional standard of rigor.*

4. Empirical evidence, numerical experimentation and probabilistic proof all can help us decide what to believe in mathematics. *Aristotelian logic isn’t necessarily always the best way of deciding.*

5. Mathematical objects are a special variety of a social-cultural-historical object. *Contrary to the assertions of certain post-modern detractors, mathematics cannot be dismissed as merely a new form of literature or religion. Nevertheless, many mathematical objects can be seen as shared ideas, like Moby Dick in literature, or the Immaculate Conception in religion.*


\(^{13}\)Especially after revision in 1997.
I entirely subscribe to points 2.–4., and with certain caveats about objective knowledge\textsuperscript{14} to points 1. and 5. In any event mathematics is and will remain a uniquely human undertaking.

This sits fairly comfortable with

\textit{The social constructivist thesis is that mathematics is a social construction, a cultural product, fallible like any other branch of knowledge.} (Paul Ernest)

I personally qualify this with “but much less fallible than most.” Associated most notably with the writings of Paul Ernest,\textsuperscript{15} social constructivism seeks to define mathematical knowledge and epistemology through the social structure and interactions of the mathematical community and society as a whole.

Many of the ideas our students—and some colleagues—take for granted took a great deal of time to gel. The Greeks were aware of the impossibility of the three classical construction problems \textsuperscript{16} and the irrationality of the golden mean was well known to the Pythagoreans.

While concerns about potential and completed infinities are very old, until the advent of the calculus with Newton and Leibnitz and the need to handle fluxions or infinitesimals, the level of need for rigour remained modest. Certainly Euclid is in its geometric domain generally a model of rigour, while Archimedes’ numerical analysis was not also equalled until the 19th century.

The need for rigour arrives in full force in the time of Cauchy and Fourier. The treacherous countably infinite processes of analysis and the limitations of formal manipulation came to the fore. It is difficult with a modern sensibility to understand how Cauchy’s proof of the continuity of pointwise limits could coexist for half a century with Fourier’s clear counter-examples originating in his theory of heat.

By the end of the 19th century Frege’s (1848-1925) attempt to base mathematics in a linguistically based \textit{logicism} had foundered on Russell and other’s discoveries of the paradoxes of naive theory. Within thirty five years Gödel and then Turing had similarly damaged both Russell and Whitehead’s and Hilbert’s programs.

Throughout the twentieth century, bolstered by the armour of abstraction, the great ship Mathematics has sailed on largely unperturbed. During the last decade of the 19th and first few decades of the 20th century the following main streams of philosophy emerged explicitly within mathematics to replace logicism, but primarily as the domain of philosophers and logicians.

\textsuperscript{14}While it is not Hersh’s intention, a superficial reading of 5. hints at a cultural relativism to which I do not subscribe.

\textsuperscript{15}In \textit{Social Constructivism As a Philosophy of Mathematics}, Ernest, an English Mathematician and Professor in the Philosophy of Mathematics Education, carefully traces the intellectual pedigree for his thesis, a pedigree that encompasses the writings of Wittgenstein, Lakatos, Davis, and Hersh among others.

\textsuperscript{16}Trisection, circle squaring and cube doubling were known by the educated to be impossible in antiquity, while the French Academy stopped accepting proofs a full two centuries before the 19th century proofs of their impossibility.
• **Platonism.** Everyman’s idealist philosophy—stuff exists and we must find it. Despite being the oldest mathematical philosophy, Platonism was only christened in 1936.

• **Formalism.** Associated mostly with Hilbert—it asserts that mathematics is invented and is best viewed as formal symbolic games without intrinsic meaning.

• **Intuitionism.** Invented by Brouwer and championed by Heyting, intuitionism asks for inarguable monadic components that can be fully analyzed and has many variants; this has interesting overlaps with recent work in cognitive psychology such as Lakoff and Nunez’ work on ‘embodied cognition’.\(^{17}\)

• **Constructivism.** Originating with Markoff and especially Kronecker (1823–1891) and refined by Bishop it finds fault with significant parts of classical mathematics. Its ‘I’m from Missouri, tell me how big it is’ sensibility is not to be confused with Paul Ernest’s ‘social constructivism’.\(^{18}\)

The last two philosophies deny the principle of the *excluded middle*: “A or not A”, and resonate with computer science—as does some of formalism. It is hard after all to run a deterministic program which does not know which disjunctive gate to follow. By contrast the battle between ‘absolutism’ and ‘fallibilism’ plays out across all four, but fallibilism perhaps lives most easily within a restrained version of intuitionism which looks for ‘intuitive arguments’ and is willing to accept that ‘a proof is what convinces’. As Lakatos shows an argument convincing a hundred years ago may well now now viewed as inadequate.

