SENSITIVITY ANALYSIS FOR OPTIMIZATION PROBLEMS WITH VARIATIONAL INEQUALITY CONSTRAINTS

YVES LUCET AND JANE J. YE

ABSTRACT. In this paper we perform sensitivity analysis for optimization problems with variational inequality constraints (OPVIC). We provide estimates for the limiting subgradient (singular limiting subgradient) of the value function in terms of the set of normal (abnormal) Coderivative (CD) multipliers for (OPVIC). For the case of optimization problem with complementarity constraints (OPCC), we proved that there always exist nonzero abnormal Non-linear Programming (NLP) multipliers. Hence the NLP multipliers may not provide useful information on the subgradients of the value function while the CD multipliers may provide tighter bounds. Applications to the sensitivity analysis of bilevel programming problems are also given.

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1. INTRODUCTION

In this paper, we consider the sensitivity analysis for the following optimization problem with variational inequality constraints (OPVIC):

\begin{align*}
\text{(OPVIC)} \quad & \text{minimize} \quad f(x, y) \\
& \text{subject to} \quad \Psi(x, y) \leq 0, (x, y) \in C \\
& \quad \quad \quad \quad \quad \quad y \in \Omega, \langle F(x, y), y - z \rangle \leq 0 \quad \forall z \in \Omega
\end{align*}

where the following basic assumptions are satisfied:

\begin{itemize}
\item \textbf{(BA):} The functions $f : \mathbb{R}^{n+m} \to \mathbb{R}$, $\Psi : \mathbb{R}^{n+m} \to \mathbb{R}^l$, and $F : \mathbb{R}^{n+m} \to \mathbb{R}^m$ are locally Lipschitz near all optimal solutions of (OPVIC); $C$ is a closed subset of $\mathbb{R}^{n+m}$ and $\Omega$ is a closed convex subset of $\mathbb{R}^m$.
\end{itemize}

There have been a lot of developments in the theory of (OPVIC). The reader is referred to recent monographs of Luo, Pang and Ralph [7] and Outrata, Kocvara and Zowe [12] for recent developments and references on various optimality conditions and computational algorithms.

Recently, constraint qualifications and necessary optimality conditions for the problem (OPVIC) have been studied in [18] by rewriting the problem as an optimization problem with a generalized equation constraint

\begin{align*}
\text{(GP)} \quad & \text{minimize} \quad f(x, y) \\
& \text{subject to} \quad \Psi(x, y) \leq 0, (x, y) \in C \\
& \quad \quad \quad \quad \quad \quad 0 \in F(x, y) + N_\Omega(y)
\end{align*}

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where
\[ N_G(y) := \begin{cases} 
\text{the normal cone of } \Omega \text{ if } y \in \Omega, \\
\emptyset \text{ otherwise}
\end{cases} \]

is the normal cone operator.

Let \((\bar{x}, \bar{y})\) be feasible for \((\text{OPVIC})\), and \(\lambda\) be a non-negative number. We define \(M^\lambda(\bar{x}, \bar{y})\), the index \(\lambda\) Codervative (CD) multiplier set corresponding to \((\bar{x}, \bar{y})\), to be the set of vectors \((\gamma, \eta)\) in \(\mathbb{R}^d \times \mathbb{R}^m\) satisfying the Fritz John type necessary optimality condition involving the Mordukhovich coderivatives for \((\text{GP})\) [18, Theorem 3.1]; that is, the vectors \((\gamma, \eta)\) such that
\[
\gamma \geq 0 \quad \text{and} \quad \langle \Psi(\bar{x}, \bar{y}), \gamma \rangle = 0, \\
0 \in \lambda \partial f(\bar{x}, \bar{y}) + \partial \langle \Psi, \gamma \rangle(\bar{x}, \bar{y}) + \partial (F, -\eta)(\bar{x}, \bar{y}) + \{0\} \times D^* N_G(\bar{y}, -F(\bar{x}, \bar{y}))(-\eta) \\
+ N_G(\bar{x}, \bar{y}),
\]
where the notation \(\langle F, -\eta \rangle\) refers to the mapping defined by \(\langle F, -\eta \rangle(x, y) := \langle F(x, y), -\eta \rangle\), \(\partial\) denotes the limiting subgradient (see Definition 1), \(N_G(\bar{x}, \bar{y})\) denotes the limiting normal cone (see Definition 2) and \(D^*\) denotes the Mordukhovich coderivative (see Definition 3). Call \(M^1(\bar{x}, \bar{y})\) the set of normal CD multipliers and \(M^0(\bar{x}, \bar{y})\) the set of abnormal CD multipliers. Then by [18, Theorem 3.1], the Fritz John type necessary optimality condition involving the Mordukhovich coderivatives can be rephrased as follows:

**Proposition 1.** If \((\bar{x}, \bar{y})\) is a local solution of \((\text{OPVIC})\) then either the set of normal CD multipliers is nonempty or there is a nonzero abnormal CD multiplier, i.e.,
\[ M^1(\bar{x}, \bar{y}) \cup (M^0(\bar{x}, \bar{y}) \setminus \{0\}) \neq \emptyset. \]

Note that the above Fritz John condition involves \(D^* N_G(\bar{y}, -F(\bar{x}, \bar{y}))(-\eta)\). By the definition of coderivatives,
\[ \xi \in D^* N_G(\bar{y}, -F(\bar{x}, \bar{y}))(-\eta) \iff (\xi, \eta) \in N_{\text{Gph } N_G}(\bar{y}, -F(\bar{x}, \bar{y})), \]
where \(\text{Gph } N_G := \{(y, u) : u \in N_G(y)\}\) is the graph of the set-valued map \(N_G\). Hence calculation of the coderivative \(D^* N_G(\bar{y}, -F(\bar{x}, \bar{y}))(-\eta)\) depends on calculation of the limiting normal cone \(N_{\text{Gph } N_G}(\bar{y}, -F(\bar{x}, \bar{y}))\). In the case where \(\Omega = \mathbb{R}^m\), (OPVIC) is an ordinary mathematical programming problem with equality, inequality and abstract constraints. The term \(D^* N_G(\bar{y}, -F(\bar{x}, \bar{y}))(-\eta)\) vanishes and the above Fritz John condition can be considered as a limiting subgradient version of the generalized Lagrange multiplier rules [1, Theorem 6.1.1] and was obtained in [8, Theorem 1(b)].

In the case where \(\Omega = \mathbb{R}^a \times \mathbb{R}^b_+\) for some non-negative integers \(a, b\) with \(a + b = m\). Let \(y = (z, u), F(x, y) = (H(x, z, u), G(x, z, u))\); Since \(N_{\mathbb{R}^a}(z) = \{0\}\) is a constant map, we have
\[ D^* N_G(\bar{y}, -F(\bar{x}, \bar{y}))(-\eta_H, -\eta_G) = \{0\} \times D^* N_{\mathbb{R}^b_+}(\bar{u}, -G(\bar{x}, \bar{z}, \bar{u}))(-\eta_G), \]
where \(N_{\text{Gph } N_{\mathbb{R}^b_+}}(\bar{u}, -G(\bar{x}, \bar{z}, \bar{u}))\) can be calculated by using the following proposition whose proof follows from [17, Proposition 2.7] and the definition of the limiting normal cones.
Proposition 2. For any \((\bar{u}, \bar{v}) \in \text{Gph } N_{\mathbb{R}^b}^+\), define

\[
L := L(\bar{u}, \bar{v}) := \{i \in \{1, 2, \ldots, b\} : \bar{u}_i > 0, \bar{v}_i = 0\}
\]

\[
I_+ := I_+(\bar{u}, \bar{v}) := \{i \in \{1, 2, \ldots, b\} : \bar{u}_i = 0, \bar{v}_i < 0\}
\]

\[
I_0 := I_0(\bar{u}, \bar{v}) := \{i \in \{1, 2, \ldots, b\} : \bar{u}_i = 0, \bar{v}_i = 0\}.
\]

Then

\[
N_{\text{Gph } N_{\mathbb{R}^b}^+}(\bar{u}, \bar{v}) = \{(\alpha, \beta) \in \mathbb{R}^{2b} : \alpha_L = 0, \beta_{I_+} = 0, \forall i \in I_0, \text{ either } \alpha_i \beta_i = 0 \text{ or } \alpha_i < 0 \text{ and } \beta_i > 0\}.
\]

In the case where \(\Omega\) is a polyhedral convex set, a calculation of the limiting normal cone to the graph of the normal cone to the set \(\Omega\) was first given in the proof of [4, Theorem 2] and stated in [13, Proposition 4.4].

