Numerical Assessment of the Stability of Reconstruction Processes for Computed Tomography

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Abstract

This paper deals with the assessment of the stability of reconstruction methods for computed tomography, including Filtered Back Projection and ‘entropy-like’ methods. For each of these methods, the influence of errors in the measured data on the reconstructed image is analyzed. A small perturbation of the data vector induces a perturbation of the reconstructed object which can be computed by means of the sensitivity matrix. Using appropriate matrix computation techniques, an upper bound on the size of the reconstruction error is determined, as well as the pattern of noise in the sinogram that will result in the largest reconstruction error and the standard deviation map in the object domain. Simulations will illustrate our analysis and demonstrate its utility in the interpretation of computed images and in the selection of reconstruction parameters.

I. INTRODUCTION

The reconstruction of images from measurements of radiation around the body of a patient have been extensively studied in the past decades, and many methods have been proposed (see [1] and references therein). All of these methods must find solutions for two major classes of problems. The first one arises from the fact that the relationship between the object to be reconstructed \( \phi \) and the physical quantity to be measured \( \psi \) is theoretically known but usually undetermined in practice\(^2\). The second class of problems consists in reconstructing an image from noisy, blurred and under-sampled data.

For several imaging techniques, such as X-ray tomography or SPECT, the simplest model of forward relationship involves the Radon transform or some of its generalizations (attenuated Radon transform, X-ray transform, etc.). We shall write symbolically “\( \psi = R\phi \)”. This (linear) equation can be shown to be the integral form of the so-called transport equation [1], when no scattering is to be accounted for. In spite of recent progress [2], inversion of the transport equation and implementation of suitable regularization schemes for it remain open problems, even if one assumes perfect knowledge of the attenuation map and phase function. Nevertheless, we believe that the mastery of the inverse problem corresponding to its simplified formulation is a necessary step towards a deeper understanding of the problem in its full complexity.

The purpose of the present paper is to introduce some concepts from applied mathematics and to demonstrate, through simulations, how these concepts can provide useful information on the stability and fidelity of reconstruction processes. The stability information includes (i) an upper bound on the noise amplification due to the reconstruction process, (ii) computation of a few critical modes (see Section V) and (iii) the standard deviation map of the reconstructed image.

Before introducing a general framework for the regularization of Radon-type inverse problems, we shall glance at the standard FBP algorithm because, unlike heuristic methods, the FBP method allows for a rigorous error analysis to be performed.

II. OVERVIEW OF THE FILTERED BACK PROJECTION METHOD

Among the essential features of the FBP method are its linearity and the fact that a certain level of resolution in the reconstructed image is imposed by means of the filtering operation. Recall that the (attenuation-free) Radon transform \( \mathcal{R} \) is defined by\(^3\)

\[
[\mathcal{R}\phi](\theta, p) \equiv \int \phi(\xi) \delta(p - (\theta, \xi)) \, d\xi.
\]

As pointed out in [3], the FBP algorithm can be derived from the following fundamental relationship:

\[
(\mathcal{R}^* g) \ast \phi = \mathcal{R}^*(g \ast R\phi),
\]

where \( \mathcal{R}^* \) denotes the adjoint of \( \mathcal{R} \), \( \ast \) is the standard 2-dimensional convolution, \( \ast \) is the radial convolution and \( g \equiv g(\theta, p) \) is some smoothing kernel (often chosen independent of \( \theta \)). The adjoint of \( \mathcal{R} \) is the Back Projection operator. It is defined by

\[
[\mathcal{R}^*\psi](\xi) \equiv \int_{S_1} \psi(\theta, \langle \theta, \xi \rangle) \, d\theta.
\]

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\(^2\)For example, in the case of SPECT, \( \phi \) is the distribution of radioactivity, \( \psi \) is the outgoing radiation as a function of an angular and position variables. The relation between these distributions involves effects such as the attenuation and scatter, which can not be determined with perfect accuracy.

\(^3\)For symbols which are not deﬁned in the text, the reader is referred to the Glossary (Section VIII).
Here, $S_1$ denotes the unit circle. Ignoring sampling issues, the FBP reconstruction is defined by

$$\tilde{\phi} \equiv R^* (g * \psi),$$

where $\psi$ represents the given Radon transform of the original object\footnote{In the discrete implementation of Equation (1), the filter must include the so-called ramp filter, in order to cope with the discrepancy between the polar and Cartesian coordinate systems.}. The FBP method therefore essentially consists in applying the adjoint of the Radon transform to smoothed projection data.