Perhaps it is only in the last 25 years, with the emergence of powerful mathematical platforms, that any approach other than a soggy Platonism and a reliance on proof and abstraction has had the tools to give it traction.

In this light, Hales’ proof of Kepler’s conjecture that the **densest way to stack spheres is in a pyramid** resolves the oldest problem in discrete geometry. It also supplies the most interesting recent example of intensively computer assisted proof, and after five years with the review process was published in the *Annals of Mathematics*—with an “only 99% checked” disclaimer.

This process has triggered very varied reactions\(^{19}\) and has provoked Thomas Hales to attempt a formal computational proof. Famous earlier examples of computer-assisted proof include *The Four Color Theorem* and *The non existence of a projective plane of order 10*. The three raise and answer quite distinct questions about computer assisted proof—both real and specious.

To make the case as to how far mathematical computation has come we trace the changes over the past half century. The 1949 computation of \(\pi\) to 2,037 places suggested by von Neumann, took 70 hours. A billion digits may now be computed in much less time on a

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\(^{17}\)“It is by logic we prove, it is by intuition that we invent” reflects the cognate views of Henri Poincaré (1854–1912)


laptop. Strikingly, it would have taken roughly 100,000 ENIACs to store the Smithsonian’s picture—as is possible thanks to 40 years of Moore’s law in action . . .

“Moore’s Law” is now taken to be the assertion that semiconductor technology approximately doubles in capacity and performance roughly every 18 to 24 months. This is an astounding record of sustained exponential progress without peer in history of technology. Additionally, mathematical tools are now being implemented on parallel platforms, providing much greater power to the research mathematician. Amassing huge amounts of processing power will not alone solve many mathematical problems. There are few math ‘Grand-challenge problems’—there is much more value in very rapid ‘Aha’s’ as can be obtained through micro-parallelism. That is, to get answers on a neurologically rapid scale.

To sum up, in light of the discussion and terms above, I now describe my self a a social-constructivist, and a computer-assisted fallibilist with constructivist leanings.

3 Our Experimental Methodology

Despite Picasso’s complaint that “computers are useless, they only give answers,” the main role of computation in mathematics is clearly to yield insight. This demands speed or equivalently substantial micro-parallelism to provide answers on a cognitively relevant scale; so that we may ask more questions while they remain in our consciousness. This is relevant for rapid verification; for validation; for proofs and especially for refutations which includes what Lakatos calls “monster barring”.

In this setting it is enough to equate parallelism with more space, speed, and to consider all computations as ‘exact’ which provides reliable answers. This now usually requires a careful hybrid of symbolic and numeric methods, such as achieved by Maple’s liaison with the NAG library. There are now excellent tools for such purposes throughout analysis, algebra, geometry and topology.

Along the way questions required by or just made natural by computing start to force out older questions and possibilities in the way described by Dewey. Additionally, what is “easy” changes: high performance computing and networking are blurring, merging disciplines and collaborators—democratizing mathematics but further challenging authentication.

Moving towards a well articulated Experimental Methodology—both in theory and practice—will take much effort. The need is premised on the assertions that Intuition is acquired—we can and must better mesh computation and mathematics. That visualization is of growing importance—in many settings even three is a lot of dimensions.

“Monster-barring” (Lakatos) and “Caging” (my own term for imposing needed restrictions in a conjecture) are often easy to enhance computationally, as for example with randomized checks of equations, linear algebra, primality or graphic checks: equalities, inequalities, areas.

Let us next recapitulate the main steps Bailey and I discuss in our two recent books:

#1. Gaining insight and intuition


#2. Discovering new patterns and relationships

#3. Graphing to expose math principles

#4. Testing and especially falsifying conjectures

#5. Exploring a possible result to see if it merits formal proof

#6. Suggesting approaches for formal proof

#7. Computing replacing lengthy hand derivations

#8. Confirming analytically derived results

4 Finding Things versus Proving Things

I now illuminate these eight steps with eight mathematical examples.