In this paper we continue the study by considering the value function \(V(p, q)\) associated with the following additively perturbed (GP):

\[
\begin{align*}
\text{GP}(p, q) & \quad \text{minimize } f(x, y) \\
& \quad \text{subject to } \Psi(x, y) + p \leq 0, (x, y) \in C \\
& \quad \qquad q \in F(x, y) + N_\Omega(y),
\end{align*}
\]

i.e.,

\[
V(p, q) := \inf \{f(x, y) : \Psi(x, y) + p \leq 0, (x, y) \in C \}
\]

\[
q \in F(x, y) + N_\Omega(y)\}.
\]

Our main result shows that, as in nonlinear programming, each limiting subgradient of the value function is a normal CD multiplier and each singular limiting subgradient is an abnormal CD multiplier i.e.

\[
\partial V(\bar{p}, \bar{q}) \subset \bigcup_{(x, y) \in \Xi(p, q)} M^1(\bar{x}, \bar{y}) \quad \text{and} \quad \partial^\infty V(\bar{p}, \bar{q}) \subset \bigcup_{(x, y) \in \Sigma(p, q)} M^0(\bar{x}, \bar{y})
\]

where \(M^\lambda(\bar{x}, \bar{y})\) is the CD multiplier set of index \(\lambda\) corresponding to \((\bar{x}, \bar{y})\) for the perturbed problem \(\text{GP}(\bar{p}, \bar{q})\), i.e., the set of \((\gamma, \eta) \in \mathbb{R}^d \times \mathbb{R}^m\) such that

\[
\begin{align*}
\gamma & \geq 0 \quad \text{and} \quad \langle \bar{g} + \Psi(\bar{x}, \bar{y}), \gamma \rangle = 0, \\
0 & \in \lambda \partial f(\bar{x}, \bar{y}) + \partial \langle \Psi(\bar{x}, \bar{y}), \gamma \rangle + \partial \langle F, -\eta \rangle((\bar{x}, \bar{y})) \{0\} \times D^* N_\Omega(\bar{g}, \bar{g}) - F(\bar{x}, \bar{y}))(-\eta) \\
& \quad + N_C(\bar{x}, \bar{y}),
\end{align*}
\]

and \(\Sigma(\bar{p}, \bar{q})\) is the set of solutions of \(\text{GP}(\bar{p}, \bar{q})\). Note that \((p, q)\) refers to the perturbed problem \(\text{GP}(p, q)\) while no subscript refers to problem (GP), i.e. \(\text{GP} = \text{GP}(0, 0)\) and \(\Sigma = \Sigma(0, 0)\).

In the case where \(\Omega = \mathbb{R}^d \times \mathbb{R}^b\), (OPVIC) becomes an optimization problem with complementarity constraints. Treating it as an ordinary nonlinear programming problem, we show that there always exist nonzero abnormal NLP multipliers. Since the nonexistence of nonzero abnormal NLP multipliers is a necessary and sufficient condition for the boundedness of the set of normal NLP multipliers [5], the set of normal NLP multipliers for (OPVIC) is either empty or unbounded. Hence the estimates for the limiting subgradients of the value function in terms of the NLP multipliers is nonexistent when the set of normal NLP multipliers is empty and may contain in a unbounded set when the set of normal NLP multipliers is not empty but unbounded. On the contrary, by rewriting (OPCC) as an optimization problem with a generalized equation constraint (GP), nonexistence of nonzero abnormal CD
multipliers is possible under very reasonable assumptions. Therefore the estimates for the subgradients of the value function in terms of the CD multipliers may be tighter than the one using the NLP multipliers. An example is given to show that in sensitivity analysis the CD multipliers may provide more useful information than the NLP multipliers. In this example, the value function is Lipschitz and the limiting subgradient of the value function is the set of CD multipliers while the set of NLP multipliers is empty. Applications to the bilevel programming problem are also given.

The following notations are used throughout the paper: $B$ denotes the open unit ball, $B(\bar{x};\delta)$ denotes the open ball centered at $\bar{x}$ with radius $\delta > 0$. For a set $E$, $\partial E$ denotes the convex hull of $E$, $\text{int} E$ and $\text{cl} E$ denote the interior and the closure of $E$ respectively. The notation $\langle a, b \rangle$ denotes the inner products of vectors $a$ and $b$.

For a differentiable function $f$, $\nabla f(\bar{x})$ denotes the gradient of $f$ at $\bar{x}$. For a vector $a \in \mathbb{R}^n$, $a_i$ denotes the $i$th component of $a$.

2. Preliminaries

The purpose of this section is to provide the background material on nonsmooth analysis which will be used later. We only give concise definitions and facts that will be needed in the paper. For more detailed information on the subject, our references are Clarke [2], Loewen [6], Rockafellar and Wets [14] and Mordukhovich [9] and [11].

First we give some definitions for various subgradients and normal cones.

**Definition 1.** Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be lower semicontinuous and finite at $\bar{x} \in \mathbb{R}^n$. The **proximal subgradient** of $f$ at $\bar{x}$ is the set defined by

$$\partial^p f(\bar{x}) = \{ v \in \mathbb{R}^n : \exists M > 0, \delta > 0 \text{ s.t. } f(x) \geq f(\bar{x}) + \langle v, x - \bar{x} \rangle + M||x - \bar{x}||^2 \forall x \in \bar{x} + \delta B \},$$

the **limiting subgradient** of $f$ at $\bar{x}$ is the set defined by

$$\partial f(\bar{x}) := \{ v \in \mathbb{R}^n : v = \lim_{\nu \to +\infty} v^\nu \text{ with } v^\nu \in \partial^p f(x^\nu) \text{ and } x^\nu \to \bar{x} \},$$

the **singular limiting subgradient** of $f$ at $\bar{x}$ is the set defined by

$$\partial^s f(\bar{x}) := \{ v \in \mathbb{R}^n : v = \lim_{\nu \to +\infty} \lambda^\nu v^\nu \text{ with } v^\nu \in \partial^p f(x^\nu) \text{ and } \lambda^\nu \downarrow 0, x^\nu \to \bar{x} \}.$$

Let $f : \mathbb{R}^n \to \mathbb{R}$ be Lipschitz near $\bar{x} \in \mathbb{R}^n$. The **Clarke generalized gradient** of $f$ at $\bar{x}$ is the set

$$\partial_C f(\bar{x}) := \text{clco} \partial f(\bar{x}).$$

For set-valued maps, the definition for a limiting normal cone leads to the definition of coderivative of a set-valued map introduced by Mordukhovich in [8].

**Definition 2.** For a closed set $C \subset \mathbb{R}^n$ and $\bar{x} \in C$, The **proximal normal cone** to $C$ at $\bar{x}$ is defined by

$$N^p_C(\bar{x}) := \{ v \in \mathbb{R}^n : \exists M > 0 \text{ s.t. } \langle v, x - \bar{x} \rangle \leq M||x - \bar{x}||^2 \forall x \in C \},$$

the **limiting normal cone** to $C$ at $\bar{x}$ is defined by

$$N_C(\bar{x}) := \{ \lim_{\nu \to \infty} v^\nu : v^\nu \in N^p_C(x^\nu), x^\nu \to \bar{x} \}.$$
Definition 3. Let $\Phi : \mathbb{R}^n \to \mathbb{R}^p$ be a set-valued map. Let $(\bar{x}, \bar{p}) \in \text{Gph} \Phi := \{ (x, p) : p \in \Phi(x) \}$, the graph of the set-valued map $\Phi$. The set-valued map $D^*\Phi(\bar{x}, \bar{p})$ from $\mathbb{R}^n$ into $\mathbb{R}^n$ defined by
\[ D^*\Phi(\bar{x}, \bar{p})(\eta) := \{ \xi \in \mathbb{R}^n : (\xi, -\eta) \in N_{\text{Gph} \Phi}(\bar{x}, \bar{p}) \}, \]
is called the Mordukhovich coderivative of $\Phi$ at $(\bar{x}, \bar{p})$.

In general, we have the following inclusions which may be strict:
\[ \partial^n f(\bar{x}) \subseteq \partial f(\bar{x}) \subseteq \partial_C f(\bar{x}). \]
In the case where $f$ is a convex function, all subgradients coincide with the subgradients in the sense of convex analysis, i.e.,
\[ \partial^n f(\bar{x}) = \partial f(\bar{x}) = \partial_C f(\bar{x}). \]
In the case where $f$ is strictly differentiable (see the definition e.g., in Clarke [1]), we have
\[ \partial f(\bar{x}) = \partial_C f(\bar{x}) = \{ \nabla f(\bar{x}) \}. \]

The following facts about the subgradients are well known.

