ON EQUVALENCE OF SOME BASIC PRINCIPLES IN VARIATIONAL ANALYSIS

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Abstract. The primary goal of this paper is to study relationships between certain basic principles of variational analysis and its applications to nonsmooth calculus and optimization. Considering a broad class of Banach spaces admitting smooth re-norms with respect to some bornology, we establish an equivalence between useful versions of a smooth variational principle for lower semicontinuous functions, an extremal principle for nonconvex sets, and an enhanced fuzzy sum rule formulated in terms of viscosity normals and subgradients with controlled ranks. Further refinements of the equivalence result are obtained in the case of a Fréchet differentiable renorm. Based on the new enhanced sum rule, we provide a simplified proof for the refined sequential description of approximate normals and subgradients in smooth spaces.

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1. INTRODUCTION

The paper concerns with some basic principles of variational analysis in Banach spaces. Variational analysis has been recognized as a fruitful area in mathematics that, on one hand, focuses on the study of optimization and related problems and, on the other hand, applies optimization, perturbation, and approximation ideas to the analysis of a broad range of problems which may not be of a variational nature. We refer the reader to the book of Rockafellar and Wets [34] for a systematic exposition of variational analysis in finite dimensions.

In this paper we deal with a large class of Banach spaces admitting a smooth renorm of any kind. For such spaces, fundamental tools of variational analysis and its applications are related to so-called smooth variational principles. Roughly speaking, the latter mean that for a given bounded below lower semicontinuous function there exists a smooth function, small in a certain sense, such that the sum of these two functions attains a local minimum at some point. The first smooth variational principle for general Banach spaces with smooth renorms was obtained by Borwein and Preiss [4]. In [5, 8, 11, 12, 15, 23, 32], one can find recent extensions and variants of this principle with further discussions and applications.

Another fruitful approach to nonsmooth calculus and optimization consists of using so-called extremal principles that can be viewed as extremal extensions of the classical separation theorem to nonconvex sets. This approach goes back to the beginning of dual-space methods in finite-dimensional variational analysis and directly provides necessary optimality conditions for nonsmooth constrained problems in terms of nonconvex normal cones and subdifferentials; see [25–27]. In infinite dimensions, a version of the extremal principle in terms of Fréchet ε-normals was first established by Kruger and Mordukhovich [22] for Banach spaces with Fréchet differentiable renorms. It was recently proved in [29] that this version is equivalent to the one in terms of Fréchet normals, holds in any Asplund space, and completely characterizes the Asplund property of a Banach space. Various variants and applications of the extremal principle can be found in [3, 5, 19, 21, 26–31, 34] and their references.

The third basic principle of variational analysis we address in this paper is related to subdifferential fuzzy sum rules. Results in this vein were first obtained by Ioffe for Dini and Fréchet ε-subdifferentials in corresponding smoothable spaces and their exact counterparts in finite dimensions; see [16, 17]. Note that we focus here not on a “full” fuzzy formula for subdifferentials of sums but on its special case concerning minimum points of a sum of two functions. In contrast to full fuzzy calculus, the latter result (sometimes called “zero” or “basic” fuzzy sum rule) is now known for most useful subdifferentials in corresponding Banach spaces; see [2, 5, 10, 14, 19, 20, 24, 31] for recent results, further references, and applications.

The main intention of this paper is to establish an equivalence between the mentioned basic principles in variational analysis under appropriate conditions. In has been already proved by Mordukhovich and Shao [29] that the extremal principle, formulated in terms
of Fréchet normals, is equivalent to the fuzzy sum rule for Fréchet subdifferentials in the case of Asplund spaces. Now we are going to verify that these rules actually transpire to be equivalent to a variant of the smooth variational principle in spaces with Fréchet differentiable renorms.

To obtain the equivalence result in more general settings, we need to employ appropriate constructions of normals and subgradients in non-Fréchet-smooth spaces. A natural generalization of the Fréchet subdifferential to smoothable spaces with respect to a weaker bornology \( \beta \) is the concept of viscosity \( \beta \)-subdifferentials \([6, 11]\) that goes back to the theory of viscosity solutions of Hamilton-Jacobi equations \([9]\). However, it turns out that viscosity \( \beta \)-subdifferentials and corresponding normal cones are too large to fit our purposes. A proper narrowing of these constructions was introduced by Borwein and Ioffe \([2]\) as \( \beta \)-normals and \( \beta \)-subdifferentials of \textit{controlled rank} (in the sense of viscosity) to provide refined representations for the approximate \( G \)-normal cone and \( G \)-subdifferential \([18]\) in spaces with \( \beta \)-smooth renorms. In what follows we use slight modifications of the subdifferential constructions in \([2]\) to establish an equivalence between the smooth variational principle and appropriate versions of the extremal principle and fuzzy sum rule in smoothable Banach spaces.

In this way we obtain new enhanced versions of the extremal principle and fuzzy sum rule in \( \beta \)-smooth spaces that are of independent interest for bornologies weaker than the Fréchet one. As an application of the enhanced fuzzy sum rule, we provide a new proof of the main representation results in \([2]\) that is a substantial simplification and clarification of the original one.

The paper is organized as follows. In Section 2 we study the constructions of \( \beta \)-normals and \( \beta \)-subdifferentials with controlled ranks. After providing the requisite preliminary material, we establish relationships between these constructions based on a new \( \beta \)-smooth version of the implicit function theorem. Section 3 is devoted to the main equivalence results and related discussions. In the concluding Section 4 we present some applications.

