Strong conical hull intersection property, bounded linear regularity, Jameson’s property (G), and error bounds in convex optimization

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Abstract

The strong conical hull intersection property and bounded linear regularity are properties of a collection of finitely many closed convex intersecting sets in Euclidean space. These fundamental notions occur in various branches of convex optimization (constrained approximation, convex feasibility problems, linear inequalities, for instance). It is shown that the standard constraint qualification from convex analysis implies bounded linear regularity, which in turn yields the strong conical hull intersection property. Jameson’s duality for two cones, which relates bounded linear regularity to property (G), is re-derived and refined. For polyhedral cones, a statement dual to Hoffman’s error bound result is obtained. A sharpening of a result on error bounds for convex inequalities by Auslender and Crouzeix is presented. Finally, for two subspaces, property (G) is quantified by the angle between the subspaces.

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1 Introduction

An intriguing list of topics

We start with an informal discussion of several topics, which appear to be quite different from each other, drawn from convex optimization and bordering fields.

Constrained interpolation In a series of papers ([7], [8], [11]), Deutsch et al. studied the problem of constrained interpolation from a convex subset or cone. One of the central ingredients in their analysis is the (strong) conical hull intersection property ("CHIP"), which captures a geometric property of the constraints. (See [12] and [10] for applications and further results.)

Projection algorithms The convex feasibility problem is often solved iteratively by projection algorithms. Rate-of-convergence results require bounded linear regularity of the constraints. (Details can be found in [4], [5], and [20].)

Duality theory for convex cones Jameson introduced in [18] a duality theory for two closed convex cones in a Banach space. He gave a generalization of the well-known fact (see, for instance, [19, Corollary 35.6]) that the sum of two closed subspaces in Banach space is closed if and only if the sum of their complements is. In our finite-dimensional setting, this linear result is trivial; for cones, however, several important questions remain unanswered. Central to Jameson’s study is the property (G).

Systems of linear inequalities In 1952, Hoffman [16] proved that the distance from an arbitrary point to the solution set of a system of linear inequalities is bounded above by a constant times the norm of the residual. This important result, which relies on the polyhedrality of the underlying system, is commonly referred to as Hoffman’s error bound result. (A good starting point to generalizations and applications is Burke and Tseng’s [6].)

Systems of convex inequalities Because of non-polyhedrality, different tools are required to tackle this natural generalization of the linear-inequality-case. Typically, existence of error bounds is guaranteed under constraint qualifications. (For additional information, see Lewis and Pang’s recent [22].)

Geometry of two subspaces The rate of convergence of von Neumann’s method of alternating projections can be formulated in terms of the angle between the subspaces. (See Deutsch’s recent survey [9].)

The aim of this paper is to reveal the somewhat surprising relationships among these topics and to demonstrate their usefulness in convex optimization.

Before we describe our main result, let us fix some notation.
Notation

Throughout the paper,

\[ X \text{ is some Euclidean space with inner product } \langle \cdot, \cdot \rangle \text{ and induced norm } \| \cdot \|. \]

The language employed follows Rockafellar’s classical [25] and/or the more recent books [14, 15] by Hiriart-Urruty and Lemaréchal. These books contain all standard facts of convex analysis.

Notation is fairly standard: We write \( B_X \) for the unit ball \( \{ x \in X : \| x \| \leq 1 \} \), \( S_X \) for the unit sphere \( \{ x \in X : \| x \| = 1 \} \), and \( B(z;r) \) for \( z + rB_X \), where \( z \in X \) and \( r \geq 0 \).

Given convex functions \( f \) and \( g \) on \( X \), the domain (conjugate function of \( f \), gradient of \( f \), subdifferential of \( f \), positive part max\{\( f,0 \)\} of \( f \), infimal convolution of \( f \) and \( g \), resp.) is denoted by \( \text{dom } f \ (f^*, \nabla f, \partial f, f^+, f \square g \), resp.). The infimal convolution \( f \square g \) is exact at a point \( x \), if the infimum \( (f \square g)(x) = \inf_{u+v=x} f(u) + g(v) \) is attained.

By a cone we mean a nonempty set in \( X \) closed under nonnegative scalar multiplication; cones thus always contain the origin.

Suppose \( S \) is a closed convex set in \( X \). The interior (resp. relative interior, boundary, closure, convex hull, convex conical hull, affine span, linear span, (negative) polar, orthogonal complement, indicator function) of \( S \) is denoted \( \text{int } S \), (resp. \( \text{ri } S \), bd \( S \), cl \( S \), conv \( S \), cone \( S \), aff \( S \), span \( S \), \( S^\ominus \), \( S^\perp \), \( i_S \)). The set \( S \) induces a distance function \( d(\cdot,S) = \| \cdot \|_{i_S} \) and the corresponding projection or nearest point mapping \( P_S \). If \( x \in S \), then the tangent cone (resp. normal cone) of \( S \) at \( x \) is denoted \( T_S(x) \) (resp. \( N_S(x) \)). Thus \( T_S(x) = \text{cl } (\text{cone } (S - x)) \) and \( N_S(x) = (S - x)^\ominus \). Finally, we use the acronyms “TFAE” (the following are equivalent) and “WLOG” (without loss of generality).

Discussion of the results

For the remainder of this introduction, let us suppose that

\[ C_1, \ldots, C_m \] are finitely many closed convex sets in \( X \) with \( C := \bigcap_i C_i \neq \emptyset \).

We define the aforementioned concepts and recall the standard constraint qualification, ubiquitous in convex optimization.

(Strong) CHIP The collection \( \{C_1, \ldots, C_m\} \) is strong CHIP (resp. CHIP), if \( N_C(x) = \sum_i N_{C_i}(x) \) (resp. \( T_C(x) = \bigcap_i T_{C_i}(x) \)), for every \( x \in C \).

Bounded linear regularity The collection \( \{C_1, \ldots, C_m\} \) is boundedly linearly regular, if for every bounded subset \( S \) of \( X \), there exists \( \kappa_S > 0 \) such that \( d(x,C) \leq \kappa \max_i d(x,C_i) \), for every \( x \in S \).

Property (G) Provided that each \( C_i \) is a cone, the collection \( \{C_1, \ldots, C_m\} \) has property (G), if there exists \( \alpha > 0 \) such that \( C \cap B_X \subseteq \alpha \sum_i (C_i \cap B_X) \).
**Standard constraint qualification** The collection \( \{C_1, \ldots, C_m\} \) satisfies the standard constraint qualification, if there exists \( r \in \{0, \ldots, m\} \) such that \( C_{r+1}, \ldots, C_m \) are polyhedral and \( \cap_{i=r} C_i \neq \emptyset \).

Our main results can be summarized as follows.

**R1** The standard constraint qualification implies bounded linear regularity.

**R2** Bounded linear regularity implies strong CHIP.

**R3** Property \( (G) \) holds for convex polyhedral cones.

The proofs are mostly rooted in Convex Analysis. Result R1 gives a very simple and natural criterion for bounded linear regularity. Consequently, several results on projection algorithms have now a broader range of applicability.

Similarly, in the area of constrained interpolation, the Result R2 recovers all previously known instances of strong CHIP.

In the context of polyhedral cones (or homogeneous linear inequalities), Result R3 is dual to Hoffman’s error bound result.

Perhaps the most interesting open question asks whether or not the converse is true in Result R2.

Our two main applications are:

**A1** a sharpening of Auslender and Crouzeix’s classical result on the existence of error bounds.

**A2** an explicit formula relating the best possible constant in the definition of property \( (G) \) to the angle (for two subspaces).

**Organization of the paper**

In Section 2, we define a norm on the sum of a collection of finitely many closed convex cones; this “cone norm” is equivalent to the given norm precisely when the collection has property \((G)\). Due to the well-known equivalence of norms on finite-dimensional vector spaces, this happens whenever the sum is linear. Convex calculus implies that the collection is boundedly linearly regular if and only if it is strong CHIP and the collection of polar cones has property \((G)\). By Hoffman’s famous error bound result, this holds for polyhedral cones. Result R3 follows. For later use, we derive technical extensions for collections of closed convex sets.

We show the following in Section 3: for a collection of finitely many closed convex intersecting sets, bounded linear regularity implies strong CHIP (Resultat R2) and hence CHIP. The converse implications remain open. The properties bounded linear regularity, strong CHIP, and CHIP are shown to be the same provided the intersection is singleton or when certain other conditions are satisfied. Examples of collections that are boundedly linearly regular (and hence strong CHIP) are presented.
The fourth section contains an entirely self-contained proof that the standard constraint qualification implies bounded linear regularity (Resultat R1).

The foregoing results are applied to systems of convex inequalities $f_i(x) \leq 0, \ldots, f_m(x) \leq 0$ in Section 5; each set $C_i$ is given by the sublevel set $\{x \in X : f_i(x) \leq 0\}$. Linear regularity (the “global version” of bounded linear regularity) of $\{C_1, \ldots, C_m\}$ follows when the weak Slater and the asymptotic constraint qualification both hold; this improves upon a result by Auslender and Crouzeix.

In the final Section 6, we focus on two closed convex cones. Jameson’s result that the two cones are boundedly linearly regular exactly when their polars have property (G) is re-derived and refined. Finally, for two subspaces, property (G) is quantified by the angle.

## 2 Property (G) and Hoffman’s error bound

### Renorming the sum of finitely many closed convex cones

**Definition 2.1** Suppose $K_1, \ldots, K_m$ are finitely many closed convex cones in $X$. Let $K := K_1 + \cdots + K_m$, $X := X^m$, $K := K_1 \times \cdots \times K_m$, and the sum operator $S$ be given by

$$X \to X : x = (x_1, \ldots, x_m) \mapsto \sum_i x_i.$$  

Suppose that ||| · ||| is a norm on $X$. Then we define

$$|||x|||_K := \min \{|||x||| : x \in K, Sx = x\}, \quad \forall x \in K.$$

**Remark 2.2** The notation $||| · |||_K$ is very concise but not entirely unambiguous, because the sum $K$ does not uniquely determine its terms. However, we hope that it will always be clear from the context which terms are meant. Note that the minimum really is a minimum, i.e., the infimum is attained (since $K \cap S^{-1}(x)$ is closed, $\forall x \in K$).

The following elementary proposition shows that $||| · |||_K$ acts like a norm on $K$; hence we refer to $||| · |||_K$ loosely as a “cone norm” on $K$.

**Proposition 2.3** Suppose $K_1, \ldots, K_m$ are finitely many closed convex cones in $X$. Let $K := \sum_i K_i$ and $x \in K$. Then:

(i) $||x|||_K \geq 0$; moreover, $||x|||_K = 0 \iff x = 0$.

(ii) For every $\lambda \in \mathbb{R}$ such that $\lambda x \in K$: $|||\lambda x|||_K = |\lambda| |||x|||_K$.

(iii) For every $y \in K$: $|||x + y|||_K \leq |||x|||_K + |||y|||_K$.

**Definition 2.4** Suppose $K_1, \ldots, K_m$ are finitely many closed convex cones in $X$. Let $K := \sum_i K_i$. Suppose further the product space $X^m$ is equipped with the norm $||| · |||$, defined as follows: for some fixed $p \in [1, +\infty]$, $|||x||| = \|\|x\|\|_p$, $\forall x = (x_i), \in X^m$. Then we write

$$\| · \|_{K,p}$$

for $||| · |||_K$, and simply $\| · \|_K$ when $p = 2$. 5
For the reader’s convenience, we write out the cases $p = 1, 2, +\infty$ explicitly. For every $x \in K$:

\[
\|x\|_K = \min \left\{ \sqrt{\sum_i \|x_i\|^2} : \text{each } x_i \in K_i, \sum_i x_i = x \right\},
\]

\[
\|x\|_{K,1} = \min \left\{ \sum_i \|x_i\| : \text{each } x_i \in K_i, \sum_i x_i = x \right\},
\]

\[
\|x\|_{K,\infty} = \min \left\{ \max_i \|x_i\| : \text{each } x_i \in K_i, \sum_i x_i = x \right\}.
\]

We omit the proof of the following elementary result.