Proposition 3. (i) A function $f : \mathbb{R}^n \to \mathbb{R}$ is Lipschitz near $\bar{x}$ and $\partial f(\bar{x}) = \{ \xi \}$ if and only if $f$ is strictly differentiable at $\bar{x}$ and the gradient of $f$ at $\bar{x}$ equals $\xi$.

(ii) A function $f : \mathbb{R}^n \to \mathbb{R}$ is Lipschitz near $\bar{x}$ if and only if $\partial f(\bar{x}) = \{ 0 \}$.

(iii) A function $f : \mathbb{R}^n \to \mathbb{R}$ is Lipschitz near $\bar{x}$ with constant $L_f$ implies that $\partial f(\bar{x}) \subseteq L_f \bar{B}$.

The following calculus rules will be useful and can be found in the references given in the beginning of this section.

Proposition 4. (See e.g. [6, Proposition 5A.4]) Let $f : \mathbb{R}^n \to \mathbb{R}$ Lipschitz near $\bar{x}$ and $g : \mathbb{R}^n \to \mathbb{R} \cup \{ +\infty \}$ be lower semicontinuous and finite at $\bar{x}$. Then
\[ \partial (f + g)(\bar{x}) \subseteq \partial f(\bar{x}) + \partial g(\bar{x}). \]
\[ \partial^\infty (f + g)(\bar{x}) \subseteq \partial^\infty g(\bar{x}). \]

Proposition 5. (See e.g. [6, Lemma 5A.3]) Let $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R} \cup \{ +\infty \}$ be lower semicontinuous and finite at $(\bar{x}, \bar{y})$. If $(\xi, 0) \in \partial^\infty f(\bar{x}, \bar{y})$ implies that $\xi = 0$, then
\[ \partial^\infty f(\bar{x}, \bar{y}) \subseteq \{ \eta : (\xi, \eta) \in \partial f(\bar{x}, \bar{y}) \text{ for some } \xi \} \]
\[ \partial^\infty f(\bar{x}, \bar{y}) \subseteq \{ \eta : (\xi, \eta) \in \partial^\infty f(\bar{x}, \bar{y}) \text{ for some } \xi \}. \]

Classical results on the value function can be found in [1, 3, 6, 14, 10] while the result we quote are from [6].

Proposition 6. [6, (b) and (d) of Theorem 5A.2] Let $g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R} \cup \{ +\infty \}$ be lower semicontinuous everywhere and finite at $(\bar{x}, \bar{a})$. Suppose $g$ is bounded below on some set $E \times O$, where $E$ is a compact neighborhood of $\bar{x}$ and $O$ is an open set containing $\bar{a}$. Define the value function $V : \mathbb{R}^n \to \mathbb{R} \cup \{ +\infty \}$ and the set of minimizers $\Sigma$ as follows:
\[ V(\alpha) := \inf \{ g(z, \alpha) : z \in E \}; \]
\[ \Sigma(\alpha) := \{ x \in E : g(x, \alpha) = V(\alpha) \}. \]
If $\Sigma(\bar{a}) \subseteq \text{int} E$, then the value function $V$ is lower semicontinuous on $O$ and the subgradients of $V$ satisfy these estimates:

$$\partial V(\bar{a}) \subseteq \{ \eta \in \mathbb{R} : (\eta, 0) \in \partial g(\bar{a}, z) \text{ for some } z \in \Sigma(\bar{a}) \}$$

$$\partial^\infty V(\bar{a}) \subseteq \{ \eta \in \mathbb{R} : (\eta, 0) \in \partial^\infty g(\bar{a}, z) \text{ for some } z \in \Sigma(\bar{a}) \}.$$ 

Our results are stated using the limiting subgradients. Alternatively, they could be derived by using the Fréchet subgradients instead of the proximal subgradients (both lead to the same limiting subgradients in finite dimensional spaces). In [14] arguments are given in favor of the former (called there the regular subgradients). In the present paper we use the proximal subgradients to provide the same framework as in [18].

3. MAIN RESULT

In order to obtain useful information on the perturbed problem $\text{GP}(p, q)$, we may assume that it has some solutions near the point of interest $(\bar{p}, \bar{q})$. This is usually done by making a growth hypothesis (see for example [3, Growth Hypothesis 3.1.1], [1, Hypothesis 6.5.1], [14, Definition 1.8]). In this paper, we make the following growth hypothesis [6, Theorem 5A.2]:

**(GH) at $(\bar{p}, \bar{q})$:** There exists $\delta > 0$ such that the set

$$\{(x, y) \in C : (p, q) \in B(\bar{p}, \bar{q}, \delta), \Psi(x, y) + p \leq 0, q \in F(x, y) + N_D(y), f(x, y) \leq r\}$$

is bounded for each $r$.

In order to apply Proposition 6, we rewrite our value function in the following form:

$$V(p, q) = \inf g(x, y, p, q)$$

where $g$ is the extended-valued function defined by

$$g(x, y, p, q) := f(x, y) + I_{\{\text{Gph } \Psi \cap (C \times \mathbb{R}^{d+m})\}}(x, y, p, q)$$

with $I_E$ being the indicator function of a set $E$ defined by

$$I_E(x) := \begin{cases} 0 & \text{if } x \in E, \\ \infty & \text{if } x \not\in E \end{cases}$$

and $\Phi$ being the set-valued map defined by

$$\Phi(x, y) = (-\Psi(x, y), F(x, y)) + \mathbb{R}_+^d \times N_D(y).$$

The growth hypothesis (GH) amounts to saying the function $g$ is level-bounded in $(x, y)$ uniformly for any $(p, q) \in B(\bar{p}, \bar{q}, \delta)$. Hence by virtue of [14, Theorem 1.9],

$$\bigcup_{(p, q) \in B(\bar{p}, \bar{q}, \delta)} \Sigma(p, q)$$

is a compact set and for all $(p, q) \in B(\bar{p}, \bar{q}, \delta)$,

$$V(p, q) = \inf \{ g(x, y, p, q) : (x, y) \in E \}$$

where $E$ is a compact set with interior containing $\bigcup_{(p, q) \in B(\bar{p}, \bar{q}, \delta)} \Sigma(p, q)$. It is clear that $g$ is lower semicontinuous everywhere and finite at any $(x, y, p, q) \in (\text{Gph } \Phi) \cap (C \times \mathbb{R}^{d+m})$. Since $f$ is Lipschitz near $(\bar{x}, \bar{y})$, $f$ is Lipschitz on $B(\bar{x}, \bar{y}; \varepsilon)$, a neighborhood of $(\bar{x}, \bar{y})$, Hence $g$ is bounded below on $B(\bar{x}, \bar{y}; \varepsilon) \times R^{d+m}$. The following result then follows immediately by applying Proposition 6.
**Proposition 7.** Under the Basic Assumption (BA) and the Growth Hypothesis (GH) at $(\bar{p}, \bar{q})$ the value function $V$ is lower semicontinuous on $B(\bar{p}, \bar{q}; \delta)$ and

$$
\partial V(\bar{p}, \bar{q}) \subseteq \bar{M}^1(\bar{x}, \bar{y}),
$$

$$
\partial^\infty V(\bar{p}, \bar{q}) \subseteq \bar{M}^0(\bar{x}, \bar{y}),
$$

where

$$
\bar{M}^1(\bar{x}, \bar{y}) := \bigcup_{(x, y) \in \Sigma(\bar{p}, \bar{q})} \{(u, v) : (0, 0, u, v) \in \partial g(\bar{x}, \bar{y}, \bar{p}, \bar{q})\},
$$

$$
\bar{M}^0(\bar{x}, \bar{y}) := \bigcup_{(x, y) \in \Sigma(\bar{p}, \bar{q})} \{(u, v) : (0, 0, u, v) \in \partial^\infty g(\bar{x}, \bar{y}, \bar{p}, \bar{q})\}.
$$

We now prove that the set $\bar{M}^1$ (respectively $\bar{M}^0$) is included in the normal multiplier set $M^1$ (respectively the abnormal multiplier set $M^0$).