1. **Pictorially comparison** of $y - y^2$ and $y^2 - y^4$ to $-y^2 \ln(y)$ when $y$ lies in the unit interval, is a much more rapid way to divine which function is larger than by using traditional analytic methods.

   The figure shows that it is clear in the later case they cross, and it is futile to try to prove one majorizes the other. In the first case, evidence is provided to motivate attempting a proof and often the picture serves to guide such a proof.

   ![Graphical comparison](image)

   **Graphical comparison of** $y - y^2$ and $y^2 - y^4$ to $-y^2 \ln(y)$ (red)

2. A **disproof and a proof.** Any modern computer algebra can tell one that

   \[0 < \int_0^1 \frac{(1-x)^4x^4}{1+x^2} \, dx = \frac{22}{7} - \pi,\]

   since the integral may be interpreted as the area under a positive curve. We are however no wiser as to why! If however we ask the same system to compute the indefinite integral, we are likely to be told that

   \[
   \int_0^t \left( \frac{1}{7} t^7 - \frac{2}{3} t^6 + \frac{5}{3} t^5 - 4 t^4 + 4 t - 4 \arctan(t) \right).
   \]
Then (1) is now rigourously established by differentiation and an appeal to the Fundamental theorem of calculus.

3. A computer discovery and a ‘proof’ of the series for arcsin(x)^2. We compute a few coefficients and observe that there is a regular power of 4 in the numerator, and integers in the denominator; or equivalently we look at arcsin((x/2)^2). The generating function package ‘gfun’ in Maple then predicts a recursion (r) for the denominators and solves it (R).

\[ s := \{ \text{seq}(1/\text{coeff}(\text{series}(\text{arcsin}(x/2)^2,x,25),x,2*n),n=1..6)) \}; \]
\[ R := \text{unapply}(\text{rsolve}((\text{op}(1, \text{listtorec}(s,r(\cdot))))\),r(\cdot))\),m);[\text{seq}(R(m),m=0..8)]; \]
yields, \( s := [4, 48, 360, 2240, 12600, 66528] \),
\[
R := m \mapsto 8 \frac{4^m \Gamma(3/2 + m)(m + 1)}{\pi^{1/2} \Gamma(1 + m)},
\]
and then \( [4, 48, 360, 2240, 12600, 66528, 336336, 1647360, 7876440] \).

We may now use Sloane’s Online Encyclopedia of Integer Sequences\(^{20}\) to reveal that the coefficients are \( R(n) = 2^n (2n) \). More precisely, sequence A002544 identifies \( R(n)/4 = \binom{2n+1}{n}(n+1)^2 \).

\[ > \text{seq}(2*n^2*\text{binomial}(2*n,n),n=1..8)]; \]
confirms this with \( [4, 48, 360, 2240, 12600, 66528, 336336, 1647360] \).

Next

\[ > S := \text{Sum}((2*x)^2*(2*n)/(2*n^2*\text{binomial}(2*n,n)),n=1..\text{infinity}); S=\text{values}(S); \]
returns
\[
\frac{1}{2} \sum_{n=1}^{\infty} \frac{(2x)^{2n}}{n^2 \binom{2n}{n}} = \arcsin(x)^2.
\]

That is, we have discovered—and proven if we trust or verify Maple’s summation algorithm—the desired Maclaurin series.

\(^{20}\)At www.research.att.com/~njas/sequences/index.html
4. Discovery without proof. Donald Knuth\textsuperscript{21} asked for a closed form evaluation of:

\[
\sum_{k=1}^{\infty} \left\{ \frac{k^k}{k! e^k} - \frac{1}{\sqrt{2 \pi k}} \right\} = -0.084069508727655 \ldots
\]

Since 2000 CE it has been easy to compute 20 or 200 digits of this sum, the ‘smart lookup’ facility in the Inverse Symbolic Calculator\textsuperscript{22} rapidly returns

\[
0.084069508727655 \approx \frac{2}{3} + \frac{\zeta(1/2)}{\sqrt{2 \pi}}.
\]

We thus have a prediction which Maple 9.5 on a 2004 laptop confirms to 100 places in under 6 seconds and to 500 in 40 seconds. Arguably we are done.\textsuperscript{23}