The discrete version of Equation (1) reads $x = By$, where $x$ the discrete representation of the reconstructed object, $B$ is the matrix corresponding to the application of the FBP algorithm and $y$ is the vector representing the noisy and sampled data. It follows that any perturbation $\delta y$ of the data, will induce a perturbation $\delta x = B \delta y$ of the FBP solution.

III. INVERSE PROBLEMS AND RECONSTRUCTION PROCESSES

Suppose we are interested in an object $\phi_0$ (such as an image or a sequence of images), but we only have access to a sampling $y_0$ of a transformation $\psi_0$ of this object:

$$L^{(o)} \xrightarrow{R} L^{(i)} \xrightarrow{S} \mathbb{R}^m$$

$$\phi_0 \mapsto \psi_0 = R \phi_0 \mapsto y_0 = S \psi_0. \quad (2)$$

Here, $L^{(o)}$ and $L^{(i)}$ denote the (infinite-dimensional) functional spaces in which the object of interest and its image by the transformation $R$ are assumed to lie, respectively, and $S$ is the sampling operator. Equation (2) is referred to as the forward relationship. In image reconstruction, the space $L^{(o)}$ is often chosen as the space of square integrable functions whose support lie in some bounded domain.

The problem consists in reconstructing an approximation of the original object, given an approximation $\tilde{y}$ of $y_0$. The difference between $\tilde{y}$ and $y_0$ is the experimental error.

For many problems of practical interest, the transformation $R$ is an invertible linear operator. In general, ill-posedness of these problems is primarily due to the fact that the inverse operator $R^{-1}$ is not continuous, so that to small variations of $\psi \in L^{(i)}$ may correspond very large variations of its inverse image $R^{-1} \psi$.

A reconstruction process can be regarded as a function which associates an object $\tilde{\phi}$ to each data vector $y$:

$$\tilde{\phi} = \mathcal{F}(y).$$

For a reconstruction process to be acceptable, $\mathcal{F}$ should meet certain requirements. Firstly, $\tilde{\phi}$ should reproduce the data to within the experimental error (fidelity); secondly, $\tilde{\phi}$ should not be too sensitive to fluctuations of the data vector (stability); thirdly, $\tilde{\phi}$ should be related to the true object in a way that makes it physically interpretable (legibility); finally, $\tilde{\phi}$ should be computable in a reasonable time (computability).

The first two requirements are always partially conflicting. Consequently, a compromise needs to be found. Keeping this in mind, it becomes clear that a reconstruction process should provide, together with a reconstructed object, some information on its fidelity and stability. Although of crucial importance, the last two requirements (legibility and computability) will not be addressed in this paper. All reconstruction methods discussed further perform reasonably well in terms of legibility and computability.

IV. REGULARIZATION SCHEME FOR COMPUTED TOMOGRAPHY

In practice, reconstructing an object directly as an element of the functional space $L^{[o]}$ may be difficult or even impossible, simply because elements of $L^{[o]}$ have infinite (vector) dimension. The first step in the definition of a regularization scheme therefore consists in choosing an appropriate interpolation basis $\{e_j\}_{j=1}^n \subset L^{[i]}$, allowing finite dimensional representation of the object. Otherwise expressed, the object to be reconstructed is confined to a finite dimensional subspace of $L^{[o]}$, and is represented by a vector $x \in \mathbb{R}^n$:

$$\phi = Tx \equiv \sum_{j=1}^n x_j e_j. \quad (3)$$

We shall call $T$ the emulation operator. The interpolation functions $e_j$ can be regarded as generalized pixels. The component vector $x$ is constrained by the data vector as shown by the following relationship:

$$\mathbb{R}^n \xrightarrow{T} L^{[o]} \xrightarrow{R} L^{[i]} \xrightarrow{S} \mathbb{R}^m$$

$$x \quad \mapsto \quad \phi = Tx \quad \mapsto \quad \psi = R \phi \quad \mapsto \quad y = S \psi.$$

We shall write $R \equiv SRTI$. It can be represented by an $(m \times n)$-matrix. When $R$ is the Radon transform, the entries of $R$ are given by the following formula:

$$R_{j,k} \equiv Re_j(\theta_k, p_1) = \int e_j(\xi) \delta(p_1 - \langle \xi, \theta_k \rangle) \, d\xi.$$

It should be noted that confining the object to $T \mathbb{R}^n$ may stabilize it, to some extent. In general, however, interpolation bases that are rich enough to allow a convenient representation of the object (in other words, interpolation bases which emulate the original object workspace $L^{[o]}$ well enough) do not sufficiently stabilize the reconstruction.