**Proposition 2.5** Suppose $K_1, \ldots, K_m$ are finitely many closed convex cones in $X$. Let $K := \sum_i K_i$. Then the “cone norms” of Definition 2.4 are all equivalent in the following sense:

\[
\|x\|_{K,\infty} \leq \|x\|_{K,p} \leq \|x\|_{K,1} = m \|x\|_{K,\infty}, \quad \forall x \in K, \forall p \in [1, +\infty].
\]

Also, $\|x\| \leq \|x\|_{K,1} \leq \sqrt{m} \|x\|_K$, $\forall x \in K$.

**Jameson’s property (G)**

**Definition 2.6** (Jameson [18]) Suppose $K_1, \ldots, K_m$ are finitely many closed convex cones in $X$. Let $K := \sum_i K_i$. We say that the collection $\{K_1, \ldots, K_m\}$ has property (G), if there exists $\alpha > 0$ such that

\[
K \cap B_X \subseteq \alpha [(K_1 \cap B_X) + \cdots + (K_m \cap B_X)].
\]

**Remark 2.7** Jameson introduced property (G) for two closed convex cones in a locally convex vector space in the early 1970s. The results he obtained in [18] hold in a quite abstract setting, their proofs require more sophisticated tools from Functional Analysis. We will reprove some of his results in our setting; our proofs are much simpler.

**Proposition 2.8** (see also [18, Proposition 5]) Suppose $K_1, \ldots, K_m$ are finitely many closed convex cones in $X$. If $\{K_1, \ldots, K_m\}$ has property (G), then $\sum_i K_i$ is closed.

**Proof.** Let $K := \sum_i K_i$ and $\alpha > 0$ such that $K \cap B_X \subseteq \alpha \sum_i (K_i \cap B_X)$. Fix an arbitrary $\bar{x} \in \text{cl} (K)$. After scaling if necessary, we assume WLOG that $\|\bar{x}\| < 1$. Then $\bar{x} \in \text{cl} (K \cap B_X) \subseteq \text{ocl} \left( \sum_i (K_i \cap B_X) \right)$. But the terms in the last sum are all compact, hence so is the set $\sum_i (K_i \cap B_X)$. It follows that $\bar{x} \in \alpha \sum_i (K_i \cap B_X) \subseteq K$. \hfill \blacksquare

**Proposition 2.9** Suppose $K_1, \ldots, K_m$ are finitely many closed convex cones in $X$. Let $K := \sum_i K_i$. Suppose further $\alpha > 0$. Then

\[
K \cap B_X \subseteq \alpha [(K_1 \cap B_X) + \cdots + (K_m \cap B_X)]
\]

if and only if $\|x\|_{K,\infty} \leq \alpha \|x\|$, $\forall x \in K$. 6
Proof. “⇒”: In view of Proposition 2.3.(ii), we can assume WLOG that \( \|x\| = 1 \). By assumption, there exist \( y_i \in K_i \cap B_X, \forall i \), such that \( x = \alpha \sum_i y_i \). Let \( x_i := \alpha y_i \in K_i, \forall i \). Then \( x = \sum_i x_i \) and hence \( \|x\|_{K,\infty} \leq \max_i \|x_i\| = \alpha \max_i \|y_i\| \leq \alpha \), as desired.

“⇐”: Suppose \( x \in K \cap B_X \). Recall that the minimum in the definition of \( \|x\|_{K,\infty} \) is attained; thus, there exist \( x_i \in K_i, \forall i \), such that \( x = \sum_i x_i \) and \( \|x\|_{K,\infty} = \max_i \|x_i\| \leq \alpha \|x\| \leq \alpha \). Let \( y_i := x_i/\alpha \in K_i \cap B_X, \forall i \). Then \( x = \alpha \sum_i y_i \) and the proof is complete. ■

Corollary 2.10 Suppose \( K_1, \ldots, K_m \) are finitely many closed convex cones in \( X \). Let \( K := \sum_i K_i \). Then the collection \( \{K_1, \ldots, K_m\} \) has property (G) if and only if there exists \( \alpha > 0 \) and \( p \in [1, +\infty) \) such that \( \|x\|_{K,p} \leq \alpha \|x\|, \forall x \in K \).

Proof. Combine Proposition 2.9 with Proposition 2.5. ■

Proposition 2.11 (see also [18, Proposition 43]) Suppose \( K_1, \ldots, K_m \) are finitely many closed convex cones in \( X \) such that \( \sum_i K_i \) is linear. Then \( \{K_1, \ldots, K_m\} \) has property (G).

Proof. Let \( K := \sum_i K_i \). Not only is \( K \) is linear but also \( \|\cdot\| \) and \( \|\cdot\|_K \) are norms on \( K \) (Proposition 2.3). Hence these two norms are equivalent ([27, Corollary 3]). By Corollary 2.10, \( \{K_1, \ldots, K_m\} \) has property (G). ■

In stark contrast to the linear case, where two arbitrary norms on \( X \) are always equivalent ([27, Corollary 3]), the next example shows that two given “cone norms” may fail to be equivalent.

Example 2.12 Let \( X := \mathbb{R}^3 \) and define

\[
x(t) := t \cdot (\cos(2\pi t), \sin(2\pi t), 1)
\]

and \( C := \text{conv} \{x(t) : t \in [0, 1]\} \).

Let \( \gamma_C \) be the gauge of \( C \) and let \( K \) be the icecream cone:

\[
K := \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 \leq x_3^2, x_3 \geq 0\}.
\]

Then \( q := \gamma_C|_K \) is a “cone norm” on \( K \), but \( \lim_{t \to 0^+} q(x(t))/\|x(t)\| = +\infty \), so \( q \) and \( \|\cdot\| \) are not equivalent “cone norms” on \( K \).

Proof. \( C \) is compact (see [25, Theorem 17.2]) and cone \( C = K \). Sublinearity of gauges together with compactness of \( C \) implies that \( q \) is a “cone norm” on \( K \). Now \( q(x(t)) = 1, \forall t \in [0, 1] \) and \( \lim_{t \to 0^+} \|x(t)\| = 0 \), hence \( \lim_{t \to 0^+} q(x(t))/\|x(t)\| = +\infty \) and the proof is complete. ■

Proposition 2.13 (see also Jameson’s [18, Theorem 2.1]) Suppose \( C_1, \ldots, C_m \) are finitely many closed convex cones in \( X \) and \( \alpha > 0 \). Let \( C := \bigcap_i C_i \) and \( K := \sum_i C_i^{\ominus} \). Then TFAE:
(i) \( \alpha \frac{1}{2} d^2(x, C) \leq \sum_i \frac{1}{2} d^2(x, C_i), \forall x \in X. \)

(ii) \( K = C^\ominus \) and \( \min \left\{ \frac{1}{2} \| x_i^* \|^2 : \text{each } x_i^* \in C_i^\ominus, \sum_i x_i^* = x^* \right\} \leq \frac{1}{\alpha} \| x^* \|^2, \forall x^* \in K. \)

(iii) \( K = C^\ominus \) and \( \| x^* \|_K \leq \frac{1}{\sqrt{\alpha}} \| x^* \|, \forall x^* \in K. \)

**Proof.** Recall that \( \text{cl}(K) \subseteq C^\ominus. \) Set \( g := \alpha \frac{1}{2} d^2(\cdot, C) = \alpha \left( \frac{1}{2} \| \cdot \|^2 + \iota_{C^\ominus} \right). \) Then \( g^* := \frac{1}{\alpha} \| \cdot \|^2 + \iota_{C^\ominus}. \) Further set \( f_i := \frac{1}{2} d^2(\cdot, C_i), \forall i, \) then \( f_i^* := \frac{1}{2} \| \cdot \|^2 + \iota_{C_i^\ominus}. \) The functions \( g, f_1, \ldots, f_m \) are closed convex proper; hence so is \( f := \sum_i f_i. \) Because each \( f_i \) is everywhere continuous, we learn that \( f^* = f_1^* \oplus \cdots \ominus f_m^*; \) moreover, this infimal convolution is exact on \( \text{dom} f^* = K \) (by [25, Theorem 16.4]). Using this, we obtain: (i) \( \iff g \leq f\iff f^* \leq g^*\iff \inf \{ \sum_i \frac{1}{2} \| x_i^* \|^2 : \text{each } x_i^* \in C_i^\ominus, \sum_i x_i^* = x^* \} \leq \frac{1}{\alpha} \| x^* \|^2, \forall x^* \in X \iff K = C^\ominus \) and \( \min \{ \sum_i \frac{1}{2} \| x_i^* \|^2 : \text{each } x_i^* \in C_i^\ominus, \sum_i x_i^* = x^* \} \leq \frac{1}{\alpha} \| x^* \|^2, \forall x^* \in K \iff (ii) \iff (iii). \)

**Remark 2.14** Using terms to be defined in Definitions 3.1 and 3.4 below, Proposition 2.13 says that "linear regularity of the original cones is equivalent to strong CHIP of the original cones and property (G) of the dual cones"; this will be refined in Theorem 6.7. Jameson’s proof is very different from the present convex analytical approach.

**Consequences of Hoffman’s error bound**

The classical Hoffman error bound result can be reformulated as follows.

**Fact 2.15** (Hoffman [16]; 1952) Suppose \( C_1, \ldots, C_m \) are finitely many convex polyhedral subsets of \( X \) with \( C := \bigcap_i C_i \neq \emptyset. \) Then there exists \( \alpha > 0 \) such that \( \alpha \frac{1}{2} d^2(x, C) \leq \sum_i \frac{1}{2} d^2(x, C_i), \forall x \in X. \)

**Proof.** See [4, Corollary 5.26].

Our reformulation of Hoffman’s result is somewhat unusual; however, it allows a direct application of Proposition 2.13. For instance, we deduce the following result, dual to Hoffman’s result.

**Corollary 2.16 (Dual of Hoffman’s error bound)** Every finite collection of convex polyhedral cones has property (G).

The next result is a useful extension of Corollary 2.16.

**Theorem 2.17** Suppose \( C_1, \ldots, C_m \) are finitely many convex polyhedral subsets of \( X \) with \( C := \bigcap_i C_i \neq \emptyset. \) Then there exists \( \beta > 0 \) such that

\[
\min \left\{ \sum_i \| y_i \| : \text{each } y_i \in N_{C_i}(c), \sum_i y_i = y \right\} \leq \beta \| y \|, \quad \forall c \in C, \forall y \in N_C(c).
\]
Proof. For brevity, let us write $T_i$ for $T_{C_i}$, $\forall i$. Note that (by [25, Corollary 23.8.1]) $N_C(c) = \sum_i N_{C_i}(c)$ and thus (by taking polars and [25, Corollary 16.4.2]) $T_C(c) = \bigcap_i T_i(c)$, $\forall c \in C$. Now consider the collection of $m$-tuples of tangent cones $T := \{(T_1(c), \ldots, T_m(c)) : c \in C\}$. Because each $C_i$ is polyhedral, [14, Examples III.5.2.6.(b)] implies that the collection $T$ is finite: there exist finitely many $c_1, \ldots, c_n \in C$ such that

$$T = \bigcup_{j \in \{1, \ldots, n\}} \{(T_1(c_j), \ldots, T_m(c_j))\}.$$

Now every tangent cone $T_i(c_j)$ is polyhedral and $T_C(c_j) = \bigcap_i T_i(c_j)$, $\forall j$, so Hoffman’s result (Fact 2.15) yields the existence of positive reals $\alpha_1, \ldots, \alpha_n$ such that

$$\alpha_j \frac{1}{2} d^2(x, T_C(c_j)) \leq \sum_i \frac{1}{2} d^2(x, T_i(c_j)), \quad \forall x \in X, \forall j \in \{1, \ldots, n\}.$$

Hence after setting $\alpha := \min\{\alpha_1, \ldots, \alpha_n\} > 0$, we obtain

$$\alpha \frac{1}{2} d^2(x, T_C(c)) \leq \sum_i \frac{1}{2} d^2(x, T_i(c)), \quad \forall x \in X, \forall c \in C.$$

Set $K(c) := \sum_i T_i(c) = \sum_i N_{C_i}(c) = N_C(c) = T_C(c)$, $\forall c \in C$. Then Proposition 2.13 results in $\|y\|_{K(c)} \leq \frac{1}{\sqrt{\alpha}} \|y\|$, $\forall c \in C$, $\forall y \in K(c)$. Use Proposition 2.5 to further get

$$\sqrt{m} \|y\|_{K(c), 1} \leq \|y\|_{K(c)} \leq \frac{1}{\sqrt{\alpha}} \|y\|, \quad \forall c \in C, \forall y \in K(c).$$

Therefore, the desired inequality holds with $\beta := \sqrt{m/\alpha}$.