By the sum rule Proposition 4 and the fact that for any closed set $E$ with $\bar{z} \in E$

$$
\partial I_E(\bar{z}) = \partial^\infty I_E(\bar{z}) = N_E(\bar{z}),
$$

we have

$$
\partial g(\bar{x}, \bar{y}, \bar{p}, \bar{q}) \subseteq \partial f(\bar{x}, \bar{y}) \times \{(0, 0)\} + N_{[\text{Gph } \Phi]} \cap (C \times \mathbb{R}^{d+m})\{\bar{x}, \bar{y}, \bar{p}, \bar{q}\},
$$

$$
\partial^\infty g(\bar{x}, \bar{y}, \bar{p}, \bar{q}) \subseteq N_{[\text{Gph } \Phi]} \cap (C \times \mathbb{R}^{d+m})\{\bar{x}, \bar{y}, \bar{p}, \bar{q}\}.
$$

Hence, we only need to compute the normal cone.

**Lemma 1.** If $(r_x, r_y, s_p, s_q) \in N_{[\text{Gph } \Phi]} \cap (C \times \mathbb{R}^{d+m})\{\bar{x}, \bar{y}, \bar{p}, \bar{q}\}$ then

$$
s_p \geq 0, \quad \langle \bar{g} + \Psi(\bar{x}, \bar{y}), s_p \rangle = 0, \quad \text{and}
$$

$$(r_x, r_y) \in \partial \langle \Phi, s_p \rangle(\bar{x}, \bar{y}) + \partial \langle F, -s_q \rangle(\bar{x}, \bar{y}) + N_C(\bar{x}, \bar{y}) +
$$

$$
\{0\} \times D^*N_{\Omega}(\bar{y}, \bar{q} - F(\bar{x}, \bar{y})).
$$

**Proof.** Step 1. Let $(\bar{x}, \bar{y}, \bar{p}, \bar{q})$ be any point in a neighborhood of $(\bar{x}, \bar{y}, \bar{p}, \bar{q})$ on which $\Phi$, $F$ are Lipschitz continuous and $(r_x, r_y, s_p, s_q) \in N_{[\text{Gph } \Phi]} \cap (C \times \mathbb{R}^{d+m})\{\bar{x}, \bar{y}, \bar{p}, \bar{q}\}$.

By the definition of proximal normal cones, there is $M > 0$ such that for all $(x, y, p, q) \in (\text{Gph } \Phi) \cap (C \times \mathbb{R}^{d+m})$

$$
\langle (r_x, r_y, s_p, s_q), (x, y, p, q) - (\bar{x}, \bar{y}, \bar{p}, \bar{q}) \rangle \leq M\| (x, y, p, q) - (\bar{x}, \bar{y}, \bar{p}, \bar{q}) \|^2.
$$

In other words, $(\bar{x}, \bar{y}, \bar{p}, \bar{q})$ is a solution to the optimization problem

minimize $\langle - (r_x, r_y, s_p, s_q), (x, y, p, q) \rangle + M\| (x, y, p, q) - (\bar{x}, \bar{y}, \bar{p}, \bar{q}) \|^2$

subject to $p + \Phi(x, y) \leq 0, (x, y) \in C$

$q \in F(x, y) + N_{\Omega}(y)$.

We now prove that the only abnormal CD multiplier for the above problem is the zero vector. Indeed, the set of abnormal CD multipliers at $(\bar{x}, \bar{y}, \bar{p}, \bar{q})$ for the above problem are the vectors $(\gamma, \eta)$ satisfying

$$
\gamma \geq 0 \quad \text{and} \quad \langle \bar{g} + \Psi(\bar{x}, \bar{y}), \gamma \rangle = 0,
$$

$$
0 \in \partial \langle \Phi, \gamma \rangle(\bar{x}, \bar{y}) \times \{\gamma, 0\} + \partial \langle F, -\eta \rangle(\bar{x}, \bar{y}) \times \{(0, \eta)\}
$$

$$
+ \{0\} \times D^*N_{\Omega}(\bar{y}, \bar{q} - F(\bar{x}, \bar{y}))(\eta) \times \{(0, 0)\} + N_C(\bar{x}, \bar{y}) \times \{(0, 0)\},
$$

which obviously coincides with the set $\{(0, 0)\}$. Applying Proposition 1, we conclude that the set of normal CD multipliers for the above problem must be nonempty.
That is, there are vectors \( \eta \in \mathbb{R}^n \) and \( \gamma \in \mathbb{R}^d \) such that

\[
\begin{align*}
\{ \gamma \geq 0 \text{ and } & \langle \bar{g} + \Psi(\bar{x}, \bar{y}), \gamma \rangle = 0 \\
0 \in & \{(r_x, r_y, s_p, s_q) \} + \partial(\Psi, \gamma)(\bar{x}, \bar{y}) \times \{(\gamma, 0) \} + \partial(F, -\eta)(\bar{x}, \bar{y}) \times \{(0, \eta) \} \\
& + \{0\} \times D^* N_\Omega(\bar{y}, \bar{q} - F(\bar{x}, \bar{y}))(\eta) \times \{(0, 0)\} + NC(\bar{x}, \bar{y}) \times \{(0, 0)\} \}.
\end{align*}
\]

That is,

\[
\begin{align*}
s_p \geq 0 \text{ and } & \langle \bar{g} + \Psi(\bar{x}, \bar{y}), s_p \rangle = 0 \\
(r_x, r_y) \in & \partial(\Psi, s_p)(\bar{x}, \bar{y}) + \partial(F, -s_q)(\bar{x}, \bar{y}) + \{0\} \times D^* N_\Omega(\bar{y}, \bar{q} - F(\bar{x}, \bar{y}))(\eta) + NC(\bar{x}, \bar{y}).
\end{align*}
\]

Step 2. Now take any \((r_x, r_y, s_p, s_q) \in N_{(G \Phi \Psi \Phi \Psi) \cap (C \times \mathbb{R}^{d+n+m})}(\bar{x}, \bar{y}, \bar{p}, \bar{q})\). Then by definition of limiting normal cones, there are sequences \((x'\nu, y'\nu, p'\nu, q'\nu) \to (\bar{x}, \bar{y}, \bar{p}, \bar{q})\) and \((r'\nu_x, r'\nu_y, s'\nu_p, s'\nu_q) \to (r_x, r_y, s_p, s_q)\) with

\[
(r'\nu_x, r'\nu_y, s'\nu_p, s'\nu_q) \in N_{(G \Phi \Psi \Phi \Psi) \cap (C \times \mathbb{R}^{d+n+m})}(x'\nu, y'\nu, p'\nu, q'\nu)
\]

By virtue of step 1,

\[
\begin{align*}
\begin{cases}
s'\nu_p \geq 0 \text{ and } & \langle \bar{g} + \Psi(x'\nu, y'\nu), s'\nu_p \rangle = 0 \\
(r'\nu_x, r'\nu_y) \in & \partial(\Psi, s'\nu_p)(x'\nu, y'\nu) + \partial(F, -s'\nu_q)(x'\nu, y'\nu) \\
& \quad + \{0\} \times D^* N_\Omega(y'\nu, q'\nu - F(x'\nu, y'\nu))(\eta) + NC(x'\nu, y'\nu).
\end{cases}
\end{align*}
\]

Since \(\Psi\) is Lipschitz near \((\bar{x}, \bar{y})\), we have

\[
\begin{align*}
\partial(\Psi, s'\nu_p)(x'\nu, y'\nu) \subseteq & \partial(\Psi, s_p)(x'\nu, y'\nu) + ||s'\nu_p - s_p||L_\Psi \text{cl} B \text{ by Proposition 4} \\
& \subseteq \partial(\Psi, s_p)(x'\nu, y'\nu) + ||s'\nu_p - s_p||L_\Psi \text{cl} B \text{ by Proposition 3},
\end{align*}
\]

where \(L_\Psi\) is the Lipschitz constant of \(\Psi\). Similarly,\[
\begin{align*}
\partial(F, -s'\nu_q)(x'\nu, y'\nu) \subseteq & \partial(F, -s_q)(x'\nu, y'\nu) + ||s'\nu_q - s_q||L_F \text{cl} B,
\end{align*}
\]

where \(L_F\) is the Lipschitz constant of \(F\). Hence, we have

\[
\begin{align*}
\begin{cases}
s'\nu_p \geq 0 \text{ and } & \langle \bar{g} + \Psi(x'\nu, y'\nu), s'\nu_p \rangle = 0 \\
(r'\nu_x, r'\nu_y) \in & \partial(\Psi, s_p)(x'\nu, y'\nu) + \partial(F, -s_q)(x'\nu, y'\nu) + ||s'\nu_p - s_p||L_\Psi \text{cl} B \\
& \quad + \{0\} \times D^* N_\Omega(y'\nu, q'\nu - F(x'\nu, y'\nu))(\eta) + NC(x'\nu, y'\nu).
\end{cases}
\end{align*}
\]

Taking limits as \(\nu \to \infty\) and using the definitions of the limiting normal cone and the limiting subgradients completes the proof.