The second step then consists in defining the component vector of the reconstructed object as the minimizer of a regularized objective function $f$:

$$x \equiv \arg \min \{ f(x) \equiv \epsilon(y - Rx) + \alpha \phi(x) \mid x \in \mathbb{R}^n \}. \quad (4)$$

In the definition of $f$, the first term forces the solution to fit the data, while the second term stabilizes it with respect to
the data vector. In other words, \( \varepsilon \) strives for fidelity, while \( \varrho \) (whose negative is often called an entropy) aims at stabilizing the reconstruction. The regularization parameter \( \alpha \) controls the relative weight of each function. In practice, \( f \) should always be a convex function having a unique minimizer, so that \( \bar{x} \) in Equation (4) is unambiguously defined. Naturally, the reconstructed object is obtained from its components via Equation (3).

In summary, the main issues regarding the definition of a regularized reconstruction process are:

- the choice of an appropriate interpolation basis;
- the definition of the regularized objective function;
- and the development of numerical tools allowing the control of both the fidelity of the reconstruction and its stability.

For a general discussion on this regularization scheme, see for example [4], Section 3.

For the purposes of this article, we have chosen the standard gate function set as the interpolation basis, which corresponds to the familiar notion of pixel. It forms an orthogonal set (with respect to the usual integral scalar product), and can easily be scaled to form an orthonormal basis (of \( \mathbb{R}^n \)). Examples of regularized objective functions will be given in the next section. A reasonable and easy to calculate fidelity estimator is provided by \( \varepsilon (y - R \bar{x}) \). The sensitivity analysis can be performed by taking the (implicit) function defining the reconstruction process and linearizing it around the solution\(^5\), as is shown in the next section.

V. COMPUTATIONAL ISSUES

A. Objective functions

From now on, we assume that \( \varepsilon \) is the squared Euclidean norm:

\[
\varepsilon_0(y - Rx) = \frac{1}{2} \| y - Rx \|^2.
\]

We list below some classical regularizers, namely the Boltzmann-Shannon, the Kullback-Leibler, the Tikhonov and the generalized Tikhonov neg-entropies, respectively:

\[
\varrho_1(x) = \sum_{j=1}^{n} x_j \ln x_j, \quad \varrho_2(x) = \sum_{j=1}^{n} x_j \ln \frac{x_j}{x_{0j}},
\]

\[
\varrho_3(x) = \frac{1}{2} \| x \|^2, \quad \varrho_4(x) = \frac{1}{2} \langle x, Qx \rangle.
\]

In the definition of \( \varrho_2 \), the \( x_{0j} \)'s are the components of some reference model of the object. In the definition of \( \varrho_4 \), \( Q \) is a symmetric positive semi-definite matrix, which must be chosen so that the resulting objective function has a unique minimizer.

It should be noticed that the regularized objective function \( f \) obtained with \( \varepsilon = \varepsilon_0 \) and \( \varrho = \varrho_4 \) gives rise to a linear reconstruction process. As a matter of fact, since the gradient of \( f \) is zero at the optimal solution \( \bar{x} \), we must have

\[
R^* (R \bar{x} - y) + \alpha Q \bar{x} = 0,
\]

which can be rewritten as

\[
\bar{x} = [R^* R + \alpha Q]^{-1} R^* y.
\]

Our purpose here is not to discuss the argumentation leading to the choice of one particular regularizer or the other. Let us simply recall that, formally, the same type of regularized solution can be derived from (i) Bayesian considerations, from (ii) a purely deterministic approach, or from (iii) the Maximum Entropy on the Mean Method (see, for example, [4] and references therein; see also [5], for an application to Computed Tomography, and [6], for a general discussion on the Maximum Entropy on the Mean). The choice of a particular interpolation basis will not be discussed either. The main objective of what follows is to provide numerical tools for the assessment of the stability of each regularizer.