We conclude this section with an extension of Theorem 2.17 to the case when nonpolyhedral sets are involved; this will turn out to be very useful in the later development.

**Theorem 2.18** Suppose $C_1, \ldots, C_m$ are finitely many closed convex sets in $X$, where, for some $r \in \{0, \ldots, m\}$, the sets $C_{r+1}, \ldots, C_m$ are polyhedral. Suppose further $z \in \bigcap_{i=1}^r \text{ri}(C_i) \cap \bigcap_{i=r+1}^m C_i \neq \emptyset$ and let $C := \bigcap_{i=1}^m C_i$. Then $N_C(x) = \sum_i N_{C_i}(x)$, $\forall x \in C$. Let $A_i := \text{aff}(C_i)$, $\forall i \in \{1, \ldots, r\}$ and $\delta > 0$ such that $A_i \cap B(z; \delta) \subseteq C_i$, $\forall i \in \{1, \ldots, r\}$. Then there exists $\beta > 0$, independent of $z$ and $\delta$, such that

$$\min\left\{\sum_{i=1}^m \|y_i\| : \text{each } y_i \in N_{C_i}(x), \sum_{i=1}^m y_i = y\right\} \leq \frac{\|x-z\|_{(1+\beta)\delta}^2}{\delta^2}\|y\|, \quad \forall x \in C, \forall y \in N_C(x).$$

**Proof.** The statement concerning the normal cones follows from [25, Corollary 23.8.1]. Let $L_i := \text{span}(C_i - z)$ so that $A_i = z + \text{span}(C_i - z) = z + L_i = C_i + L_i$, $\forall i \in \{1, \ldots, r\}$. Define further

$$D_0 := \bigcap_{i=1}^r A_i \cap \bigcap_{i=r+1}^m C_i \quad \text{and} \quad D_i := C_i + L_i, \quad \forall i \in \{1, \ldots, r\}.$$

After verification of $(C_i + L_i) \cap (C_i + L_i) = C_i$, $\forall i \in \{1, \ldots, r\}$, it is easy to see that

$$C = \bigcap_{i=0}^r D_i.$$
Let us fix momentarily $i \in \{1, \ldots, r\}$ and $b \in B_X$. Write $b = l_i + l_i^+$, where $l_i \in L_i$ and $l_i^+ \in L_i^+$. Then $l_i, l_i^+ \in B_X$. Hence $z + \delta b = (z + \delta l_i) + \delta l_i^+ \in (B(z; \delta) \cap A_i) + L_i^+ \subseteq C_i + L_i^+$. Thus $B(z; \delta) \subseteq D_i$, $\forall i \in \{1, \ldots, r\}$ and so $z \in D_0 \cap \bigcap_{i=1}^r \text{int}(D_i)$, which in turn implies (by [25, Corollary 23.8.1] again)

$$N_C(x) = \sum_{i=0}^r N_{D_i}(x), \quad \forall x \in C.$$ 

For the remainder of the proof, fix $x \in C$ and $y \in N_C(x) = \sum_{i=1}^m N_{C_i}(x) = \sum_{i=0}^r N_{D_i}(x)$. Obtain $\bar{y}_i \in N_{D_i}(x)$, $\forall i \in \{0, \ldots, r\}$ such that $y = \sum_{i=0}^r \bar{y}_i$. Then $\langle x, \bar{y}_i \rangle = \sup \langle D_i, \bar{y}_i \rangle \geq \sup \langle B(z; \delta), \bar{y}_i \rangle = \langle z, \bar{y}_i \rangle + \delta \|\bar{y}_i\|, \forall i \in \{1, \ldots, r\}$ and $\langle x, \bar{y}_0 \rangle = \sup \langle D_0, \bar{y}_0 \rangle \geq \langle z, \bar{y}_0 \rangle$. Hence

$$\langle x, y \rangle = \sum_{i=0}^r \langle x, \bar{y}_i \rangle \geq \sum_{i=0}^r \langle z, \bar{y}_i \rangle + \delta \sum_{i=1}^r \|\bar{y}_i\| = \langle z, y \rangle + \delta \sum_{i=1}^r \|\bar{y}_i\|,$$

which implies

(*)

$$\sum_{i=1}^r \|\bar{y}_i\| \leq \frac{1}{\delta} \langle x - z, y \rangle \leq \frac{\|x - z\|}{\delta} \|y\|$$

and further

(**)

$$\|\bar{y}_0\| = \left\| y - \sum_{i=1}^r \bar{y}_i \right\| \leq \|y\| + \sum_{i=1}^r \|\bar{y}_i\| \leq \frac{\|x - z\| + \delta}{\delta} \|y\|.$$ 

Now $A_1, \ldots, A_r, C_{r+1}, \ldots, C_m$ are polyhedral and their intersection equals $D_0$. By Theorem 2.17, there exist $\beta > 0$ (depending only on $A_1, \ldots, A_r, C_{r+1}, \ldots, C_m$) and

$$\bar{y}_i \in N_{A_i}(x) = L_i^+, \quad \forall i \in \{1, \ldots, r\}, \quad y_i \in N_{C_i}(x), \quad \forall i \in \{r+1, \ldots, m\}$$

such that

(***)

$$\sum_{i=1}^r \bar{y}_i + \sum_{i=r+1}^m y_i = \bar{y}_0 \quad \text{and} \quad \sum_{i=1}^r \|\bar{y}_i\| + \sum_{i=r+1}^m \|y_i\| \leq \beta \|\bar{y}_0\|.$$ 

Define $y_i := \bar{y}_i + y_i$, $\forall i \in \{1, \ldots, r\}$. On the one hand, $C_i \subseteq D_i$, and so $\bar{y}_i \in N_{D_i}(x) \subseteq N_{C_i}(x)$, $\forall i \in \{1, \ldots, r\}$. On the other hand, $C_i \subseteq A_i$, and so $y_i \in N_{A_i}(x) \subseteq N_{C_i}(x)$, $\forall i \in \{1, \ldots, r\}$. Altogether,

$$y_i \in N_{C_i}(x), \quad \forall i \in \{1, \ldots, m\}.$$ 

Also, using (***),

$$\sum_{i=1}^m y_i = \sum_{i=1}^r (\bar{y}_i + \bar{y}_i) + \sum_{i=r+1}^m y_i = \bar{y}_0 + \sum_{i=1}^r \bar{y}_i = y.$$ 

Therefore, invoking (*),(**), and (***)

$$\sum_{i=1}^m \|y_i\| \leq \sum_{i=1}^r (\|\bar{y}_i\| + \|\bar{y}_i\|) + \sum_{i=r+1}^m \|y_i\| \leq \frac{\|x - z\|}{\delta} \|y\| + \beta \|\bar{y}_0\|$$

$$\leq \left(\frac{\|x - z\|}{\delta} + \beta \frac{\|x - z\| + \delta}{\delta}\right) \|y\|. \blacksquare$$
3 Strong CHIP and bounded linear regularity

Strong CHIP ...

**Definition 3.1** ([11, Deutsch et al.’s Definition 2.3] and [7, Chui et al.’s Definition on page 38]) Suppose $C_1, \ldots, C_m$ are finitely many closed convex sets in $X$ with $C := \bigcap_{i=1}^m C_i \neq \emptyset$ and $x \in C$. Then we say the collection $\{C_1, \ldots, C_m\}$ is strong CHIP at $x$ (resp. is CHIP at $x$), if $N_C(x) = \sum_i N_{C_i}(x)$ (resp. if $T_C(x) = \bigcap_i T_{C_i}(x)$). We say that the collection $\{C_1, \ldots, C_m\}$ is strong CHIP (resp. is CHIP), if $\{C_1, \ldots, C_m\}$ is strong CHIP (resp. CHIP) at every point in $C$.

By taking polars, it follows immediately that strong CHIP at $x$ implies CHIP at $x$.

The next proposition compares strong CHIP to CHIP.

**Proposition 3.2** (see also [11, Deutsch et al.’s Lemma 2.4]) Suppose $C_1, \ldots, C_m$ are finitely many closed convex sets in $X$ with $C := \bigcap_{i=1}^m C_i \neq \emptyset$. Then TFAE:

(i) $\{C_1, \ldots, C_m\}$ is strong CHIP.

(ii) $\partial(\sum_i \iota_{C_i}) = \sum_i \partial \iota_{C_i}$.

(iii) $\{C_1, \ldots, C_m\}$ is CHIP and $\sum_i N_{C_i}(x)$ is closed, $\forall x \in C$.

**Proof.** “(i) $\Leftrightarrow$ (ii)”: obvious. “(i) $\Leftrightarrow$ (iii)”: use [25, Corollary 16.4.2].

**Remark 3.3** We can loosely say that a collection of closed convex sets with nonempty intersection is strong CHIP (resp. CHIP), if the normal cone (resp. tangent cone) to the intersection is always the sum (resp. the intersection) of the individual normal cones (resp. tangent cones).

The properties strong CHIP and CHIP are pivotal in the study of constrained approximation problems; see [7], [8], [10], [11], and [12]. Recall that the following inclusions always hold:

$$\text{cl} \left( \sum_i N_{C_i}(x) \right) \subseteq N_C(x) \quad \text{and} \quad T_C(x) \subseteq \bigcap_i T_{C_i}(x), \quad \forall x \in C.$$ 

In $\mathbb{R}^2$, CHIP and strong CHIP are the same: indeed, all cones in $\mathbb{R}^2$ are polyhedral and so the sum of polyhedral cones is (polyhedral and hence) closed.

... and bounded linear regularity

**Definition 3.4** ([4, Definition 5.6]) Suppose $C_1, \ldots, C_m$ are finitely many closed convex sets in $X$ with $C := \bigcap_i C_i \neq \emptyset$. Then $\{C_1, \ldots, C_m\}$ is boundedly linearly regular, if for every bounded subset $S$ of $X$, there exists $\kappa_S > 0$ such that

$$d(x, C) \leq \kappa_S \max_i d(x, C_i), \quad \forall x \in S.$$ 

If there exists $\kappa > 0$ such that $d(x, C) \leq \kappa \max_i d(x, C_i)$, $\forall x \in X$, then $\{C_1, \ldots, C_m\}$ is called linearly regular. It is clear that linear regularity implies bounded linear regularity.
**Remark 3.5** Bounded linear regularity of \( \{C_1, \ldots, C_m\} \) is a quantitative version of a very intuitive idea: “closeness to all sets \( C_i \) implies closeness to their intersection”. This concept is of fundamental importance in the study of projection methods; see [3], [4, Section 5], and [2, Chapters 4 and 5].

**Theorem 3.6 (bounded linear regularity implies strong CHIP)** Suppose \( C_1, \ldots, C_m \) are finitely many closed convex sets in \( X \) with \( C := \bigcap_i C_i \neq \emptyset \). Consider the following three conditions.

(i) \( \{C_1, \ldots, C_m\} \) is boundedly linearly regular.

(ii) \( \{C_1, \ldots, C_m\} \) is strong CHIP.

(iii) \( \{C_1, \ldots, C_m\} \) is CHIP.

Then: (i) \( \Rightarrow \) (ii) \( \Rightarrow \) (iii).

**Proof.** “(i)\( \Rightarrow \)(ii)”: Fix \( x \in C \) and \( x^* \in \partial C(x) = N_C(x) \). In view of Proposition 3.2 and Remark 3.3, it suffices to show that \( x^* \in \sum_i \partial C_i(x) = \sum_i N_{C_i}(x) \). We may assume WLOG \( x^* \neq 0 \) and then let \( \hat{x}^* := x^*/\|x^*\| \). Then \( \hat{x}^* \in N_C(x) \cap B_X = \partial d(\cdot, C)(x) \); see [14, Example VI.3.3]. Thus

\[
\langle \hat{x}^*, y - x \rangle \leq d(y, C) - d(x, C) = d(y, C), \quad \forall y \in X.
\]

On the other hand, \( \{C_1, \ldots, C_m\} \) is boundedly linearly regular; we thus obtain \( \kappa > 0 \) such that

\[
d(y, C) \leq \kappa \sum_i d(y, C_i), \quad \forall y \in B(x; 1).
\]

Altogether \( \langle \hat{x}^*, y - x \rangle \leq \kappa \sum_i d(y, C_i), \forall y \in B(x; 1) \). It follows that

\[
\hat{x}^*/\kappa \in \partial(\sum_i d(\cdot, C_i) + i_{B(x; 1)})(x) = \sum_i (N_{C_i}(x) \cap B_X);
\]

consequently, \( x^* \in \sum_i N_{C_i}(x) \). “(ii)\( \Rightarrow \)(iii)”: Proposition 3.2. 