All in all, we proved the following result.

**Theorem 1.** Assume \((GH)\) and \((BA)\) hold. Then the value function \(V\) is lower semicontinuous on \(B(\bar{p}, \bar{q}, \delta)\) and

\[
\begin{align*}
\partial V(\bar{p}, \bar{q}) \subset \bigcup_{(x, y) \in \Xi(p, q)} M^1(\bar{x}, \bar{y}) \text{ and } \partial^\infty V(\bar{p}, \bar{q}) \subset \bigcup_{(x, y) \in \Xi(p, q)} M^0(\bar{x}, \bar{y}).
\end{align*}
\]

The above estimates may not be useful in the case where \(\partial V(\bar{p}, \bar{q})\) is empty. The following consequence of Theorem 1 and Proposition 3 provides conditions which rule out this possibility.
Corollary 1. If \( \bigcup_{(x,y) \in \Sigma(p,q)} M^0(\bar{x},\bar{y}) = \{0\} \) then \( V(\bar{p},\bar{q}) \) is finite and Lipschitz near \((\bar{p},\bar{q})\) with

\[
\emptyset \neq \partial V(\bar{p},\bar{q}) \subset \bigcup_{(x,y) \in \Sigma(p,q)} M^1(\bar{x},\bar{y}).
\]

In addition if \( \bigcup_{(x,y) \in \Sigma(p,q)} M^1(\bar{x},\bar{y}) = \{\xi\} \) then \( V \) is strictly differentiable and \( \nabla V(\bar{p},\bar{q}) = \xi \).

We now consider non-additive perturbations of the kind

\[
\tilde{G}\tilde{P}(\alpha) \quad \text{minimize} \quad f(x,y,\alpha)
\]
subject to
\[
\Psi(x,y,\alpha) \leq 0, (x,y) \in C
\]
\[
0 \in F(x,y,\alpha) + N_\Omega(y).
\]

where \( f : \mathbb{R}^{n+m+c} \to \mathbb{R}, \Psi : \mathbb{R}^{n+m+c} \to \mathbb{R}^d \) are Lipschitz near \((\bar{x},\bar{y},\bar{\alpha})\). The value function for the above perturbed problem is defined as

\[
\tilde{V}(\alpha) := \{ f(x,y,\alpha) : \Psi(x,y,\alpha) \leq 0, (x,y) \in C, 0 \in F(x,y,\alpha) + N_\Omega(y) \}.
\]

It is easy to see that we can turn the non-additive perturbations into additive perturbations by adding an auxiliary variable:

\[
\tilde{G}\tilde{P}(\alpha) \quad \text{minimize} \quad f(x,y,z)
\]
subject to
\[
\Psi(x,y,z) \leq 0, (x,y,z) \in C \times \mathbb{R}
\]
\[
0 \in F(x,y,z) + N_\Omega(y)
\]
\[
-z + \alpha = 0,
\]

which is the partially perturbed problem of the fully perturbed problem:

\[
G\tilde{P}(p,q,\alpha) \quad \text{minimize} \quad f(x,y,z)
\]
subject to
\[
\Psi(x,y,z) + p \leq 0, (x,y,z) \in C \times \mathbb{R}
\]
\[
q \in F(x,y,z) + N_\Omega(y)
\]
\[
-z + \alpha = 0.
\]

By Theorem 1, if the fully perturbed problem \( G\tilde{P}(p,q,\alpha) \) satisfies the growth hypothesis (GH) at \((0,0,\bar{\alpha})\), then the value function \( V(p,q,\alpha) \) defined by

\[
V(p,q,\alpha) := \inf \{ f(x,y,z) : \Psi(x,y,z) + p \leq 0, (x,y,z) \in C \times \mathbb{R}^c, q \in F(x,y,z) + N_\Omega(y), -z + \alpha = 0 \}
\]

is lower semicontinuous on \( B(0,0,\bar{\alpha};\delta) \) and

\[
\partial V(0,0,\bar{\alpha}) \subset \bigcup_{(x,y,\alpha) \in \Sigma(0,0,\alpha)} M^0(\bar{x},\bar{y},\bar{\alpha})
\]
\[
\partial^\infty V(0,0,\bar{\alpha}) \subset \bigcup_{(x,y,\alpha) \in \Sigma(0,0,\alpha)} M^0(\bar{x},\bar{y},\bar{\alpha}),
\]

where \( M^\lambda(\bar{x},\bar{y},\bar{\alpha}) \) is the set of index \( \lambda \) CD multipliers for problem \( G\tilde{P}(p,q,\alpha) \) at \((0,0,\bar{\alpha})\), i.e., vectors \((\gamma,\eta,\xi)\) in \( \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R} \) satisfying

\[
\begin{cases}
\gamma \geq 0 \text{ and } \langle \Psi(\bar{x},\bar{y},\bar{\alpha}), \gamma \rangle = 0,
\end{cases}
\]
\[
\begin{cases}
0 \in \lambda \partial f(\bar{x},\bar{y},\bar{\alpha}) + \partial(\Psi,\gamma)(\bar{x},\bar{y},\bar{\alpha}) + \partial(F,-\eta)(\bar{x},\bar{y},\bar{\alpha})
\end{cases}
\]
\[
\begin{cases}
+ \{0\} \times D^*N_\Omega(\bar{y},-F(\bar{x},\bar{y},\bar{\alpha}))(-\eta) \times \{0\} - \{(0,0,\xi)\} + N_C(\bar{x},\bar{y}) \times \{0\}.
\end{cases}
\]
For any \((0,0,\xi) \in \partial^\infty V(0,0,\alpha)\), we have \((0,0,\xi) \in M^0(\tilde{x},\tilde{y},\tilde{\alpha})\) for some point \((\tilde{x},\tilde{y},\tilde{\alpha}) \in \Sigma(0,0,\alpha)\). Therefore,
\[(0,0,\xi) \in N_C(\tilde{x},\tilde{y}) \times \{0\}\]
which implies that \(\xi = 0\). By Proposition 5, we have
\[
\partial_0 V(0,0,\tilde{\alpha}) \subseteq \{\xi : (\gamma, \eta, \xi) \in \partial V(0,0,\tilde{\alpha}) \text{ for some } (\gamma, \eta)\}
\]
\[-\infty \subseteq \{\xi : (\gamma, \eta, \xi) \in \partial^\infty V(0,0,\tilde{\alpha}) \text{ for some } (\gamma, \eta)\}.
\]
Consequently, noticing that \(\tilde{V}(\alpha) = V(0,0,\alpha)\), we have proved the following theorem:

**Theorem 2.** Assume that the growth assumption (GH) for Problem GP\((p,q,\alpha)\) is satisfied at \((0,0,\tilde{\alpha})\). Then the value function \(\tilde{V}\) is lower semicontinuous near \(\tilde{\alpha}\) and
\[
\partial^\infty \tilde{V}(\tilde{\alpha}) \subseteq \bigcup_{(x,y) \in E(\alpha)} \{\xi : (\gamma, \eta, \xi) \in M^1(\tilde{x},\tilde{y},\tilde{\alpha})\}
\]
\[
\partial^\infty \tilde{V}(\tilde{\alpha}) \subseteq \bigcup_{(x,y) \in E(\alpha)} \{\xi : (\gamma, \eta, \xi) \in M^0(\tilde{x},\tilde{y},\tilde{\alpha})\}
\]
where \(M^\lambda(\tilde{x},\tilde{y},\tilde{\alpha})\) is the set of index \(\lambda\) multipliers for problem GP\((p,q,\alpha)\) at \((0,0,\tilde{\alpha})\) and \(\tilde{\Sigma}(\tilde{\alpha}) = \Sigma(0,0,\tilde{\alpha})\) is the set of solutions of problem \(\tilde{GP}(\tilde{\alpha})\).

**Corollary 2.** In addition to the assumptions in Theorem 2, assume that \(f, \Psi, F\) are \(C^1\) at \((\tilde{x},\tilde{y})\), then
\[
\partial^\infty \tilde{V}(\tilde{\alpha}) \subseteq \bigcup_{(x,y) \in E(\alpha)} \{\xi : (\gamma, \eta, \xi) \in M^\lambda(\tilde{x},\tilde{y})\}
\]
\[
\partial^\infty \tilde{V}(\tilde{\alpha}) \subseteq \bigcup_{(x,y) \in E(\alpha)} \{\xi : (\gamma, \eta, \xi) \in M^\lambda(\tilde{x},\tilde{y})\}
\]
where \(\tilde{M}^\lambda(\tilde{x},\tilde{y})\) is the set of index \(\lambda\) CD multipliers for problem \(\tilde{GP}(\tilde{\alpha})\) and \(M^\top\) denotes the transpose of a matrix \(M\).