Recall that the reconstructed object is \( \mathcal{T} \bar{x} \), where \( \bar{x} \) is the solution of the following optimization problem:

\[
(P_0) \quad \min \{ \varepsilon(y - Rx) + \alpha \varrho(x) \mid x \in \mathbb{R}^n \}.
\]

Constraints may be included in the above problem. For example, each component of \( x \) may be required to be nonnegative, reflecting the non-negativity of the object to be reconstructed\(^6\). In general, \( x \) is constrained to lie in some convex set \( C \subset \mathbb{R}^n \) corresponding to the prior knowledge of the object. Notationally, this can be incorporated in \((P_0)\) by replacing \( \varrho \) with

\[
\varrho_C(x) \equiv \left\{ \begin{array}{ll} \varrho(x) & \text{if } x \in C, \\ +\infty & \text{otherwise.} \end{array} \right.
\]

B. Assessing the stability of the reconstruction process

Our aim is now to obtain a first order approximation of the relationship between perturbations of the data vector and corresponding perturbations of the reconstructed object. Henceforth, we assume that both \( \varepsilon \) and \( \varrho \) are twice continuously differentiable on their domain. All objective functions involving \( \varepsilon_0 \) and \( \varrho_i \) \((i = 1, \ldots, 4)\) satisfy this assumption. Notice however that, in general, this assumption does not apply to the constrained case. The reason for this is that, even if \( \varrho \) is twice continuously differentiable, \( \varrho_C \) will not be so in practice.

\(^5\)This is of course possible provided that \( f \) possesses certain smoothness properties.

\(^6\)Depending on the nature of the interpolation basis, the constraint \( \varphi \geq 0 \) may give rise to different constraints on the component-vector \( x \). However, these constraints are always convex, since they can be written as \( x \in \mathcal{L}^{(0)} \), where \( \mathcal{L}^{(0)} = \{ \varphi \in L^{(0)} \mid \varphi \geq 0 \} \).
Recall that, for \( \sigma \) to solve Problem (P0), we must have
\[
\nabla f(\bar{x}) = -R^* \nabla \varepsilon (y - R \bar{x}) + \alpha \nabla g(\bar{x}) = 0. \tag{5}
\]
If \( y \) is replaced by \( y + \delta y \), the same condition must hold with \( \bar{x} \) replaced by the perturbed solution \( \bar{x} + \delta \bar{x} \):
\[
-R^* \nabla \varepsilon (y + \delta y - R(\bar{x} + \delta \bar{x})) + \alpha \nabla g(\bar{x} + \delta \bar{x}) = 0. \tag{6}
\]
Keeping only the first order terms in the development of Equation (6) yields (using Equation (5))
\[
R^* H_e R + \alpha H_\varepsilon \delta \bar{x} = R^* \delta y, \tag{7}
\]
in which we have denoted \( H_e \) and \( H_\varepsilon \) the Hessian of \( \varepsilon \) at \( y - R \bar{x} \) and the Hessian of \( g \) at \( \bar{x} \), respectively. We therefore obtain
\[
\delta \bar{x} = S \delta y \quad \text{with} \quad S \equiv \left[ R^* H_e R + \alpha H_\varepsilon \right]^{-1} R^* H_e. \tag{8}
\]
The \((m \times n)\)-matrix \( S \) is referred to as the sensitivity matrix. Here, \( H_{\phi_0} \) is the identity matrix and \( S \) becomes
\[
S = \left[ R^* R + \alpha H_\varepsilon \right]^{-1} R^*.
\]
On denoting \( H_\varepsilon \) the Hessian matrix of \( \phi_1 \) at \( \sigma \), we can easily see that \( H_1 \) and \( H_2 \) are both equal to the diagonal matrix whose diagonal entries are the inverses of the components of \( \sigma \), that \( H_3 \) is the identity matrix and that \( H_4 = Q \). Note in particular that the sensitivity matrices of quadratic regularizers (Tikhonov and generalized Tikhonov) do not depend on the optimal solution. In these cases, sensitivity analysis can be performed \textit{a priori} (i.e. before actually reconstructing the object).

The first order perturbation of the reconstructed object is then given by \( \delta \bar{\phi} = \mathbb{I} \delta \bar{x} = \mathbb{I} S \delta y \).

From Equation (8), various types of error analysis can be performed. In particular, the (spectral) norm of \( S \) will provide an upper bound on the size of the reconstruction error. As a matter of fact we have for all \( \delta y \),
\[
\| \delta \bar{x} \| \leq \| S \| \cdot \| \delta y \|,
\]
and equality holds when \( \delta y \) is a singular vector corresponding to the highest singular value \( \sigma_1 \) of \( S \). Recall that
\[
\sigma_1 = \| S \| \equiv \sup \left\{ \| S x \| \mid \| x \| \leq 1 \right\}.
\]
Note that provided that the interpolation basis is orthonormal, the squared Euclidean norm of \( \delta \bar{x} \) coincides with the squared \( L_2 \)-norm of \( \delta \bar{\phi} \) for all \( \phi = \mathbb{I} x \), we have
\[
\| \phi \|_2^2 = \int |\phi|^2 = \int |\mathbb{I} x|^2 = \int \sum_{j=1}^n \sum_{k=1}^n x_j x_k e_j e_k = \sum_{j=1}^n x_j^2.
\]
Slightly more insightful is the computation of singular vectors associated with the highest singular values, which we shall refer to as critical modes of the reconstruction. They can be interpreted as artifact images which may have corrupted the reconstruction.