5. A striking conjecture with no known proof strategy is: for \(N = 1, 2, \cdots\)

\[
\sum_{n>m>o>p>0} (-1)^n \frac{1}{n^2 m^2 o^2 p} = \zeta(2, 1, 1, 1, \ldots)
\]

Explicitly, the first two cases are

\[
\sum_{n>m>0} \frac{(-1)^n}{n^2 m} = \sum_{n>0} \frac{1}{n^3} \quad \sum_{n>m>o>p>0} \frac{(-1)^n}{n^2 m o^2 p} = \sum_{n>m>0} \frac{1}{n^3 m^3}.
\]

The notation should now be clear. Such alternating sums are called multi-zeta values (MZV) and positive ones are called Euler sums after Euler who first studied them seriously. There is abundant evidence amassed since it was found in 1996. For example, very recently Petr Lisonek checked the first 85 cases to 1000 places in about 41 HP hours with only the predicted round-off error. And the case \(N=163\) was checked in about ten hours.

This is the only identification of its type of an Euler sum with a distinct MZV and we have no idea why it is true. Any similar MZV proof has been highly non-trivial. Can even just the case \(n = 2\) be proven symbolically as has been the case for \(n = 1\)? \(\Box\)

\textsuperscript{21}Posed as MAA Problem 10832, November 2002.

\textsuperscript{22}At www.cecm.sfu.ca/projects/ISC/ISCmain.html

\textsuperscript{23}In this case one can also be lead by the computer to very satisfactory computer assisted proof, [6].
6. **What You Draw is What You See.** It is hard to see how the structure revealed in the pictures above would be seen other than through graphically data mining. Note the different shapes of the holes around the various roots of unity. The striations are unexplained but all re-computations expose them! And the fractal structure is provably there. Nonetheless different ways of measuring the stability of the calculations reveal somewhat different features.

7. **Visual Dynamics.** In recent continued fraction work, Crandall and I needed to study the dynamical system $t_0 := t_1 := 1$:

$$t_n \leftarrow \frac{1}{n} t_{n-1} + \omega_{n-1} \left( 1 - \frac{1}{n} \right) t_{n-2},$$

where $\omega_n = a^2, b^2$ for $n$ even, odd respectively, are two unit vectors. Think of this as a black box which we wish to scientifically examine. Numerically all one sees is $t_n \to 0$ slowly. Pictorially we learn significantly more. If the iterates are plotted with changing colour, it is clear that they spiral roman-candle like in to the origin:

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25 We plot all complex zeroes of polynomials with only 0 and 1 as coefficients up to a given degree. As the degree increases some of the holes fill in.

26 ... “Then felt I like a watcher of the skies, when a new planet swims into his ken.” (Chapman’s Homer)
Scaling by $\sqrt{n}$, and coloring even and odd iterates, fine structure appears. We now observe, predict and validate that the outcomes depend on whether or not one or both of $a$ and $b$ are roots of unity (that is, rational multiples of $\pi$).

The attractors for various $|a| = |b| = 1$

It took my coauthors and me, over a year and 100 pages to convert this intuition into a rigorous formal proof, [7]. Indeed, the results are technical and delicate enough that I have more faith in the facts than in the finished argument. I am not alone.

Carl Friedrich Gauss, who drew (carefully) and computed a great deal, once noted, I have the result, but I do not yet know how to get it.\textsuperscript{27} An excited young Gauss writes: “A new field of analysis has appeared to us, self-evidently, in the study of functions etc.” (October 1798). It had and the consequent proofs pried open the doors of much modern elliptic function and number theory.

# 8 Consider the unsolved Problem 10738 in the 1999 American Mathematical Monthly:

**Problem:** For $t > 0$ let

$$m_n(t) = \sum_{k=0}^{\infty} h^n \exp(-t) \frac{t^k}{k!}$$

\textsuperscript{27}Likewise, the quote has so far escaped exact isolation!
be the $n$th moment of a Poisson distribution with parameter $t$. Let $c_n(t) = m_n(t)/n!$.

Show

a) $\{m_n(t)\}_{n=0}^{\infty}$ is log-convex for all $t > 0$.

b) $\{c_n(t)\}_{n=0}^{\infty}$ is not log-concave for $t < 1$.

c$^*$) $\{c_n(t)\}_{n=0}^{\infty}$ is log-concave for $t \geq 1$.