4. Relations with classical results in nonlinear programming

How does the above approach relate to the usual nonlinear programming results? In general (OPVIC) is not a nonlinear programming problem. However in the case where \(\Omega = R^a \times R^b\) with \(a,b\) non-negative numbers and \(a+b = m\), (OPVIC) is a nonlinear programming problem and we can compare our results with the classical results in nonlinear programming. Indeed, let \(y = (z,u)\) and \(F = (H,G)\) where \(H(x,z,u) : R^{n+m} \to R^a, G(x,z,u) : R^{n+m} \to R^b\). (OPVIC) becomes an optimization problem with complementarity constraints:

(OPCC) \[
\begin{aligned}
&\text{minimize} \quad f(x,z,u), \\
&\text{subject to} \quad \Psi(x,z,u) \leq 0, (x,z,u) \in C \\
&H(x,z,u) = 0 \\
&u \geq 0, G(x,z,u) \geq 0, \\
&\langle u, G(x,z,u) \rangle = 0
\end{aligned}
\]
which is a nonlinear programming problem
\[
\text{(NLP)} \quad \text{minimize} \quad f(x, z, u) \\
\text{subject to} \quad g(x, z, u) \leq 0 \\
\quad h(x, z, u) = 0, \\
\quad (x, z, u) \in C
\]
with \( g(x, z, u) := (\Psi(x, z, u), -G(x, z, u), -u), \) and \( h(x, z, u) := \langle u, G \rangle(x, z, u). \)

For simplicity, we first consider (OPCC) with \( d = a = 0 \) and \( C = \mathbb{R}^{n+m}. \) That is, we consider the following (OPCC) without equality, inequality and abstract constraints:
\[
\text{(OPCC)} \quad \text{minimize} \quad f(x, z, u), \\
\text{subject to} \quad u \geq 0, \ G(x, z, u) \geq 0, \\
\quad \langle u, G(x, z, u) \rangle = 0.
\]

We assume that \( f, G \) are \( C^1 \) around the point \((\bar{x}, \bar{z}, \bar{u})\). The Fritz John optimality condition for the above nonlinear programming problem is the existence of \( \lambda \geq 0, \mu \in \mathbb{R}, r_G \in \mathbb{R}^b, r_u \in \mathbb{R}^b \) not all zero such that
\[
(1) \quad r_G \geq 0, r_u \geq 0 \text{ and } \langle r_G, G(\bar{x}, \bar{z}, \bar{u}) \rangle = 0, \langle r_u, \bar{u} \rangle = 0.
\]
and
\[
0 = \lambda \nabla f(\bar{x}, \bar{z}, \bar{u}) - \nabla G(\bar{x}, \bar{z}, \bar{u})^\top r_G - \{(0, 0, r_u)\} + \mu \nabla \langle u, G \rangle(\bar{x}, \bar{z}, \bar{u})
\]
Using the sum rule and product rule, we have
\[
\nabla \langle u, G \rangle(\bar{x}, \bar{z}, \bar{u}) = \{(0, 0, G(\bar{x}, \bar{z}, \bar{u}))\} + \nabla G(\bar{x}, \bar{z}, \bar{u})^\top \bar{u}.
\]
Therefore
\[
(2) \quad 0 = \lambda \nabla f(\bar{x}, \bar{z}, \bar{u}) + \nabla G(\bar{x}, \bar{z}, \bar{u})^\top (\mu \bar{u} - r_G) + \{(0, 0, \mu G(\bar{x}, \bar{z}, \bar{u}) - r_u)\}
\]
\[
\quad r_G \geq 0, r_u \geq 0 \text{ and } \langle r_G, G(\bar{x}, \bar{z}, \bar{u}) \rangle = 0, \langle r_u, \bar{u} \rangle = 0.
\]

On the other hand, (OPCC) is equivalent to the optimization problem with a generalized equation constraint:
\[
\min \quad f(x, z, u) \\
\text{s.t.} \quad 0 \in G(x, z, u) + N_{R^b}(u).
\]
The index \( \lambda \) CD multipliers are vectors \( \eta \in R^b \) such that
\[
(3) \quad 0 \in \lambda \nabla f(\bar{x}, \bar{z}, \bar{u}) - \nabla G(\bar{x}, \bar{z}, \bar{u})^\top \eta + \{(0, 0)\} \times D^* N_{R^b}(\bar{u}, -G(\bar{x}, \bar{z}, \bar{u}))(\eta).
\]
Comparing equations (2) and (3), we see that we would like to prove that \( \alpha := \mu G(\bar{x}, \bar{z}, \bar{u}) - r_u \in D^* N_{R^b}(\bar{u}, -G(\bar{x}, \bar{z}, \bar{u}))(\eta) \) with \( -\eta = \mu \bar{u} - r_G. \) Or equivalently,
\[
(\alpha, \eta) \in N_{Gph R^b}(\bar{u}, -G(\bar{x}, \bar{z}, \bar{u})).
\]

Note that by Proposition 2,
\[
N_{Gph N_{R^b}}(\bar{u}, -G(\bar{x}, \bar{z}, \bar{u})) = \{ (\alpha, \eta) : \alpha_L = 0, \eta_{I_0} = 0, \text{ and } \forall i \in I_0, \text{ either } \alpha_i \eta_i = 0 \text{ or } (\alpha_i < 0 \text{ and } \eta_i > 0) \}
\]
with
\[
L := L(\bar{x}, \bar{z}, \bar{u}) := \{i \in \{1, 2, \ldots, m\} : \bar{u}_i > 0, G(\bar{x}, \bar{z}, \bar{u})_i = 0\}, \\
I_i := I_i(\bar{x}, \bar{z}, \bar{u}) := \{i \in \{1, 2, \ldots, m\} : \bar{u}_i = 0, G(\bar{x}, \bar{z}, \bar{u})_i > 0\}, \\
I_0 := I_0(\bar{x}, \bar{z}, \bar{u}) := \{i \in \{1, 2, \ldots, m\} : \bar{u}_i = 0, G(\bar{x}, \bar{z}, \bar{u})_i = 0\}.
\]
We consider the following cases

Case $i \in I_1$: then $G_i(\bar{x}, \bar{z}, \bar{u}) = 0$ and $r_i^u = 0$. So $\alpha_i = \mu G_i(\bar{x}, \bar{z}, \bar{u}) - r_i^u = 0$.

Case $i \in I_2$: then $\eta_i = 0$, $G_i(\bar{x}, \bar{z}, \bar{u}) > 0$ and $r_i^G = 0$. So $\eta_i = \mu \bar{u} - r_i^G = 0$.

Case $i \in I_3$: then $u_i = 0$, $G_i(\bar{x}, \bar{z}, \bar{u}) = 0$ and $\alpha_i = \mu G_i(\bar{x}, \bar{z}, \bar{u}) - r_i^u = -r_i^u$ and $\eta_i = -\mu \bar{u} + r_i^G = r_i^G$. So either $\alpha_i \eta_i = 0$ or $\alpha_i \eta_i = r_i^u r_i^G \neq 0$. In the later case $\alpha_i = -r_i^u < 0$ and $\eta_i = r_i^G > 0$.

Hence $(\alpha, \eta) \in N_{G, \text{ph}}(\bar{u}, -G(\bar{x}, \bar{z}, \bar{u}))$.

In the general case where $d$ or $a$ is not zero and the functions involved are Lipschitz only, the proof is exactly similar by using the sum rule and the product rule for the Clarke generalized gradients. Consequently, we have established the relationship between the necessary optimality conditions for problem (OPCC) using the nonlinear programming approach and our approach as stated in the following proposition.