**Converse implications**

In this subsection, we discuss conditions guaranteeing that CHIP implies bounded linear regularity.

**Proposition 3.7** Suppose \( C_1, \ldots, C_m \) are finitely many closed convex sets in \( X \) with \( C := \bigcap_i C_i \neq \emptyset \). Suppose further that \( \{C_1, \ldots, C_m\} \) is not boundedly linearly regular. Then there exist: (i) a sequence \( (c_n) \) in \( C \) converging to some \( \bar{c} \in C \), (ii) a point \( q \in S_X \cap N_C(\bar{c}) \), (iii) sequences \( (y_{i,n}) \) such that \( y_{i,n} \in T_{C_i}(c_n), \forall n \) and \( y_{i,n} \to q, \forall i \). Moreover, if each \( C_i \) is a cone, then we can additionally arrange that (iv) \( \bar{c} \in S_X \).
**Proof.** Because \( \{C_1, \ldots, C_m\} \) is not boundedly linearly regular, there exists a bounded sequence \((x_n)\) in \( C \) such that

\[
(*) \quad d(x_n, C) > n \max_i d(x_n, C_i), \quad \forall n.
\]

(Digression. Suppose momentarily that each \( C_i \) is a cone. Then \((*)\) shows that \( x_n \neq 0, \forall n \). Because projections onto cones are positively homogeneous, we can replace each \( x_n \) by \( x_n/\|x_n\| \) to get a new sequence lying entirely in \( S_X \) and still satisfying \((*)\). This will prove the “Moreover” part. End of Digression.) Now \((x_n)\) is bounded, hence so is \((d(x_n, C))\). By \((*)\), \( \max_i d(x_n, C_i) \to 0 \). This implies that \( d(x_n, C) \to 0 \) (by either a straight-forward proof by contradiction or [4, Proposition 5.4.(iii)]). Hence all cluster points of \((x_n)\) lie in \( C \). For brevity, set \( c_n := P_C(x_n) \) and \( c_{i,n} := P_{C_i}(x_n), \forall n, \forall i \). Then \( 0 < d(x_n, C) = \|x_n - c_n\| \) and \( d(x_n, C_i) = \|x_n - c_{i,n}\|, \forall n, \forall i \). After passing to a subsequence if necessary, we assume WLOG that \((x_n)\) converges to some \( \bar{c} \in C \) and that \((d(x_n - c_n)/\|x_n - c_n\|)\) converges to some \( q \in S_X \). Hence \((c_n)\) and each \((c_{i,n})\) converge to \( \bar{c} \), too. Set \( t_n := \|x_n - c_n\| > 0, \forall n \). Then \( t_n \to 0 \). We have, using \( d(x_n, C_i)/d(x_n, C) \leq 1/n \to 0 \) from \((*)\),

\[
y_{i,n} := \frac{c_{i,n} - c_n}{\|x_n - c_n\|} = \frac{c_{i,n} - x_n}{\|x_n - c_n\|} + \frac{x_n - c_n}{\|x_n - c_n\|} \to q, \quad \forall i.
\]

Since \( c_n + t_n y_{i,n} = c_{i,n} \in C_i \), we obtain \( y_{i,n} \in T_{C_i}(c_n), \forall n, \forall i \). Also, \( (c - c_n, x_n - c_n)/\|x_n - c_n\| \leq 0, \forall n, \forall c \in C \) (using \( c_n = P_C(x_n) \)) and a well-known property of projections; see, for instance, Proposition 4.1.(i) below. Taking limits yields \( q \in N_C(\bar{c}) \). Therefore, we have constructed objects that possess all announced properties. \( \blacksquare \)

**Definition 3.8** Suppose \( C_1, \ldots, C_m \) are finitely many closed convex sets in \( X \) with \( C := \bigcap_i C_i \neq \emptyset \) and \( \bar{x} \in C \). We say that the collection \( \{C_1, \ldots, C_m\} \) is intersection-closed at \( x \), if whenever \((c_n)\) is a sequence in \( C \) converging to \( \bar{x} \) and \((y_{i,n})\) are sequences converging to \( y_i \), with \( y_{i,n} \in T_{C_i}(c_n), \forall n, \forall i \), then necessarily \( y_i \in T_{C_i}(\bar{x}), \forall i \). If \( D \) is a closed nonempty subset of \( C \) and \( \{C_1, \ldots, C_m\} \) is intersection-closed at every point in \( D \), then we say that \( \{C_1, \ldots, C_m\} \) is intersection-closed on \( D \).

**Definition 3.9** (see also [17, Section 7.D] or [26, Definition 7.5]) Suppose \( C \) is a closed convex set in \( X \) with \( \text{int} \ (C) \neq \emptyset \) and \( \bar{x} \in \text{bd} \ (C) \). We say that \( C \) is smooth at \( \bar{x} \), if \( T_C(\bar{x}) \) is a halfspace; equivalently, if \( N_C(\bar{x}) \) is a ray. If \( C \) is smooth at \( \bar{x} \), then \( N_C(\bar{x}) = [0, \infty[ d_C(\bar{x}) \), for some unique vector \( d_C(\bar{x}) \in S_X \), called the normal direction of \( C \) at \( \bar{x} \). Suppose \( D \) is a closed nonempty subset of \( \text{bd} \ (C) \). If \( C \) is smooth at every \( x \in D \), then we say that \( C \) is smooth on \( D \). If there exists some \( \epsilon > 0 \) such that \( C \) is smooth on \( C \cap B(\bar{x}; \epsilon) \), then we say that \( C \) is locally smooth at \( \bar{x} \). Finally, if \( C \) is locally smooth at every \( x \in D \), then we say that \( C \) is locally smooth on \( D \).

**Proposition 3.10** Suppose \( C \) is a closed convex set in \( X \) with \( \text{int} \ (C) \neq \emptyset \) and \( D \) is a closed subset of \( \text{bd} \ (C) \) on which \( C \) is smooth. Then the mapping \( d_C|_D : D \to S_X : x \mapsto d_C(x) \) is continuous.
**Proof.** Suppose \((x_n)\) is a sequence in \(D\) converging to some point \(\bar{x} \in D\). Let \(d_n := d_C(x_n) \in S_X, \forall n\). We have to show that \((d_n)\) converges to \(d_C(\bar{x})\). Let \(d'\) be an arbitrary cluster point of \((d_n)\). Then \(d' \in S_X\) and there exists a subsequence \((d_{n'}))\) of \((d_n)\) that converges to \(d'\). Now \(d_{n'} \in N_C(x_{n'}), \forall n, \forall c \in C\). Taking limits along \((n')\) yields \(\langle c - x_{n'}, d' \rangle \leq 0, \forall c \in C\). Thus \(d' \in N_C(\bar{x}) \cap S_X = \{d_C(\bar{x})\}\). Therefore, the entire sequence \((d_n)\) converges to \(d_C(\bar{x})\) and the proof is complete. \(\blacksquare\)

**Proposition 3.11** Suppose \(C_1, \ldots, C_m\) are finitely many closed convex sets in \(X\) with \(C := \bigcap_i C_i \neq \emptyset\) and \(c \in C\). If \(C_i\) is locally smooth at \(c\) or \(T_{C_i}|C\) does not change on a neighborhood of \(c, \forall i\), then \(\{C_1, \ldots, C_m\}\) is intersection-closed at \(c\).

**Proof.** Let \((c_n)\) be a sequence in \(C\) converging to \(c\) and \((y_{i,n})\) be a sequence converging to some \(y_i\) such that \(y_{i,n} \in T_{C_i}(c_n), \forall i, \forall n\). Fix \(i \in \{1, \ldots, m\}\). If \(T_{C_i}|C\) is constant on a neighborhood of \(c\), then \(y_i \in T_{C_i}(c)\) and we are done (tangent cones are closed). Otherwise, \(C_i\) is locally smooth at \(c\). Let \(d_{i,n} := d_{C_i}(c_n)\) be the normal direction of \(C_i\) at \(c_n, \forall n\). Then \(y_{i,n} \in T_{C_i}(c_n) = N_{C_i}(c_n) = [0, +\infty[|d_{i,n}| \Rightarrow \langle y_{i,n}, d_{i,n} \rangle \leq 0, \forall n\). Let \(d_i := d_{C_i}(c)\). Then, by Proposition 3.10, \(d_i = \lim_n d_{i,n}, \forall i\). Thus \(\langle y_{i,n}, d_i \rangle \leq 0, \text{i.e., } y_i \in N_{C_i}(c) = T_{C_i}(c)\). Altogether, \(\{C_1, \ldots, C_m\}\) is intersection-closed at \(c\). \(\blacksquare\)

**Proposition 3.12** Suppose \(C_1, \ldots, C_m\) are finitely many closed convex cones in \(X\) such that \(C := \bigcap_i C_i\) is a ray: \(C = [0, +\infty[|c\), for some \(c \in C \setminus \{0\}\). Then \(\{C_1, \ldots, C_m\}\) is intersection-closed at \(c\).

**Proof.** Suppose \((c_n)\) is a sequence in \(C\) converging to \(c\). WLOG \(c_n = p_n c, \forall n\), where \((p_n)\) is a sequence of positive reals. By Proposition 6.2, \(T_{C_i}(c_n) = T_{C_i}(p_n c) = \text{cl} (C_i + \mathbb{R} p_n c) = \text{cl} (C_i + \mathbb{R} c) = T_{C_i}(c), \forall n\). Hence \(\{C_1, \ldots, C_m\}\) is intersection-closed at \(c\). \(\blacksquare\)

**Theorem 3.13** Suppose \(C_1, \ldots, C_m\) are finitely many closed convex sets in \(X\) with \(C := \bigcap_i C_i \neq \emptyset\). Suppose further that \(\{C_1, \ldots, C_m\}\) is intersection-closed on \(C\). In particular, this holds when (i) \(C\) is singleton; or (ii) each \(C_i\) is smooth on \(C\). Then \(\{C_1, \ldots, C_m\}\) is boundedly linearly regular if and only if it is CHIP.

**Proof.** By Theorem 3.6, bounded linear regularity implies CHIP, so we have to show that CHIP implies bounded linear regularity. We do so by contradiction: assume \(\{C_1, \ldots, C_m\}\) is CHIP but not boundedly linearly regular. Then, by Proposition 3.7, there exist a sequence \((c_n)\) in \(C\) converging to some \(c \in C\), a point \(q \in S_X \cap N_C(c)\), sequences \((y_{i,n})\) such that \(y_{i,n} \in T_{C_i}(c_n), \forall n\) and \(y_{i,n} \rightarrow q, \forall i\).

Now \(\{C_1, \ldots, C_m\}\) is intersection-closed, whence \(q \in T_{C_i}(c), \forall i\). Since \(\{C_1, \ldots, C_m\}\) is also CHIP, we conclude \(q \in T_C(c)\). On the other hand, \(q \in N_C(c) \cap S_X\). Altogether, we contradict \(N_C(c) \cap T_C(c) = \{0\}\). The “In particular” case follows from Proposition 3.11. \(\blacksquare\)
Theorem 3.14 Suppose $C_1, \ldots, C_m$ are finitely many closed convex cones in $X$ and let $C := \bigcap_i C_i$. Suppose further that $\{C_1, \ldots, C_m\}$ is intersection-closed on $C \cap S_X$. In particular, this holds when (i) each $C_i$ is a subspace or locally smooth on $C \cap S_X$; or (ii) $C$ is a ray or linear. Then $\{C_1, \ldots, C_m\}$ is boundedly linearly regular if and only if it is CHIP.

Proof. The main statement follows similarly to the proof of Theorem 3.13 (use the “cone” version of Proposition 3.7). The “In particular” part follows from Proposition 3.11, Proposition 3.12, Corollary 6.8, and Remark 6.10 below. ■

Open Problem 3.15 The following questions remain open. Does CHIP imply strong CHIP? Does strong CHIP imply bounded linear regularity?