Solution. (a) Neglecting the factor of $\exp(-t)$ as we may, this reduces to

$$\sum_{k,j \geq 0} \frac{(jk)^{n+1} t^k}{k!j!} \leq \sum_{k,j \geq 0} \frac{(jk)^n t^k}{k!j!} \frac{k^2 + j^2}{2},$$

and this now follows from $2jk \leq k^2 + j^2$.

(b) As

$$m_{n+1}(t) = t \sum_{k=0}^{\infty} (k+1)^n \exp(-t) \frac{t^k}{k!},$$

on applying the binomial theorem to $(k+1)^n$, we see that $m_n(t)$ satisfies the recurrence

$$m_{n+1}(t) = t \sum_{k=0}^{n} \binom{n}{k} m_k(t), \quad m_0(t) = 1.$$ 

In particular for $t = 1$, we obtain the sequence

$$1, 1, 2, 5, 15, 52, 203, 877, 4140, \ldots.$$ 

These are the 0.00,0.00,1.00 Bell numbers as was discovered again by consulting Sloane’s Encyclopedia which can also tell us that, for $t = 2$, we have the generalized Bell numbers, and gives the exponential generating functions. Inter alia, an explicit computation shows that

$$t \frac{1 + t}{2} = c_0(t) c_2(t) \leq c_1(t)^2 = t^2$$

exactly if $t \geq 1$, which completes (b).

Also, preparatory to the next part, a simple calculation shows that

$$\sum_{n \geq 0} c_n t^n = \exp\left(t(e^n - 1)\right). \quad (2)$$

(c$^*$) We appeal to a recent theorem due to E. Rodney Canfield, which proves the lovely and quite difficult result below.

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28 A sequence $\{a_n\}$ is log-convex if $a_{n+1}a_{n-1} \geq a_n^2$, for $n \geq 1$ and log-concave when the sign is reversed.

29 The Bell numbers were known earlier to Ramanujan — an example of Stigler’s Law of Eponymy!

30 The '*' indicates this was the unsolved component.

31 A search in 2001 on MathSciNet for “Bell numbers” since 1995 turned up 18 items. This paper showed up as number 10. Later, Google found it immediately!
Theorem 1 If a sequence $1, b_1, b_2, \cdots$ is non-negative and log-concave then so is the sequence $1, c_1, c_2, \cdots$ determined by the generating function equation

$$\sum_{n \geq 0} c_n u^n = \exp \left( \sum_{j \geq 1} b_j \frac{u^j}{j} \right).$$

Using equation (2) above, we apply this to the sequence $b_j = t/(j-1)!$ which is log-concave exactly for $t \geq 1$.

Quite unusually, the given solution to (c) was the only one received by the Monthly. The reason might well be that it relied on the following sequence of steps:

(Question Posed) $\Rightarrow$ Computer Algebra System $\Rightarrow$ Interface $\Rightarrow$

Search Engine $\Rightarrow$ Digital Library $\Rightarrow$ Hard New Paper $\Rightarrow$ (Answer)

Now if only we could automate this!

Jacques Hadamard, describes the role of proof as well as anyone—and most persuasively given his 1896 proof of the Prime number theorem an inarguable apex of rigorous analysis.

"The object of mathematical rigor is to sanction and legitimize the conquests of intuition, and there was never any other object for it."  

(Jacques Hadamard)

5 Conclusions

We return to the matter of what it takes to persuade an individual to adopt new methods and drop time honoured ones. Aptly, we may start by consulting Kuhn on the matter of paradigm shift:

"The issue of paradigm choice can never be unequivocally settled by logic and experiment alone. \cdots{} in these matters neither proof nor error is at issue. The transfer of allegiance from paradigm to paradigm is a conversion experience that cannot be forced."  

(Thomas Kuhn)

The pragmatist philosopher John Dewey agrees

"Old ideas give way slowly; for they are more than abstract logical forms and categories. They are habits, predispositions, deeply engrained attitudes of aversion and preference. \cdots{} Old questions are solved by disappearing, evaporating, while new questions corresponding to the changed attitude of endeavor and preference take their place"  

(John Dewey)

\footnote{Hadamard quoted at length in E. Borel, Lecons sur la theorie des fonctions, 1928.}

\footnote{In Who Got Einstein's Office? The answer is Arne Beurling.}

\footnote{John Dewey, The Influence of Darwin on Philosophy, 1910.}
Planck like Dewey has famously remarked on the difficulty of such paradigm shifts:

> And Max Planck, surveying his own career in his Scientific Autobiography, sadly remarked that “a new scientific truth does not triumph by convincing its opponents and making them see the light, but rather because its opponents eventually die, and a new generation grows up that is familiar with it.” (Albert Einstein)

This transition is certainly already apparent. It is certainly rarer to find a mathematician under thirty who is unfamiliar with at least one of Maple, Mathematica or MatLab, than it is to one over sixty five who is really fluent.