**Proposition 8.**

If $(\gamma, \eta_H, r_G, r_u, \mu) \in M^\lambda_{NLP}(\bar{x}, \bar{z}, \bar{u})$ then $(\gamma, \eta) \in M^\lambda_{CD}(\bar{x}, \bar{z}, \bar{u})$ with $\eta := \eta_H \mu \bar{u} - r_G$ for all $\lambda \geq 0$, where $M^\lambda_{NLP}(\bar{x}, \bar{z}, \bar{u})$ is the index $\lambda$ NLP multiplier set corresponding to $(\bar{x}, \bar{z}, \bar{u})$, i.e., the set of vectors $(\gamma, \eta_H, r_G, r_u, \mu) \in \mathbb{R}^{d+a+b+b}$ such that

\[
\begin{cases}
\gamma \geq 0, r_G \geq 0, r_u \geq 0 \quad \text{and} \quad \langle \Psi(\bar{x}, \bar{z}, \bar{u}), \gamma \rangle = 0, \langle r_G, G(\bar{x}, \bar{z}, \bar{u}) \rangle = 0, \langle r_u, \bar{u} \rangle = 0 \\
0 \in \lambda \partial f(\bar{x}, \bar{z}, \bar{u}) + \partial \Psi(\bar{x}, \bar{z}, \bar{u})^\top \gamma + \partial H(\bar{x}, \bar{z}, \bar{u})^\top \eta_H - \partial C G(\bar{x}, \bar{z}, \bar{u})^\top r_G - \{0, 0\} + \mu \partial C(\bar{u}, G)(\bar{x}, \bar{z}, \bar{u}) + N_C(\bar{x}, \bar{z}, \bar{u})
\end{cases}
\]

and $M^\lambda_{CD}(\bar{x}, \bar{z}, \bar{u})$ is the index $\lambda$ CD multiplier set corresponding to $(\bar{x}, \bar{z}, \bar{u})$ with the limiting subgradient of $G$ replaced by the Clarke subgradients, i.e., the set of vectors $(\gamma, \eta_H, \eta_G) \in \mathbb{R}^{d+a+b+b}$ such that

\[
0 \in \lambda \partial f(\bar{x}, \bar{z}, \bar{u}) + \partial \Psi(\bar{x}, \bar{z}, \bar{u})^\top \gamma + \partial H(\bar{x}, \bar{z}, \bar{u})^\top \eta_H - \partial C G(\bar{x}, \bar{z}, \bar{u})^\top \eta_G + \{0, 0\} + D^* N_R^\lambda(\bar{u}, -G(\bar{x}, \bar{z}, \bar{u}))(\eta_G) + N_C(\bar{x}, \bar{z}, \bar{u}).
\]

Since the CD multipliers in $M^\lambda_{CD}(\bar{x}, \bar{y})$ are functions of the NLP multipliers, it is clear that the NLP multiplier set $M^\lambda_{NLP}(\bar{x}, \bar{y})$ being nonempty would imply that the CD multiplier set $M^\lambda_{CD}(\bar{x}, \bar{y})$ is nonempty and it is possible that $M^0_{NLP}(\bar{x}, \bar{y}) \neq \{0\}$ while $M^0_{CD}(\bar{x}, \bar{y}) = \{0\}$. In other words, it is more likely to have no normal NLP multipliers than to have no normal CD multipliers. Here is such an example.

**Example.** Consider the following (OPCC):

\[
\begin{align*}
\text{minimize} & \quad -y \\
\text{subject to} & \quad x - y = 0 \\
& \quad x \geq 0, y \geq 0, xy = 0
\end{align*}
\]

where $x \in \mathbb{R}$ and $y \in \mathbb{R}$.

It is clear that the only feasible solution is $(0, 0)$ and hence the only optimal solution is $(0, 0)$. The index $\lambda$ NLP multipliers $(\gamma_1, \gamma_2, \eta_1, \eta_2)$ at $(0, 0)$ satisfy

\[
\begin{cases}
0 = \lambda(0, -1) - (\gamma_1, 0) - (0, \gamma_2) + \eta_1(1, -1) + \eta_2(0, 0) \\
\gamma_1 \geq 0, \gamma_2 \geq 0.
\end{cases}
\]

It is clear that any $(\gamma_1, \gamma_2, \eta_1, \eta_2) = (0, 0, 0, \eta_2)$ with $\eta_2 \neq 0$ is a nonzero (NLP) abnormal multiplier and there is no (NLP) normal multiplier. Hence $M^0_{NLP}(0, 0) =$
\{(0, 0, 0)\} \times (-\infty, +\infty) \neq \{(0, 0, 0)\} \text{ and } M^1_{NLP}(0, 0) = \emptyset. \text{ However we can rewrite (4) as}

\text{minimize } f(x, y) \\
\text{subject to } 0 \in F(x, y) + N_{\mathbb{R} \times \mathbb{R}^+}(x, y)

\text{with } f(x, y) := -y, \text{ and } F(x, y) := (x - y, x). \text{ The index } \lambda \text{ CD multipliers } (\eta_H, \eta_G) \text{ at } (0, 0) \text{ satisfy}

0 \in \lambda(0, -1) - \eta_H(1, -1) - \eta_G(1, 0) + \{0\} \times D^* N_{\mathbb{R}^+}(0, 0)(-\eta_G)

\text{or equivalently}

\eta_H + \eta_G = 0 \\
(\lambda - \eta_H, \eta_G) \in N_{\text{Gph. } N_{\mathbb{R}^+}}(0, 0).

\text{Taking } \lambda = 0, \text{ the above inclusion becomes}

\eta_H = -\eta_G \\
(-\eta_H, \eta_G) \in N_{\text{Gph. } N_{\mathbb{R}^+}}(0, 0)

\text{which implies that } \eta_G = \eta_H = 0 \text{ by virtue of Proposition 2. While taking } \lambda = 1, \\
\text{we have } \eta_H = -\eta_G \text{ and}

(1 - \eta_H, \eta_G) \in N_{\text{Gph. } N_{\mathbb{R}^+}}(0, 0)

\text{which implies that either } \eta_H = 1, \eta_G = -1 \text{ or } \eta_G = \eta_H = 0. \text{ So } M^1(0, 0) = \\
\{(0, 0)\} \cup \{(1, -1)\} \text{ and } M^0(0, 0) = \{(0, 0)\}.

\text{What is the effect of nonexistence of normal NLP multipliers? Looking at the value function for (4) provides a better understanding of what happens. Consider}

V(q_1, q_2) := \inf \{-y : (q_1, q_2) \in (x - y, x) + N_{\mathbb{R} \times \mathbb{R}^+}(x, y)\}

= \inf \{-y : q_2 - x \leq 0, -y \leq 0, (q_2 - x)y = 0, x - y - q_1 = 0\}

\text{The feasible set of the perturbed problem still reduces to one point. In fact,}

\begin{align*}
\begin{cases}
\Sigma(q_1, q_2) = \{(q_2, q_2 - q_1)\} \text{ and } V(q_1, q_2) = q_1 - q_2 & \text{if } q_2 \geq q_1 \\
\Sigma(q_1, q_2) = \{(q_1, 0)\} \text{ and } V(q_1, q_2) = 0 & \text{if } q_2 \leq q_1.
\end{cases}
\end{align*}

\text{So } V(p, q) = \min(0, q_1 - q_2) \text{ is Lipschitz continuous everywhere.}

\text{Note that the inclusions of Theorem 1 is an equality here: } \partial V(0, 0) = M^1(0, 0) = \\
\{(0, 0)\} \cup \{(1, -1)\} \text{ and } \partial^* V(0, 0) = M^0(0, 0) = \{(0, 0)\}.

\text{On the other hand, if we use the nonlinear programming formulation we shall be considering the non-additive perturbations:}

\widehat{GP}(\alpha) \min \quad f(x, y, \alpha) \\
\text{st.} \quad \Psi_1(x, y, \alpha) \leq 0 \\
\Psi_2(x, y, \alpha) \leq 0 \\
H_1(x, y, \alpha) = 0 \\
H_2(x, y, \alpha) = 0
where $\alpha = (q_1, q_2)$ and

\[
\begin{align*}
    f(x, y, \alpha) &= -y \\
    \Psi_1(x, y, \alpha) &= q_2 - x \\
    \Psi_2(x, y, \alpha) &= -y \\
    H_1(x, y, \alpha) &= (q_2 - x)y \\
    H_2(x, y, \alpha) &= x - y - q_1.
\end{align*}
\]

According to Corollary 2, if the growth hypothesis (GH) were satisfied, then

\[
\begin{align*}
    \partial V(0, 0) &\subseteq \{ \gamma_1(0, 1) + \eta_2(-1, 0) : (\gamma_1, \gamma_2, \eta_1, \eta_2) \in M^1_{NLP}(0, 0) \} \\
    \partial^\infty V(0, 0) &\subseteq \{ \gamma_1(0, 1) + \eta_2(-1, 0) : (\gamma_1, \gamma_2, \eta_1, \eta_2) \in M^0_{NLP}(0, 0) \}.
\end{align*}
\]

But this is not possible since $M^0_{NLP}(0, 0) = \emptyset$. Indeed (GH) is not satisfied for this example.