Examples

Several conditions sufficient for bounded linear regularity can be found in [3, Section 4], [4, Section 5], and [2, Chapter 4 and 5]. In view of Theorem 3.6, we thus immediately obtain criteria for strong CHIP.

It is most satisfying that the standard constraint qualification yields bounded linear regularity. (For a self-contained proof, see Corollary 4.7 below.)

Corollary 3.16 Suppose $C_1, \ldots, C_m$ are finitely many closed convex sets in $X$, where $C_{r+1}, \ldots, C_m$ are polyhedral, for some $r \in \{0, \ldots, m\}$. Suppose further $\bigcap_{i=1}^r \text{ri}(C_i) \cap \bigcap_{i=r+1}^m C_i \neq \emptyset$. Then $\{C_1, \ldots, C_m\}$ is boundedly linearly regular and hence strong CHIP.

Proof. The collection $\{C_1, \ldots, C_m\}$ is boundedly linearly regular ([2, Theorem 5.6.2]) and hence strong CHIP (Theorem 3.6). ■

It is worth noting two special cases.

Example 3.17 (see also [11, Theorems 3.6 and 3.12]) Suppose $A$ is a linear operator from $X$ to some Euclidean space $Y$, and $C_1$ is a closed convex subset in $X$. Let $C_2 := A^{-1}(b)$ and suppose $C_1 \cap C_2 \neq \emptyset$. Then $\{C_1, C_2\}$ is boundedly linearly regular and hence strong CHIP, whenever

(i) $C_1$ is polyhedral or

(ii) $b \in \text{ri}A(C_1)$.

Example 3.18 A collection of finitely many subspaces of $X$ is boundedly linearly regular and strong CHIP.

Remark 3.19 With care, some of the results of this subsection generalize to the case when $X$ is an infinite-dimensional Hilbert space: Example 3.17 remains true without change (use [4, Corollary 5.26] and [2, Proposition 4.7.1]). Example 3.18 is no longer true in infinite-dimensional Hilbert space: in fact, a collection $\{C_1, \ldots, C_m\}$ of finitely many closed subspaces is boundedly linearly regular if and only if it is strong CHIP if and only if $\sum_i C_i^\perp$ is closed; see [4, Theorem 5.19].
4 Bounded linear regularity via Convex analysis

Work in this sections culminates in a self-contained convex-analytical proof of Corollary 3.16.

**Proposition 4.1** Suppose $S$ is a closed convex nonempty set in $X$ and let $x, x^* \in X$. Then:

(i) $x - P_S(x) \in N_S(P_S(x))$.

(ii) $\frac{1}{2}d^2(x, S) + \frac{1}{2}\|x^*\|^2 + i_S^*(x^*) \geq \langle x^*, x \rangle$.

(iii) $x^* \in N_S(x)$ if and only if $x \in S$ and $\langle x^*, x \rangle = i_S^*(x^*)$.

**Proof.** (i) is a well-known property of the projection; see [14, Theorem III.3.1.1]. (ii): Note that $\frac{1}{2}d^2(x, S) = \frac{1}{2}\|x - P_S(x)\|^2$, hence $(\frac{1}{2}d^2(x, S))^*(x^*) = \frac{1}{2}\|x^*\|^2 + i_S^*(x^*)$. The statements follow from the Fenchel/Young inequality; see [25, Theorem 23.5]. (iii): Follows from the characterization of equality in the Fenchel/Young inequality applied to the function $i_S$.

**Theorem 4.2** Suppose $C_1, \ldots, C_m$ are finitely many closed convex sets in $X$ with $C := \bigcap_i C_i \neq \emptyset$. Suppose further $x \in X$ with

$$T_C(P_C(x)) = \bigcap_i T_{C_i}(P_C(x))$$

and there exists $\lambda > 0$ such that for every $y \in \sum_i N_{C_i}(P_C(x))$,

$$\min \left\{ \sum_i \|y_i\|^2 : \text{each } y_i \in N_{C_i}(P_C(x)), \sum_i y_i = y \right\} \leq \lambda \sum_i \|d(x, C_i)\|^2.$$ 

Then $\sum_i N_{C_i}(P_C(x)) = N_C(P_C(x))$ and $d(x, C) \leq \lambda \cdot \sum_i d(x, C_i)$.

**Proof.** “$\sum_i N_{C_i}(P_C(x)) = N_C(P_C(x))$”: On the one hand, the assumption on the tangent cones yields (take polars), $N_C(P_C(x)) = \text{cl} \left( \sum_i N_{C_i}(P_C(x)) \right)$. On the other hand, by the assumption on the minimum and Corollary 2.10, the collection $\{N_{C_1}(P_C(x)), \ldots, N_{C_m}(P_C(x))\}$ has property (G). Thus, by Proposition 2.8, $\sum_i N_{C_i}(P_C(x))$ is closed.

“$d(x, C) \leq \lambda \sum_i d(x, C_i)$”: By Proposition 4.1.(i), $x - P_C(x) \in N_C(P_C(x))$. By the preceding statement on the normal cones, there exists $y_i \in N_{C_i}(P_C(x))$, $\forall i$, such that $x - P_C(x) = \sum_i y_i$ and $\sum_i \|y_i\|^2 \leq \lambda \sum_i \|x - P_C(x)\|^2$. Also, by Proposition 4.1.(iii), $\langle y_i, P_C(x) \rangle = i_{C_i}(y_i)$. Using the last displayed inequality and Proposition 4.1.(ii), we obtain the inequalities in the following chain of equalities and inequalities.

$$\frac{1}{2}d^2(x, C) = \frac{1}{2}\|x - P_C(x)\|^2$$

$$= \langle x - P_C(x), x - P_C(x) \rangle - \frac{1}{2}\|x - P_C(x)\|^2$$

$$= \langle x - P_C(x), \sum_i y_i \rangle - \frac{1}{2}\|x - P_C(x)\|^2$$

$$= \sum_i \langle x, y_i \rangle - \sum_i \langle P_C(x), y_i \rangle - \frac{1}{2}\|x - P_C(x)\|^2$$

$$= \sum_i \langle x, y_i \rangle - \sum_i i_{C_i}^*(y_i) - \frac{1}{2}\|x - P_C(x)\|^2$$

$$\leq \sum_i \langle x, y_i \rangle - \sum_i i_{C_i}^*(y_i) - \frac{1}{2}\lambda \sum_i \|y_i\|^2$$

$$= \frac{1}{\lambda} \sum_i \langle \lambda^2 x, y_i \rangle - i_{C_i}^*(\lambda y_i) - \frac{1}{2}\|y_i\|^2$$

$$\leq \frac{1}{\lambda} \sum_i \frac{1}{2}d^2(\lambda^2 x, \lambda y_i C_i)$$

$$= \sum_i \frac{1}{2}d^2(x, C_i). \quad \blacksquare$$
The next example shows that the assumption on the intersection of the tangent cones in Theorem 4.2 is important.

**Example 4.3** Let $X := \mathbb{R}^2$, $C_1 := B((1, 0); 1)$ and $C_2 := B((-1, 0); 1)$. Then $C := C_1 \cap C_2 := \{0\}$. Hence $PC(x) = 0$, $\forall x \in X$. Now $T_{C_1}(0) = [0, +\infty[ \times \mathbb{R}$, $T_{C_2}(0) = ]-\infty, 0[, \mathbb{R}$, and $T_C(0) = \{0\}$. Hence $T_{C_1}(0) \cap T_{C_2}(0) = \{0\} \times \mathbb{R}$ $\supset \{0\} = T_C(0)$, *i.e.*, the condition on the intersection of the tangent cones in Theorem 4.2 is not satisfied and $\{C_1, C_2\}$ is not CHIP. On the other hand, the condition on the minimum is satisfied with $\lambda_x = 1$, $\forall x \in X$, but the first conclusion of Theorem 4.2 fails. Now consider $x := (0, r)$, where $r > 0$. Then $d(x, C) = d(x, C) = \sqrt{1 + r^2} - 1$ and $d(x, C) = r$. Since $d(x, C) / (d(x, C_1) + d(x, C_2)) = r/(2\sqrt{1 + r^2} - 2) \to +\infty$, the second conclusion of Theorem 4.2 is not true either.

**Applications**

**Corollary 4.4** Suppose $C_1, \ldots, C_m$ are finitely many closed convex sets in $X$ with $C := \bigcap_i C_i \neq \emptyset$. Suppose further $TC(c) = \bigcap_i TC_i(c)$, $\forall c \in C$ and there exists $\lambda > 0$ such that

$$\min \{\sum_i \|y_i\|^2 : \text{each } y_i \in N_{C_i}(c), \sum_i y_i = y\} \leq \lambda^2 \|y\|^2, \quad \forall c \in C, \forall y \in N_C(c).$$

Then $d(x, C) \leq \lambda \sum_i d(x, C_i), \forall x \in X$.

**Proof.** Theorem 4.2. ■

**Remark 4.5** Corollary 4.4 says that “if the tangent cones of $\{C_1, \ldots, C_m\}$ are CHIP and the collection of normal cones has property (G) (uniformly on $C$), then the collection $\{C_1, \ldots, C_m\}$ is linearly regular.”

**Theorem 4.6** Suppose $C_1, \ldots, C_m$ are finitely many closed convex sets in $X$, where, for some $r \in \{0, \ldots, m\}$, the sets $C_{r+1}, \ldots, C_m$ are polyhedral. Suppose further there exists $z \in \bigcap_{i=0}^r ri(C_i) \cap \bigcap_{i=r+1}^m C_i$ and let $C := \bigcap_{i=1}^m C_i$. Then there exists $\gamma > 0$, depending on $z$, such that

$$d(x, C) \leq \gamma (\|PC(x) - z\| + 1) \sum_i d(x, C_i), \quad \forall x \in X.$$

**Proof.** Let $\delta > 0$ such that $\text{aff} \ (C_i) \cap B(z; \delta) \subset C_i, \forall i \in \{1, \ldots, r\}$. Then Theorem 2.18 yields $\beta > 0$, independent of $z$ and $\delta$, such that for every $c \in C$ and $y \in N_C(c)$,

$$\min \{\sum_{i=1}^m \|y_i\| : \text{each } y_i \in N_{C_i}(c), \sum_{i=1}^m y_i = y\} \leq \frac{x^2(1+\beta) + \delta^2}{\delta} \|y\|.$$  

For an arbitrary but fixed $x \in X$, define $\lambda_x := (\|PC(x) - z\|(1 + \beta) + \delta)/\delta$. Then for every $y \in N_C(PC(x)) = \sum_i N_{C_i}(PC(x))$,

$$\min \{\sum_{i=1}^m \|y_i\|^2 : \text{each } y_i \in N_{C_i}(PC(x)), \sum_{i=1}^m y_i = y\} \leq \lambda_x^2 \|y\|^2.$$  

Theorem 4.2 implies

$$d(x, C) \leq \frac{x^2(1+\beta) + \delta^2}{\delta} \sum_i d(x, C_i), \quad \forall x \in X;$$

therefore, $\gamma := \max\{(1 + \beta)/\delta, \beta\}$ does the job. ■
Corollary 4.7 (standard constraint qualification implies bounded linear regularity)
Suppose $C_1, \ldots, C_m$ are finitely many closed convex sets in $X$, where, for some $r \in \{0, \ldots, m\}$, the sets are polyhedral. Suppose further $\bigcap_{i=1}^{r} \text{ri}(C_i) \cap \bigcap_{i=r+1}^{m} C_i \neq \emptyset$. Then for every $\rho > 0$, there exists $\lambda_\rho > 0$ such that $d(x, C) \leq \lambda_\rho \sum_i d(x, C_i)$, $\forall x \in \rho B_X$.

**Proof.** Pick an arbitrary $z \in \bigcap_{i=1}^{r} \text{ri}(C_i) \cap \bigcap_{i=r+1}^{m} C_i$. By Theorem 4.6, there exists $\gamma > 0$ such that $d(x, C) \leq \gamma(\|P_C(x) - z\| + 1) \sum_i d(x, C_i)$, $\forall x \in X$. Therefore, $\lambda_\rho := \gamma(\rho + \|z\| + 1)$ does the job. ■

Corollary 4.8 Suppose $C_1, \ldots, C_m$ are finitely many closed convex sets in $X$, where, for some $r \in \{0, \ldots, m\}$, the sets $C_{r+1}, \ldots, C_m$ are polyhedral. Suppose further $\bigcap_{i=1}^{r} \text{ri}(C_i) \cap \bigcap_{i=r+1}^{m} C_i \neq \emptyset$ and $C := \bigcap_{i=1}^{m} C_i$ is bounded. Then there exists $\lambda > 0$ such that $d(x, C) \leq \lambda \sum_i d(x, C_i)$, $\forall x \in X$.