In his ‘23’ “Mathematische Probleme” lecture to the Paris International Congress in 1900, Hilbert writes\(^{35}\)

> Moreover a mathematical problem should be difficult in order to entice us, yet not completely inaccessible, lest it mock our efforts. It should be to us a guidepost on the mazy path to hidden truths, and ultimately a reminder of our pleasure in the successful solution. (David Hilbert, 1900)

Note the primacy given by a most exacting researcher to discovery and to truth over proof and rigor. More controversially and most of a century later, Greg Chaitin invites us to be bolder and act more like physicists.

> I believe that elementary number theory and the rest of mathematics should be pursued more in the spirit of experimental science, and that you should be willing to adopt new principles... And the Riemann Hypothesis isn’t self-evident either, but it’s very useful. A physicist would say that there is ample experimental evidence for the Riemann Hypothesis and would go ahead and take it as a working assumption. ... We may want to introduce it formally into our mathematical system.\(^ {36} \) (Greg Chaitin, 1994)

Much as I admire the challenge of Greg Chaitin’s statement, I am not yet convinced that it is helpful to add axioms as opposed to proving conditional results that start “Assuming the continuum hypothesis” or emphasize that “without assuming the Riemann hypothesis we are able to show.” Most important is that we lay our cards on the table. We should explicitly and honestly indicate when we believe our tools to be heuristic, we should carefully indicate why we have confidence in our computations—and where our uncertainty lies—and the like.

On that note, Hardy is supposed to have commented—somewhat dismissively—that Landau, a great German number theorist, would never be the first to prove (RH), but that if someone else did so then Landau would have the best possible proof shortly after. I certainly

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\(^{35}\)See the late Ben Yandell’s fine account of the Hilbert Problems and their solvers in The Honors Class, AK Peters, 2002. The written lecture is considerably longer and further ranging that the one delivered in person.

\(^{36}\)A like purposed article is in the 2004 Mathematical Intelligencer.
hope that a more experimental methodology will better value independent replication and honour the first transparent proof\textsuperscript{37} of Fermat’s last theorem as much as Andrew Wiles’ monumental proof. Hardy also commented that he did his best work past forty. Inductive, accretive, tool-assisted mathematics certainly allows brilliance to be supplemented by experience and—as in my case—stands to further undermine the notion that one does ones best mathematics young.

5.1 Last Words

To reprise, I hope to have made convincing arguments that the traditional deductive accounting of Mathematics is a largely ahistorical caricature—Euclid’s millennial sway not withstanding.\textsuperscript{38} Above all, mathematics is primarily about secure knowledge not proof, and that while the aesthetic is central, we must put much more emphasis on notions of supporting evidence and attend to the reliability of witnesses.

Proofs are often out of reach—but understanding, even certainty, is not. Certainly computer packages can make concepts accessible. A short list includes linear relation algorithms, Galois theory, Groebner bases, etc. While progress is made “one funeral at a time”\textsuperscript{39}, in Thomas Wolfe’s words “you can’t go home again” and as the co-inventor of the Fast Fourier transform properly observed

\begin{quote}
\textit{Far better an approximate answer to the right question, which is often vague, than the exact answer to the wrong question, which can always be made precise.}

(J. W. Tuckey, 1962)
\end{quote}

References

1. J.M. Borwein, P.B. Borwein, R. Girgensohn and S. Parnes, “Making Sense of Experimental Mathematics,” Mathematical Intelligencer, 18, (Fall 1996), 12–18.\textsuperscript{40} [CECM 95:032]


\textsuperscript{37}If such should such exist and be discovered or invented.

\textsuperscript{38}Most quotations are stored at jborwein/quotations.html

\textsuperscript{39}This grim version of Planck’s comment is sometimes attribute to Niels Bohr

\textsuperscript{40}All references are lodged at www.cecm.sfu.ca/preprints.


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