The sensitivity matrix also enables us to see how statistical features are carried from the data domain to the object domain. For example, if the covariance matrix of the noise \( C_y \equiv \text{Cov}(\delta y) \) is available, we can obtain the covariance matrix of the corresponding perturbation in the object domain as follows:
\[
C_\sigma \equiv \text{Cov}(\delta \bar{x}) = SC_y S^*.
\]
Here, \( S^* \) is the transpose of \( S \). Each diagonal element of \( C_\sigma \) is the variance of the corresponding pixel value. The square roots of these elements are the components of the standard deviation map. This will also be illustrated in the next section.

VI. NUMERICAL EXPERIMENTS

In this section, the above considerations are illustrated by means of 2D numerical experiments. Systematic study of regularization techniques for Radon-type inverse problems is deferred to separate publication.

From a given test object \( \phi_0 \), we compute samples of its Radon transform over a \((16 \times 96)\)-grid in the \((\theta, p)\)-plane. Both \( \phi_0 \) and the corresponding sinogram \( y_0 \) are shown in Figure 1.

![Figure 1: Left: original object; right: corresponding sinogram;](image-url)

In Figure 2, \((64 \times 64)\)-reconstructions using Tikhonov’s and Boltzmann-Shannon’s regularizers (see caption) are shown. All these reconstructions where obtained from the noise-free sinogram displayed in Figure 1. All four reconstructions appear to be of similar quality. However, we shall see in the discussion below that in the presence of noise the reconstruction procedures used to generate the top two images will perform very differently. The non-negativity constraint used to obtain the bottom left image makes it more difficult to analyze the reconstruction error. This image was included only to illustrate the effect of the non-negativity constraint on the Tikhonov reconstruction.

Figures 3, 4 and 5 show the influence of the regularization parameter \( \alpha \) on the fidelity and the stability for both Tikhonov and Boltzmann-Shannon regularizers. The fidelity is defined here as the number \( \beta \equiv \| y - R \sigma \| \). The stability is assessed according to the analysis presented in Section V. For each value of \( \alpha \),
\[
\sigma \equiv \| S \| \| y \| = \sigma_1 \| y \| \| \sigma \|
\]
is computed. The value of \( \sigma \) gives the maximum amplification
of the relative error, as shown by the following inequality:
\[
\frac{\|\delta \hat{x}\|}{\|\hat{x}\|} \leq \sigma \frac{\|\delta y\|}{\|y\|}.
\]
As expected, increasing \(\alpha\) causes \(\beta\) to increase and \(\sigma\) to decrease. The curves displaying \(\sigma\) versus \(\alpha\) can be used in practice for the selection of the regularization parameter. Indeed, \(\alpha\) may be chosen so that \(\beta\) is approximately equal to the expected norm of the noise.

These curves are also useful for the comparison of the behaviors of different regularizers in terms of the trade-off between stability and fidelity. For a desired \(\beta\), the most efficient regularizer is the one for which \(\sigma\) is smallest. (This doesn’t mean that it should necessarily be selected, since other criteria such as legibility and computability, must be considered.)

Figure 6 shows calculated values of the norm of the FBP sensitivity matrix for two different filters and a range of cutoff frequencies. This graph illustrates the well-known fact that low cutoff frequencies improve stability and therefore reduce the influence of noise.

In Figures 7 and 8, we give examples of error amplification. The critical perturbations were obtained by computing the highest singular value of the sensitivity matrix and corresponding singular vectors. Perturbed reconstructions were obtained by adding critical and random perturbations. These perturbations were scaled to obtain a relative error of 10% in...
The data domain. These reconstructions clearly illustrate the importance of stability of reconstruction processes: insufficient regularization may result in very high amplification of the noise by the reconstruction process.

Figure 7: Top: critical perturbation in the data domain (left) and corresponding perturbation in the object domain (right), for 'Tikhonov' reconstruction with \( \alpha = 0.7 \) \((\sigma \approx 11.7)\); bottom: critically perturbed sinogram (left) and corresponding reconstruction. The latter reconstruction must be compared to the top left image in Figure 2.