**Proof.** Similarly to the proof of Corollary 4.7. ■

Remark 4.9 Corollary 4.8 can be proved differently by using Corollary 4.7 and Lewis’s observation that bounded linear regularity plus a bounded intersection yields linear regularity (see [2, Theorems 4.2.6 and 5.2.3]).

5 Applications to convex inequalities

Throughout this section, we assume that $f_1, \ldots, f_m$ are finitely many functions on $X$ that are convex and finite everywhere on $X$.

Let $C_i := \{x \in X : f_i(x) \leq 0\}$, $\forall i$. We assume $C := \bigcap_i C_i$, *i.e.*, the system of convex inequalities

\[(*) \quad f_1(x) \leq 0, f_2(x) \leq 0, \ldots, f_m(x) \leq 0\]

has at least one solution.

Systems of convex inequalities have been receiving much attention lately; for more information, the reader is referred to Li’s [23], Klatte and Li’s [21], Lewis and Pang’s [22], and the many references therein.

To demonstrate the usefulness of the results of the previous sections, we present some selected applications to systems of convex inequalities.

Basic constraint qualification and strong CHIP

**Definition 5.1** ([14, Section VII.2.2]) Suppose $x \in C$. Then the system of convex inequalities (*) satisfies the basic constraint qualification at $x$, if $N_C(x) = \text{cone} \{\partial f_i(x) : f_i(x) = 0\}$.  

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Remark 5.2 • The basic constraint qualification is formulated in terms of the normal cone of \( C \). There exists an equivalent "dual" condition, called Abadie’s constraint qualification, which is formulated in terms of the tangent cone of \( C \); see Li’s [23] for more. • If \( x \in C \) and \( f_i(x) = 0 \), then cone \( (\partial f_i(x)) \subseteq N_C(x) \) (with equality if \( x \) is not a minimizer of \( f \); see [25, Corollary 23.7.1]).

Definition 5.1 and Remark 3.3 thus immediately imply the following result.

Proposition 5.3 Suppose \( x \in C \) and the system of convex inequalities (*) satisfies the basic constraint qualification at \( x \). Then \( \{C_1, \ldots, C_m\} \) is strong CHIP at \( x \).

Note that strong CHIP is a purely geometric property, while the basic constraint qualification depends on the analytic representation of each \( C_i \). The next example shows that strong CHIP is genuinely less restrictive than the basic constraint qualification.

Example 5.4 Let \( X := \mathbb{R}^2 \), \( f_1(x_1, x_2) := x_1^2 \), and \( f_2(x_1, x_2) := x_2^2 \). Then \( C_1 := \{x \in X : f_1(x) \leq 0\} = \{0\} \times \mathbb{R} \) and \( C_2 := \{x \in X : f_2(x) \leq 0\} = \mathbb{R} \times \{0\} \), which implies \( N_{C_1}(0) = \mathbb{R} \times \{0\} \) and \( N_{C_2}(0) = \{0\} \times \mathbb{R} \). Let \( C := C_1 \cap C_2 = \{0\} \), then \( N_C(0) = X \). On the one hand, \( N_{C_1}(0) + N_{C_2}(0) = N_C(0) \), so \( \{C_1, C_2\} \) is strong CHIP. On the other hand, \( \nabla f_1(0) = 0 = \nabla f_2(0) \), so the system of convex inequalities \( f_1(x) \leq 0, f_2(x) \leq 0 \) does not satisfy the basic constraint qualification at 0.

The next proposition shows that strong CHIP and the basic constraint qualification do coincide, provided the subdifferentials are well-behaved.

Proposition 5.5 Suppose \( x \in C \) and \( N_{C_i}(x) = \text{cone}(\partial f_i(x)) \), for each \( i \) with \( f_i(x) = 0 \). Then the system of convex inequalities (*) satisfies the basic constraint qualification if and only if the collection \( \{C_1, \ldots, C_m\} \) is strong CHIP at \( x \).

Proof. “⇒”: Proposition 5.3. “⇐”: if \( f_i(x) < 0 \), then \( x \in \text{int} C_i \) and so \( N_{C_i}(x) = \{0\} \). Let \( I := \{i : f_i(x) = 0\} \). Then \( N_C(x) = \sum_{i \in I} N_{C_i}(x) = \sum_{i \in I} \text{cone}(\partial f_i(x)) = \text{cone} \{\partial f_i(x) : i \in I\} \), and therefore (*) satisfies the basic constraint qualification at \( x \). ■

Definition 5.6 ([14, Definition VII.2.2.3]) The system of convex inequalities (*) satisfies the weak Slater assumption, if there exists some \( \tilde{x} \in C \), called a weak Slater point, such that for every \( i \), \( f_i \) is affine or \( f_i(\tilde{x}) < 0 \).

Corollary 5.7 Suppose the system of convex inequalities (*) satisfies the weak Slater assumption. Then it satisfies the basic constraint qualification at every point in \( C \) and \( \{C_1, \ldots, C_m\} \) is strong CHIP.

Proof. By [14, Section VII.2.2], (*) satisfies the basic constraint qualification at every point in \( C \). Apply Proposition 5.3. ■
Asymptotic constraint qualification and error bounds

**Definition 5.8** (Auslender and Crouzeix [1]) We say that the system of convex inequalities (*) satisfies the asymptotic constraint qualification, if the following is satisfied: “Suppose \((x_n)\) is a sequence in \(C\) with \(\|x_n\| \to +\infty\). Let \(I\) be a subset of \(\{1, \ldots, m\}\) such that \(f_i(x_n) = 0\) eventually, \(\forall i \in I\). Let \(g_{i,n} \in \partial f_i(x_n), \forall n, \forall i \in I\). Define \(I_{\text{bounded}} := \{i \in I : (g_{i,n}) \text{ is bounded}\}\) and \(I_{\text{unbounded}} := I \setminus I_{\text{bounded}}\). Suppose \((k_n)\) is a subsequence of \((n)\) such that for every \(i \in I\): \((g_{i,k_n})\) converges, if \(i \in I_{\text{bounded}}\); \((g_{i,k_n}/\|g_{i,k_n}\|)\) converges, if \(i \in I_{\text{unbounded}}\). Denote the limit by \(g_i, \forall i \in I\). Then necessarily \(0 \not\in \text{conv} \{g_i : i \in I\}\).”

**Remark 5.9** • Note that if \(C\) is bounded, then the asymptotic constraint qualification holds trivially. • More on the asymptotic constraint qualification can be found in Auslender and Crouzeix’s [1] as well as in Klatte and Li’s recent study [21].

The distinction between bounded subgradient sequences and unbounded subgradient sequences in Definition 5.8 can be removed as follows.

**Proposition 5.10** The system of convex inequalities (*) satisfies the asymptotic constraint qualification if and only if the following is satisfied: “Suppose \((x_n)\) is a sequence in \(C\) with \(\|x_n\| \to +\infty\). Let \(I\) be a subset of \(\{1, \ldots, m\}\) such that \(f_i(x_n) = 0\) eventually, \(\forall i \in I\). Let \(g_{i,n} \in \partial f_i(x_n), \forall n, \forall i \in I\). Suppose \((k_n)\) is a subsequence of \((n)\) such that \((g_{i,k_n}/(1 + \|g_{i,k_n}\|))\) converges to \(\hat{g}_i, \forall i \in I\). Then necessarily \(0 \not\in \text{conv} \{\hat{g}_i : i \in I\}\).”

**Proof.** This follows from the (easy) fact that whenever \(z_1, \ldots, z_m \in X\) and \(t_1, \ldots, t_m > 0\), then: \(0 \in \text{conv} \{z_1, \ldots, z_m\}\) if and only if \(0 \in \text{conv} \{t_1z_1, \ldots, t_mz_m\}\). ■

**Theorem 5.11** (sharpening of Auslender and Crouzeix’s result) Suppose the system of convex inequalities (*) satisfies the weak Slater assumption and the asymptotic constraint qualification. Then the collection \(\{C_1, \ldots, C_m\}\) is linearly regular: there exists \(\gamma > 0\) such that

\[
d(x, C) \leq \gamma \sum_i d(x, C_i), \quad \forall x \in X.
\]

**Proof.** **Step 1.** The collection \(\{C_1, \ldots, C_m\}\) is strong CHIP (by Corollary 5.7).

**Step 2.** \(N_{C_i}(x) = \begin{cases} \text{cone} (\partial f_i(x)), & \text{if } f_i(x) = 0; \\ \{0\}, & \text{otherwise}, \end{cases} \quad \forall x \in C, \forall i \in \{1, \ldots, m\}.\)

If \(f_i(x) = 0\), then the result follows from the weak Slater assumption combined with [25, Corollary 23.7.1]. Otherwise, \(f_i(x) < 0\) in a neighborhood of \(x\) (by continuity of \(f_i\) on \(X\)). **Step 2** thus holds.

**Step 3.** There exists \(\gamma > 0\) such that \(\forall x \in C, \forall y \in N_C(x) = \sum_i N_{C_i}(x)\):

\[
\min \left\{ \sum_i \|y_i\|^2 : \text{each } y_i \in N_{C_i}(x), \sum_i y_i = y \right\} \leq \gamma^2 \|y\|^2.
\]

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Suppose to the contrary that Step 3 is not true. Then there exist sequences \((x_n)\) in \(C\) and \((y_n)\) with \(y_n \in N_C(x_n), \ \forall n\), such that

\[
\min \left\{ \sum_i \|y_i\|^2 : \text{each } y_i \in N_{C_i}(x_n), \sum_i y_i = y_n \right\} > n^2 \|y_n\|^2 > 0.
\]

We claim that \(\|x_n\| \to +\infty\): indeed, if \((\circ)\) held on a bounded subsequence of \((x_n)\), then we would contradict Theorem 2.18 (use the weak Slater point for the system of convex inequalities \((\ast)\) and recall Proposition 2.5). After normalizing (the \(N_{C_i}\) are cones), we assume WLOG that \(y_n \in S_X, \ \forall n\). Let \(y_{i,n} \in N_{C_i}(x_n), \ \forall i, \text{ such that } \sum_i y_{i,n} = y_n, \ \forall n\). Then \((\circ)\) yields

\[
\sum_i \|y_{i,n}\| > n \to +\infty.
\]

After passing to a subsequence if necessary, we assume WLOG that the sets \(\{i : y_{i,n} \neq 0\}\) all are all the same, say \(I\). Then, using Step 2, we see that \(f_i(x_n) = 0\) and obtain \(\mu_{i,n} > 0\) such that \(y_{i,n} = \mu_{i,n} g_{i,n}\), for some \(g_{i,n} \in \partial f_i(x_n), \ \forall n, \forall i \in I\). Set \(\hat{g}_{i,n} := g_{i,n}/(1 + \|g_{i,n}\|), \ \forall n, \forall i \in I\). Because \(\|\hat{g}_{i,n}\| \leq 1\), we get

\[
\|y_{i,n}\| \leq \mu_{i,n}(1 + \|g_{i,n}\|), \ \forall n, \forall i \in I.
\]

Define

\[
\lambda_{i,n} := \frac{\mu_{i,n}(1 + \|g_{i,n}\|)}{\sum_{j \in I} \mu_{j,n}(1 + \|g_{j,n}\|)}, \ \forall i \in I, \forall n.
\]

After passing to another subsequence if necessary, we assume WLOG that \(\lambda_{i,n} \to \lambda_i\) and that \(\hat{g}_{i,n} \to \hat{g}_i, \ \forall i \in I\). Note that \(\sum_{i \in I} \lambda_{i,n} = 1\) and \(\lambda_i \geq 0, \ \forall i \in I\). Using \((\circ\circ)\), \((\circ\circ\circ)\), and \(\|y_{i,n}\| = 1, \ \forall n\), we deduce

\[
\sum_{i \in I} \lambda_i \hat{g}_i \leftarrow \sum_{i \in I} \lambda_{i,n} \hat{g}_{i,n} = \frac{1}{\sum_{j \in I} \mu_{j,n}(1 + \|g_{j,n}\|)} \cdot y_n \to 0.
\]

Therefore, \(0 = \sum_{i \in I} \lambda_i \hat{g}_i\), which contradicts the characterization of the asymptotic constraint qualification in Proposition 5.10. Step 3 thus holds true.