In the above example, $M^0_{NLP}(\Sigma) \neq \{0\}$ while $M^0(\Sigma) = \{0\}$. In fact it is not just a coincidence that $M^0_{NLP}(\Sigma) \neq \{0\}$. Ye et al. showed that the Mangasarian-Fromowitz constraint qualification never holds for optimization problems with complementarity constraints with smooth functions and no abstract constraint \cite{Ye} which is equivalent to saying that $M^0_{NLP}(\Sigma) \neq \{0\}$ (see e.g. \cite{Ye} and \cite{Ye1}). In the case where there is an abstract constraint, the nonexistence of nonzero abnormal multiplier is in general a weaker assumption than the Mangasarian-Fromowitz constraint qualification (see e.g. \cite{Ye} and \cite{Ye1}). We now prove that there always exist nonzero abnormal NLP multipliers as long as a complementarity constraint is active.

**Proposition 9.** Assume $(\bar{x}, \bar{z}, \bar{u}) \in \mathbb{R}^{n+m}$ is a feasible solution of (OPCC) with at least one active complementarity constraint: there is $i$ such that $G_i(\bar{x}, \bar{z}, \bar{u}) = 0$. Then $M^0_{NLP}(\bar{x}, \bar{z}, \bar{u}) \setminus \{0\} \neq \emptyset$.

**Proof.** The point $(\bar{x}, \bar{z}, \bar{u})$ is obviously a solution to the following optimization problem:

\[
\begin{align*}
    \min & \quad \langle u, G(x, z, u) \rangle \\
    \text{s.t.} & \quad u \geq 0, G(x, z, u) \geq 0.
\end{align*}
\]

By the generalized multiplier rule, there exists $\mu \geq 0, r_u \in \mathbb{R}^d, r_G \in \mathbb{R}^c$ not all zero such that

\[
\begin{align*}
    0 &\in \mu \partial \langle u, G \rangle(\bar{x}, \bar{z}, \bar{u}) - \{(0, 0, r_u)\} - \partial G(\bar{x}, \bar{z}, \bar{u})^\top r_G \\
    \langle \bar{u}, r_u \rangle &= 0, \langle r_G, G(\bar{x}, \bar{z}, \bar{u}) \rangle = 0
\end{align*}
\]

Therefore taking $\gamma = 0, \eta_H = 0, (\gamma = 0, \eta_H = 0, r_G, r_u, \mu)$ is a nonzero NLP abnormal multiplier of (OPCC).

What is the impact of the existence of nonzero abnormal NLP multipliers? For simplicity of notations, consider the value function for (OPCC) without equality, inequality and abstract constraints where all functions are assumed to be smooth.

\[
\begin{align*}
    V(q) &:= \inf \{ f(x, y) : q \in F(x, y) + N_{\mathbb{R}^n}(y) \} \\
    &= \inf \{ f(x, y) : q - F(x, y) \leq 0, y \geq 0, \langle q - F(x, y), y \rangle = 0 \}.
\end{align*}
\]
If the growth hypothesis is satisfied, then
\[
\partial V(0) \subseteq \bigcup_{(x,y) \in \Sigma} M^1(x,y)
\]
\[
\partial^\infty V(0) \subseteq \bigcup_{(x,y) \in \Sigma} M^0(x,y),
\]
where \(M^\lambda(x,y)\) is the set of index \(\lambda\) CD multipliers at \((x,y)\) and \(\Sigma\) is the set of solutions of (OPCC). Under very reasonable assumptions (see [18] for such assumptions), it is possible the set of abnormal CD multipliers \(\bigcup_{(x,y) \in \Sigma} M^0(x,y)\) contains only the zero vector and one may conclude that the value function is Lipschitz near 0 and \(M^1(x,y)\) is a bounded set for any \((x,y) \in \Sigma\).

On the other hand, according to Corollary 2, if the growth hypothesis (GH) is satisfied, then
\[
(5)\quad \partial V(0) \subseteq \bigcup_{(x,y) \in \Sigma} \{\gamma_F + \mu y : (\gamma_F, \gamma_y, \mu) \in M_{NLP}^1(x,y)\}
\]
\[
\partial^\infty V(0) \subseteq \bigcup_{(x,y) \in \Sigma} \{\gamma_F + \mu y : (\gamma_F, \gamma_y, \mu) \in M_{NLP}^0(x,y)\},
\]
where \(M_{NLP}^\lambda(x,y)\) are the NLP multipliers \(\gamma_F, \gamma_y \in \mathbb{R}^n_+, \mu \in \mathbb{R}\) satisfying
\[
0 \in \lambda \nabla f(x,y) - \nabla F(x,y)\gamma_F + (0, -\gamma_y) - \mu \nabla \langle F, y \rangle(x,y)
\]
\[
\langle \gamma_F, F(x,y) \rangle = 0, \langle \gamma_y, y \rangle = 0.
\]
Since the nonexistence of nonzero abnormal NLP multipliers is a sufficient condition for the nonemptiness and boundedness of the set of normal NLP multipliers and is also a necessary condition in the case where the equality constraints are smooth (see [5]), \(M_{NLP}^1(x,y)\) is always unbounded and hence the estimates for the limiting subgradients of the value function may be too loose in the case where the set \(\{\gamma_F + \mu y : (\gamma_F, \gamma_y, \mu) \in M_{NLP}^1(x,y)\}\) in the right hand side of (5) is unbounded and \(M^1(x,y)\) is bounded.

5. Applications to bilevel programming

One of the motivations to consider optimization problems with complementarity constraints is to solve the following bilevel programming problems:

\[
\text{(BLPP)} \quad \begin{array}{ll}
\text{minimize} & f(x,z) \\
\text{subject to} & z \in S(x), \Psi(x,z) \leq 0, (x,z) \in C
\end{array}
\]

where \(S(x)\) is the solution of the lower-level problem

\[
\begin{array}{ll}
\text{minimize} & g(x,z) \\
\text{subject to} & \psi(x,z) \leq 0,
\end{array}
\]

where \(f : \mathbb{R}^{n+a} \to \mathbb{R}, g : \mathbb{R}^{n+a} \to \mathbb{R}^p, \psi : \mathbb{R}^{n+a} \to \mathbb{R}^q, \Psi : \mathbb{R}^{n+a} \to \mathbb{R}^d\). Under suitable convexity assumptions, we can replace the lower problem by its KKT conditions. As in [15], we find that any \((x,z)\) is solution of (BLPP) if and only if
there is $u$ such that $(x, z, u)$ is solution of the following problem

minimize $f(x, z)$
subject to $\psi(x, z) \leq 0$ and $u \geq 0$,
$\langle \psi(x, z), u \rangle = 0$,
$\nabla_z g(x, z) + \nabla_z \psi(x, z)^T u = 0$,
$\Psi(x, z) \leq 0, (x, z) \in C$

which is a (OPCC). Perturbing the complementarity constraints $\psi(x, z) \leq 0$ by $\psi(x, z) + p \leq 0, u \geq 0$ by $-u + \alpha \leq 0$ and $\langle \psi(x, z), u \rangle = 0$ by $\langle \psi(x, z), u \rangle + \beta = 0$ is not interesting in the bilevel programming context since these relations come from the KKT conditions and not from some input data subject to noise. Instead the following perturbations are practical:

BLPP($\alpha$) minimize $f(x, z, \alpha)$
subject to $z \in S(x, \alpha), \Psi(x, z, \alpha) \leq 0, (x, z) \in C$

where $S(x, \alpha)$ is the solution of the lower-level problem

minimize $g(x, z, \alpha)$
subject to $\psi(x, z, \alpha) \leq 0$.

Under suitable assumptions BLPP($\alpha$) is equivalent to

minimize $f(x, z, \alpha)$
subject to $\psi(x, z, \alpha) \leq 0$ and $u \geq 0$,
$\langle \psi(x, z, \alpha), u \rangle = 0$,
$\nabla_z g(x, z, \alpha) + \nabla_z \psi(x, z, \alpha)^T u = 0$,
$\Psi(x, z, \alpha) \leq 0, (x, z) \in C$.

which is also equivalent to

minimize $f(x, z, \alpha)$
subject to $0 \in -\psi(x, z, \alpha) + N_{\mathbb{R}^m}(u),$
$\nabla_z g(x, z, \alpha) + \nabla_z \psi(x, z, \alpha)^T u = 0$,
$\Psi(x, z, \alpha) \leq 0, (x, z) \in C$.

Hence Theorem 2 allows us to compute the limiting subgradients of $V$ by the CD multipliers for the above problem. However since there is always a nonzero abnormal NLP multiplier for the above problem, no useful information are provided in terms of NLP multipliers.

References