Figure 8: Same as Figure 7, with \( \alpha = 10 \) \((\sigma \approx 3.3)\). Again, compare the perturbed reconstruction (bottom right) to the top right image in Figure 2.

Finally, in Figure 10, we show standard deviation maps obtained from a model of covariance matrix for the noise. They correspond to Tikhonov reconstructions, with \( \alpha = 0.7 \) and \( \alpha = 10 \), and their computations were performed according to the analysis presented in Section V. The standard deviation map in the data domain (Figure 9) was chosen to be proportional to the square root of the corresponding value in the simulated \textit{noise free} sinogram (shown in Figure 1); the norm of this standard deviation map is equal to 10\% of the norm of the noise free sinogram.

Figure 9: Standard deviation map in the data domain: the value in each bin is proportional to the square root of the corresponding value in the simulated \textit{noise free} sinogram (shown in Figure 1); the norm of this standard deviation map is equal to 10\% of the norm of the noise free sinogram.

Figure 10: Standard deviation maps in the object domain corresponding the standard deviation map shown in Figure 9, for \( \alpha = 0.7 \) (left) and \( \alpha = 10 \) (right).

VII. CONCLUSION

We have presented some techniques for assessing the stability of reconstruction processes for Computed Tomography. These techniques can be applied to the Filtered Back Projection as well as methods belonging to the general framework outlined in Section IV (i.e. entropy-like methods). We have obtained graphs displaying the maximal noise amplification as a function of the regularization parameter. We have also presented examples of critical mode analysis, and obtained standard deviation maps for the reconstructed object from a given model of standard deviation map in the data domain. All of these results were based on the computation of
the sensitivity matrix (see Section V).

In this paper, the numerical experiments were carried out with a forward relationship involving the pure Radon transform. However, we emphasize that the same analysis can be applied to more general forward relationships, as long as the latter remains linear. This analysis provides quantitative information for the comparison of different methods will allow the informed selection of reconstruction parameters and techniques. In addition, interpretation of reconstructed images will benefit from quantitative information about the reconstruction error.

VIII. Glossary

\[ \mathcal{R} \] Radon transform
\[ \mathcal{R}^* \] Adjoint of \( \mathcal{R} \)
\( \mathcal{I} \) Emulator
\( \mathcal{S} \) Sampling operator
\( L^{(o)} \) Object workspace
\( L^{(i)} \) Image of \( L^{(o)} \) by \( \mathcal{R} \)
\( \xi \) Position variable (object domain)
\( (\theta, p) \) Angular and position variable (data domain)
\( S_1 \) Unit circle
\( \phi \) Generic element of \( L^{(o)} \)
\( \phi_0 \) Original object
\( \hat{\phi} \) Reconstructed object
\( \psi \) Generic element of \( L^{(i)} \)
\( \psi_0 \) Image of \( \phi_0 \) by \( \mathcal{R} \)
\( y \) Generic element of \( \mathbb{R}^n \)
\( y_0 \) Image of \( \psi_0 \) by \( \mathcal{S} \) (ideal data)
\( \bar{y} \) Measured data

\( \mathcal{F} \) Reconstruction process
\( \{\epsilon_j\} \) Interpolation basis
\( \mathcal{R} \) Forward matrix: \( \mathcal{R} = S \mathcal{R} \mathcal{I} \)
\( \mathcal{R}^* \) Adjoint (transpose) of \( \mathcal{R} \)
\( x \) Generic element of \( \mathbb{R}^n \)
\( f \) Regularized objective function
\( \varepsilon \) Fit function
\( \alpha \) Regularization parameter
\( \varnothing \) Regularizer
\( C \) Convex set representing physical constraints
\( \bar{x} \) Minimizer of \( f \)
\( \delta y \) Perturbation in the data domain
\( \delta \bar{x} \) Perturbation of \( \bar{x} \) induced by \( \delta y \)
\( H_{\varepsilon} \) Hessian of \( \varepsilon \) at \( y = R\bar{x} \)
\( H_y \) Hessian of \( \varnothing \) at \( \bar{x} \)
\( S \) Sensitivity matrix
\( S^* \) Adjoint of \( S \)
\( C_y \) Covariance matrix of \( \delta y \)
\( C_x \) Covariance matrix of \( \delta \bar{x} \)
\( \ast \) 2-dimensional convolution operator
\( \ast \) Radial convolution operator
\( \delta(\cdot) \) Dirac delta function

IX. References