**Final Step.** The result now follows from Steps 1, 3, and Corollary 4.4. ■

The following result is well-known in the area of error bounds (and implicitly contained in [21] and [22], for instance). We include it for completeness as it will allow us to give a self-contained proof of Auslender and Crouzeix’s main result.

**Proposition 5.12** Suppose \(f\) is a convex function that is everywhere finite on \(X\). Let \(S := \{x \in X : f(x) \leq 0\}\) be nonempty. Suppose further \(f\) is affine or there exists a Slater point \(\hat{x} \in X\) with \(f(\hat{x}) < 0\). If the convex inequality \(f(x) \leq 0\) satisfies the asymptotic constraint qualification, then there exists \(\alpha > 0\) such that

\[
d(x, S) \leq \alpha f^+(x), \quad \forall x \in X.
\]
Proof. We assume WLOG that $S \neq X$ (otherwise, we are done). Then, even in the affine case, there exists $\hat{x} \in X$ such that $f(\hat{x}) < 0$. By [14, Proposition VI.1.3.3], we have

$$\text{bd} (S) = \{ x \in X : f(x) = 0 \}$$

Claim: There exists $\epsilon > 0$ such that $\| \partial f(x) \| \geq \epsilon$, $\forall x \in \text{bd} (S)$.

Assume to the contrary the existence of a sequence $(x_n)$ with $f(x_n) = 0$, $\forall n$, and a sequence $(g_n)$ with $g_n \in \partial f(x_n)$, $\forall n$ and $g_n \to 0$. If $(x_n)$ possesses a cluster point, say $\bar{x}$, then it follows that $f(\bar{x}) = 0$ and $0 \in \partial f(\bar{x})$. But $\bar{x}$ cannot be a minimizer of $f$, since $f(\hat{x}) < 0$. Hence $\|x_n\| \to +\infty$. But this contradicts the asymptotic constraint qualification. The claim is thus verified.

Now fix an arbitrary $x \in X \setminus S$ and let $s := P_S(x) \in \text{bd} (S)$ so that $\| x - s \| = d(x, S)$. By Proposition 4.1, $x - s \in N_S(s)$. By [25, Corollary 23.7.1], $N_S(s) = x - s = rg$, for some $g \in \partial f(s)$ and $r > 0$. Hence, using the Claim,

$$r\epsilon \| x - s \| \leq r\| g \| \| x - s \| = \| x - s \|^2 = \langle rg, x - s \rangle$$

$$= r \langle g, x - s \rangle \leq r (f(x) - f(s)) = rf(x).$$

In particular, $d(x, S) = \| x - s \| \leq \frac{1}{\epsilon} f(x)$ and so $\alpha := \frac{1}{\epsilon}$ does the job. \[\blacksquare\]

**Corollary 5.13** (Auslander and Crouzeix’s [1, Theorem 2]) Suppose the system of convex inequalities (*) satisfies the weak Slater assumption and the asymptotic constraint qualification. Then there exists $\beta > 0$ such that

$$d(x, C) \leq \beta \sum_i f_i^+(x) \quad \forall x \in X.$$  

**Proof.** On the one hand, by Theorem 5.11, there exists $\gamma > 0$ such that $d(x, C) \leq \gamma \sum_i d(x, C_i)$, $\forall x \in X$. On the other hand, each $f_i$ by itself satisfies the weak Slater condition and the asymptotic constraint qualification; in particular, by Proposition 5.12, there exists $\alpha_i > 0$ such that $d(x, C_i) \leq \alpha_i f_i^+(x)$, $\forall x \in X, \forall i$. Altogether, $\beta := \max_i \gamma \alpha_i$ does the job. \[\blacksquare\]

**The constrained case**

As a last application to error bounds, we briefly discuss the constrained case:

As before, we assume that $f_1, \ldots, f_m$ are finitely many functions on $X$ that are convex and finite everywhere, and we let $C_i := \{ x \in X : f_i(x) \leq 0 \}, \forall i$. In addition, let $C_0$ be a closed convex set in $X$ with $C := \bigcap_{i=0}^m C_i \neq \emptyset$. That is, the system of constrained convex inequalities

$$f_1(x) \leq 0, \ldots, f_m(x), \leq 0, x \in C_0$$

has at least one solution.

We require a special case of a result due to Robinson; for completeness, we include its brief proof.
Proposition 5.14 ([24, Section 3]) Suppose $f$ is a closed convex function on $X$ and there exists a Slater point $\hat{x} \in X$ with $f(\hat{x}) < 0$. Let $S := \{x \in X : f(x) \leq 0\}$. Then

$$d(x, S) \leq \frac{\|x - \hat{x}\|}{-f(\hat{x})} f^+(x), \quad \forall x \in X.$$ 

Proof. Fix an arbitrary $x \in X$. We assume WLOG that $f(x) > 0$. Then $\lambda := f(x)/(f(x) + (-f(\hat{x}))) \in ]0, 1[$. It is easy to check that $s := (1 - \lambda)x + \lambda \hat{x} \in S$. Hence

$$d(x, S) \leq \|x - s\| = \lambda \|x - \hat{x}\| = \frac{f(x)}{f(x) + (-f(\hat{x}))}\|x - \hat{x}\| \leq \frac{\|x - \hat{x}\|}{-f(\hat{x})} f(x).$$

\[\blacksquare\]

Theorem 5.15 Consider the constrained system of convex inequalities (**). Suppose there exists a point $\hat{x} \in C_0$ such that $f_i(\hat{x}) < 0$, $\forall i \in \{1, \ldots, m\}$. Then for every $\rho > 0$, there exists $\beta_\rho > 0$ such that

$$d(x, C) \leq \beta_\rho (d(x, C_0) + \sum_{i=1}^m f_i^+(x)), \quad \forall x \in \rho B_X.$$ 

Proof. By [25, Corollary 6.3.2], there exists some $z$ (close to $\hat{x}$) such that $z \in \text{ri}(C_0) \cap \bigcap_{i=1}^m C_i$. On the one hand, by Theorem 4.6, there exists $\gamma > 0$ (depending on $z$) such that

$$d(x, C) \leq \gamma (\|P_C(x) - z\| + 1)(d(x, C_0) + \sum_{i=1}^m d(x, C_i)), \quad \forall x \in X.$$ 

On the other hand, by Proposition 5.14,

$$d(x, C_i) \leq \frac{\|x - \hat{x}\|}{-f_i(\hat{x})} f_i^+(x), \quad \forall i \in \{1, \ldots, m\}, \forall x \in X.$$ 

Altogether, using $\|P_C(x) - z\| \leq \|x - z\| \leq \rho + \|z\|$, we are done by letting

$$\beta_\rho := \gamma (\rho + \|z\| + 1) \max\{1, (\rho + \|\hat{x}\|)/\min_{i \in \{1, \ldots, m\}} (-f_i(\hat{x}))\} \quad \blacksquare$$

Remark 5.16 Suppose that $m = 1$. Then the conclusion of Theorem 5.15 also follows from Lewis and Pang’s [22, Proposition 3]. Note that Lewis and Pang pose a certain boundedness assumption on $\partial f_1$, which is trivially satisfied if $f_1$ is everywhere finite (use [4, Corollary 7.9]).

6 More on cones and subspaces

Two closed convex cones

Fact 6.1 ([25, Corollary 16.4.2]) Suppose $A$ and $B$ are two closed convex cones in $X$. Then $\text{cl} (A^\ominus + B^\ominus) = (A \cap B)^\ominus$.

Proposition 6.2 Suppose $K$ is a closed convex cone in $X$. Then

$$N_K(x) = K^\ominus \cap \{x\}^\perp \quad \text{and} \quad T_K(x) = \text{cl} (K + \mathbb{R}x), \quad \forall x \in K.$$
Proof. Pick \( x^* \in N_K(x) \), i.e., \( \langle x^*, k - x \rangle \leq 0, \forall k \in K \). If we let \( k := k' + x \), where \( k' \in K \) and hence \( k \in K \), we learn that \( \langle x^*, k' \rangle \leq 0, \forall k' \in K \) and thus \( x^* \in K^\ominus \). Furthermore, if we let \( k := 2x \in K \), then \( \langle x^*, x \rangle \leq 0 \); and if \( k := 0 \), then \( \langle x^*, -x \rangle \leq 0 \). Altogether, \( N_K(x) \subseteq K^\ominus \cap \{x\}^\perp \). The reverse inclusion is even simpler and the normal cone formula thus follows. The tangent cone formula follows by taking polars and invoking Fact 6.1.

Proposition 6.3 Suppose \( A \) and \( B \) are two closed convex cones in \( X \). Then

\[
(A^\ominus + B^\ominus) \cap \{x\}^\perp = [A^\ominus \cap \{x\}^\perp] + [B^\ominus \cap \{x\}^\perp], \quad \forall x \in A \cap B.
\]

Proof. Clearly, the right-hand side is contained in the left-hand side. Let \( z \) be a member of the left-hand side. Then \( z = a^\ominus + b^\ominus \), where \( a^\ominus \in A^\ominus \), \( b^\ominus \in B^\ominus \), and \( \langle z, x \rangle = 0 \). Now \( x \in A \cap B \); hence \( \langle x, a^\ominus \rangle \leq 0 \) and \( \langle x, b^\ominus \rangle \leq 0 \). Altogether, \( 0 = \langle x, z \rangle = \langle x, a^\ominus \rangle + \langle x, b^\ominus \rangle \leq 0 + 0 = 0 \). Hence \( a^\ominus \) and \( b^\ominus \) belong to \( \{x\}^\perp \), which yields \( z = a^\ominus + b^\ominus \in [A^\ominus \cap \{x\}^\perp] + [B^\ominus \cap \{x\}^\perp] \).

Proposition 6.4 Suppose \( A \) and \( B \) are two closed convex cones in \( X \). Then TFAE:

(i) \( \{A, B\} \) is strong CHIP.

(ii) \( \{A, B\} \) is strong CHIP at \( 0 \): \( (A \cap B)^{\ominus} = A^\ominus + B^\ominus \).

(iii) \( A^\ominus + B^\ominus \) is closed.

Proof. “(i)\( \Rightarrow \)(ii)”: \( (A \cap B)^\ominus = N_{(A \cap B)}(0) = N_A(0) + N_B(0) = A^\ominus + B^\ominus \). “(ii)\( \Leftrightarrow \)(iii)”: Fact 6.1. “(i)\( \Leftarrow \)(iii)”: Fix an arbitrary \( x \in A \cap B \). We have to show that \( N_A(x) + N_B(x) = N_{A \cap B}(x) \); equivalently, by Proposition 6.2, that \( [A^\ominus \cap \{x\}^\perp] + [B^\ominus \cap \{x\}^\perp] = (A \cap B)^{\ominus} \cap \{x\}^\perp \); equivalently, by Fact 6.1, that \( [A^\ominus \cap \{x\}^\perp] + [B^\ominus \cap \{x\}^\perp] = \text{cl} (A^\ominus + B^\ominus) \cap \{x\}^\perp \). The last condition is satisfied by assumption (iii) and Proposition 6.3.

Proposition 6.5 Suppose \( A \) and \( B \) are two closed convex cones in \( X \) such that \( A \cap B \) is linear. Then \( \{A, B\} \) is strong CHIP.

Proof. \( A \cap B \) linear \( \Rightarrow \) \( \text{cl} (A^\ominus + B^\ominus) \) is linear (Fact 6.1) \( \Rightarrow A^\ominus + B^\ominus \) is linear ([25, Theorem 6.3]) \( \Rightarrow A^\ominus + B^\ominus \) is closed \( \Rightarrow \) \( \{A, B\} \) is strong CHIP (Proposition 6.4).

The next example shows the importance of the assumption on linearity in Proposition 6.5.

Example 6.6 Let \( X := \mathbb{R}^3 \), \( A := \{(x_1, x_2, x_3) \in X : x_1^2 + x_2^2 \leq x_3^2, x_3 \geq 0\} \), and \( B := \{(x_1, x_2, x_3) : x_1 = x_3\} \). Then \( \{A, B\} \) fails to be CHIP at every nonzero point in the intersection of \( A \cap B \).

We now re-derived and refine Jameson’s main duality result.
Theorem 6.7 (see also [18, Theorem 2.1]) Suppose $A$ and $B$ are two closed convex cones in $X$. Then TFAE:

(i) $\{A, B\}$ is linearly regular.

(ii) $\{A, B\}$ is boundedly linearly regular.

(iii) $\{A, B\}$ is strong CHIP and $\{A^\ominus, B^\ominus\}$ has property (G).

(iv) $A^\ominus + B^\ominus$ is closed and $\{A^\ominus, B^\ominus\}$ has property (G).

(v) $\{A^\ominus, B^\ominus\}$ has property (G).

Proof. “(i)$\Rightarrow$(ii)”: obvious. “(i)$\iff$(ii)”: easy to verify (use homogeneity or [3, Theorem 3.17]). “(i)$\iff$(iii)”: Proposition 2.13 (see also Remark 2.14). “(iii)$\iff$(iv)”: Proposition 6.4. “(iv)$\Rightarrow$(v)”: obvious. “(iv)$\iff$(v)”: Proposition 2.8. □

Corollary 6.8 Suppose $A$ and $B$ are two closed convex cones in $X$ such that $A \cap B$ is linear. Then $\{A, B\}$ is linearly regular and strong CHIP, $A^\ominus + B^\ominus$ is closed, and $\{A^\ominus, B^\ominus\}$ has property (G).

Proof. Theorem 6.7 and Proposition 6.5. □

Corollary 6.9 Suppose $A$ and $B$ are two convex polyhedral cones in $X$. Then $\{A, B\}$ is linearly regular and strong CHIP, $A^\ominus + B^\ominus$ is closed, and $\{A^\ominus, B^\ominus\}$ has property (G).

Proof. Theorem 6.7 and Fact 2.15. □

Remark 6.10 All results in this subsection remain valid for finitely many closed convex cones; some will generalize to infinite-dimensional Hilbert space (see Jameson’s [18]).

Two subspaces and their angle

Suppose $A$ and $B$ are two subspaces of $X$. Then, by Proposition 2.11, the collection $\{A, B\}$ has property (G). By Corollary 2.10, there exists $\alpha > 0$ such that $\|x\|_{A \cap B} \leq \alpha \|x\|$, $\forall x \in X$. In this subsection, we identify the “best” possible $\alpha$, the square of which we define now.

Definition 6.11 Suppose $A$ and $B$ are two subspaces of $X$. Define

$$\beta(A, B) := \sup_{x \in (A \cap B)^\perp} \min_{u \in A^\perp, v \in B^\perp, u + v = x} \|u\|^2 + \|v\|^2.$$ 

The following notion of the angle between two subspaces goes back to 1937.
**Definition 6.12** (Friedrichs [13]) Suppose $A$ and $B$ are two subspaces of $X$. The angle between $A$ and $B$ is the angle in $[0, \pi/2]$ whose cosine is defined by

$$c(A, B) := \sup \{ \langle a, b \rangle : a \in A \cap (A \cap B)^\perp \cap B_X, b \in B \cap (A \cap B)^\perp \cap B_X \}.$$ 

**Remark 6.13** Because $X$ is finite-dimensional, we conclude: • the supremum in Definition 6.12 is attained and hence a maximum; • thus the angle is always bigger than 0; equivalently, the cosine is always less than 1. For more information on the angle, the reader is referred to Deutsch’s excellent survey [9].

**Proposition 6.14** Suppose $A$ and $B$ are two subspaces of $X$. Then:

(i) $A = B$ if and only if $A^\perp \cap (A + B) = \{0\}$ and $B^\perp \cap (A + B) = \{0\}$.

(ii) if $A = B \subsetneq X$, then $\beta(A, B) = \frac{1}{2}$.

**Proof.** (i): “⇒” is trivial. We prove the contrapositive of “⇐”: assume $A \neq B$. Then $A + B \not\subseteq A$ or $A + B \not\subseteq B$. WLOG $A + B \not\subseteq A$. Pick $s \in (A + B) \setminus A$ and write $s = P_A s + P_A \perp s$. Then $P_A \perp s$ is nonzero and contained in $A^\perp$. Also, $P_A \perp s = s - P_A s \in (A + B) - A = A + B$. Altogether, $P_A \perp s \in A^\perp \cap (A + B) \setminus \{0\}$ and (i) is proven. (ii): pick an arbitrary $x \in A^\perp = B^\perp = (A \cap B)^\perp$. Then $\min_{y \in A^\perp} \|x - y\|^2 + \|y\|^2$ is attained at $y = \frac{1}{2}x$ (use Convex Calculus) with value $\frac{1}{2}\|x\|^2$. Because $A^\perp$ contains nonzero vectors, we deduce $\beta(A, B) = \frac{1}{2}$. □

**Theorem 6.15** (property (G) and the angle) Suppose $A$ and $B$ are two subspaces of $X$. Then $\beta(A, B) \leq 1/(1 - c(A, B))$; more precisely:

$$\beta(A, B) = \begin{cases} 
0, & \text{if } A = B = X; \\
\frac{1}{2}, & \text{if } A = B \subsetneq X; \\
\frac{1}{1 - c(A, B)}, & \text{otherwise}. 
\end{cases}$$

**Proof.** Recall that $(A \cap B)^\perp = A^\perp + B^\perp$. Fix an arbitrary $x \in (A \cap B)^\perp \cap B_X$. Then there exist $u \in A^\perp$ and $v \in B^\perp$ such that $x = u + v$. For brevity, let $S := A + B$ and $S^\perp = (A + B)^\perp = A^\perp \cap B^\perp$. Write $u = P_S^{-1}(u) + P_S(u)$ and $v = P_S^{-1}(v) + P_S(v)$. Then $P_S(u) = u - P_S^{-1}(u) \in A^\perp - (A^\perp \cap B^\perp) \subseteq A^\perp$ and similarly $P_S(v) \in B^\perp$. Thus

$$P_S(u) \in A^\perp \cap (A + B) \text{ and } P_S(v) \in B^\perp \cap (A + B).$$

By [9, Lemma 2.10.2.(b)], $\langle -P_S(u), P_S(v) \rangle \leq c(A^\perp, B^\perp) \|P_S(u)\| \|P_S(v)\|$. Since $c(A, B) = c(A^\perp, B^\perp)$ ([9, Theorem 2.16]), we get $\langle -P_S(u), P_S(v) \rangle \leq \frac{1}{2} c(A, B) \|P_S(u)\|^2 + \|P_S(v)\|^2$, which easily leads to

$$\|P_S(u) + P_S(v)\|^2 \geq (1 - c(A, B)) \|P_S(u)\|^2 + \|P_S(v)\|^2.$$
Using this, $0 \leq c(A, B) < 1$, and Pythagoras (twice), we obtain the following chain of inequalities:

$$
\|x\|^2 = \|u + v\|^2 = \|P_S(u + v) + P_{S^\bot}(u + v)\|^2 \\
= \|P_S(u) + P_S(v)\|^2 + \|P_{S^\bot}(u + v)\|^2 \\
\geq (1 - c(A, B))(\|P_S(u)\|^2 + \|P_S(v)\|^2) + \|P_{S^\bot}(u + v)\|^2 \\
\geq (1 - c(A, B))(\|P_S(u)\|^2 + \|P_S(v)\|^2 + \|P_{S^\bot}(u + v)\|^2) \\
= (1 - c(A, B))(\|P_S(u)\|^2 + \|P_S(v) + P_{S^\bot}(u + v)\|^2).
$$

Now $P_S(u) \in A^\bot$ and $P_S(v) + P_{S^\bot}(u + v) \in B^\bot$, hence

$$
\min_{u' \in A^\bot, v' \in B^\bot, u' + v' = x} \|u'\|^2 + \|v'\|^2 \leq \|P_S(u)\|^2 + \|P_S(v) + P_{S^\bot}(u + v)\|^2 \\
\leq \frac{1}{1 - c(A, B)}\|x\|^2 \leq \frac{1}{1 - c(A, B)}.
$$

Since we chose $x \in (A \cap B)^\bot \cap B_X$ arbitrarily, we deduce

$$
\beta(A, B) \leq \frac{1}{1 - c(A, B)}; \\
\text{“more precisely”}: \text{Case 1:} A = B = X. \text{ Then } A^\bot = B^\bot = (A \cap B)^\bot = \{0\}, \text{ which yields } \beta(A, B) = 0. \text{ Case 2:} \text{ Proposition 6.14.(ii). } \text{ Case 3:} A \neq B. \text{ Note that } (A^\bot \cap B^\bot)^\bot = A + B. \\
\text{Since } c(A, B) = c(A^\bot, B^\bot) \text{ (again [9, Theorem 2.16]) and since } (A^\bot \cap (A + B) \cap B_X) \times (B^\bot \cap (A + B) \cap B_X) \text{ is compact, there exist } \bar{u} \in A^\bot \cap (A + B) \cap B_X \text{ and } \bar{v} \in B^\bot \cap (A + B) \cap B_X \text{ such that} \\
\langle -\bar{u}, \bar{v} \rangle = c(A^\bot, B^\bot) = c(A, B). \\
\text{Now } A \neq B. \text{ Hence, in view of Proposition 6.14.(i), we assume WLOG that} \\
\|\bar{u}\|^2 + \|\bar{v}\|^2 > 0.
$$

Suppose $u \in A^\bot$ and $v \in B^\bot$ are such that $u + v = \bar{u} + \bar{v}$. Then $u - \bar{u} = \bar{v} - v \in A^\bot \cap B^\bot = (A + B)^\bot$, hence $\langle \bar{u}, u - \bar{u} \rangle = 0$ and $\langle \bar{v}, v - \bar{v} \rangle = 0$. This implies $\|u\| \geq \|\bar{u}\|$ and $\|v\| \geq \|\bar{v}\|$ (expand $\|u\|^2 = \|(u - \bar{u}) + \bar{u}\|^2$). Thus

$$
\min_{u \in A^\bot, v \in B^\bot, u + v = \bar{u} + \bar{v}} \|u\|^2 + \|v\|^2 = \|\bar{u}\|^2 + \|\bar{v}\|^2.
$$

It follows that $\|\bar{u}\|^2 + \|\bar{v}\|^2 \leq \beta(A, B)\|\bar{u} + \bar{v}\|^2$; equivalently, $-2\beta(A, B)\langle \bar{u}, \bar{v} \rangle \leq (\beta(A, B) - 1)(\|\bar{u}\|^2 + \|\bar{v}\|^2)$. By definition of $\bar{u}$ and $\bar{v}$, we obtain

$$
2\beta(A, B)c(A, B) \leq (\beta(A, B) - 1)(\|\bar{u}\|^2 + \|\bar{v}\|^2).
$$

In the last inequality, the left-hand side is nonnegative and the factor $(\|\bar{u}\|^2 + \|\bar{v}\|^2)$ on the right-hand side is strictly positive, hence $\beta(A, B) \geq 1 > 0$. After dividing by $2\beta(A, B)$ and recalling that both $\bar{u}$ and $\bar{v}$ belong to $B_X$, we obtain

$$
c(A, B) \leq \frac{\beta(A, B) - 1}{2\beta(A, B)}(\|\bar{u}\|^2 + \|\bar{v}\|^2) \leq \frac{\beta(A, B) - 1}{\beta(A, B)}.
$$
equivalently, $\beta(A, B) \geq 1/(1 - c(A, B))$ (since $c(A, B) < 1$). ■

Theorem 6.15 allows us to write concisely: $c(A, B) = (1 - 1/\beta(A, B))^+$. We can now use the angle between the two subspaces to estimate the quantity in the definition of bounded linear regularity.

**Corollary 6.16** Suppose $A$ and $B$ are two subspaces of $X$. Then:

$$d(x, A \cap B) \leq \sqrt{\beta(A, B)}(d(x, A) + d(x, B)) \leq \frac{d(x, A) + d(x, B)}{\sqrt{1 - c(A, B)}}, \quad \forall x \in X.$$

**Proof.** Theorem 6.15 and Proposition 2.13. ■

**Remark 6.17** We do not know how Theorem 6.15 generalizes to more than two subspaces. Definition 6.11 has an obvious analog. As for the angle between more than two subspaces, one could employ the definition of the angle suggested in [5, Definition 3.7.5]. Finally, Theorem 6.15 generalizes to infinite-dimensional Hilbert space.

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**References**