Our notation is standard. For any Banach space \( X \) we denote its norm by \( \| \cdot \| \) and the dual space by \( X^* \) with the canonical pairing \( \langle \cdot, \cdot \rangle \). As usual, \( B \) and \( B^* \) stand for the unit closed balls in the space and dual space in question. The symbol \( B_r(x) \) denotes the closed ball with center \( x \) and radius \( r \); \( \text{cl}^* \) signifies the weak-star topological closure in \( X^* \). In this paper we consider various multifunctions \( \Phi \) from \( X \) into the dual space \( X^* \). For such objects, the expression

\[
\limsup_{x \to \bar{x}} \Phi(x)
\]

always connotes the \textit{sequential} Kuratowski-Painlevé upper limit with respect to the norm topology in \( X \) and the weak-star topology in \( X^* \), i.e.,

\[
\limsup_{x \to \bar{x}} \Phi(x) := \{ x^* \in X^* | \exists \text{ sequences } x_k \to \bar{x} \text{ and } x_k \rightharpoonup x^* \text{ with } x_k^* \in \Phi(x_k) \text{ for all } k = 1, 2 \ldots \}.
\]

If \( \varphi : X \to \bar{\mathbb{R}} := [-\infty, \infty] \) is an \textit{extended-real-valued} function, then, as usual,

\[
\text{dom} \varphi := \{ x \in X \text{ with } |\varphi(x)| < \infty \}, \quad \text{epi} \varphi := \{(x, \mu) \in X \times \mathbb{R} | \mu \geq \varphi(x)\}.
\]
In this case, \( \limsup \varphi(x) \) and \( \liminf \varphi(x) \) denote the upper and lower limits of such (scalar) functions in the classical sense. Depending on context, the symbols \( x \xrightarrow{\varphi} \bar{x} \) and \( x \xrightarrow{\Omega} \bar{x} \) mean, respectively, that \( x \to \bar{x} \) with \( \varphi(x) \to \varphi(\bar{x}) \) and \( x \to \bar{x} \) with \( x \in \Omega \).

2. NORMALS AND SUBDIFFERENTIALS OF CONTROLLED RANK

Let \( X \) be a real Banach space. Recall that a bornology \( \beta \) on \( X \) is a family of bounded and centrally symmetric subsets of \( X \) whose union is \( X \), which is closed under multiplication by positive scalars and is directed upwards (i.e., the union of any two members of \( \beta \) is contained in some member of \( \beta \)). The most important bornologies are those formed by all bounded sets (the Fréchet bornology), weak compact sets (the weak Hadamard bornology), compact sets (the Hadamard bornology), and finite sets (the Gâteaux bornology); see [4, 12, 32] for more details and references.

Let us denote by \( X_\beta^* \) the dual space \( X^* \) of \( X \) endowed with the topology of uniform convergence on \( \beta \)-sets. It is well known that the latter convergence agrees with the norm convergence in \( X^* \) when \( \beta \) is the Fréchet bornology, and with the weak-star convergence in \( X^* \) when \( \beta \) is the Gâteaux bornology. In the cases of Hadamard and weak Hadamard bornologies, \( X_\beta^* \) corresponds to \( X^* \) endowed with the bounded weak-star and Mackey topologies, respectively.

Every bornology generates a certain concept of differentiability. Given an extended-real-valued function \( \varphi \) on \( X \) is said to be \( \beta \)-differentiable at \( x \in \text{dom} \varphi \) with \( \beta \)-derivative \( \nabla_\beta \varphi(x) \in X^* \) provided that

\[
  t^{-1}(\varphi(x + tv) - \varphi(x) - t(\nabla_\beta \varphi(x), v)) \to 0
\]

as \( t \to 0 \) uniformly in \( v \in V \) for every \( V \in \beta \). We say that \( \varphi \) is \( \beta \)-smooth around \( x \) if it is \( \beta \)-differentiable at each point of a neighborhood of \( x \) and \( \nabla_\beta \varphi : X \to X_\beta^* \) is continuous on this neighborhood. Clearly the \( \beta \)-smoothness of \( \varphi \) implies that \( \nabla_\beta \varphi : X \to X^* \) is continuous around \( x \) with respect to the norm topology on \( X \) and the weak-star topology on \( X^* \). When \( \beta = F \) is the Fréchet bornology, one has more: \( \nabla_F \varphi : X \to X^* \) is norm-to-norm continuous around \( x \). This is an essential specific characteristic of the Fréchet case used in the sequel.

Now we define the main constructions of generalized differentiation considered in this paper.

2.1. Definition. (i) Let \( \Omega \) be a nonempty subset of the Banach space \( X \). Given \( x \in \text{cl} \Omega \), we say that \( x^* \) is a \( \beta \)-normal of rank \( \lambda > 0 \) to \( \Omega \) at \( x \) if there exist a neighborhood \( U \) of \( x \) and a \( \beta \)-smooth function \( g : U \to \mathbb{R} \) such that \( g \) is Lipschitz continuous on \( U \) with modulus \( \lambda \), \( g(u) \leq 0 \) for all \( u \in \Omega \cap U \), \( g(x) = 0 \), and \( x^* = \nabla_\beta g(x) \). The collection of all these \( \beta \)-normals of controlled rank is denoted by \( N^\lambda_\beta(x; \Omega) \).

(ii) Let \( \varphi \) be an extended-real-valued function on \( X \). Its (viscosity) \( \beta \)-subdifferential of rank \( \lambda \) at \( x \in \text{dom} \varphi \) is the set \( \partial_\beta^\lambda \varphi(x) \) of all \( x^* \in X^* \) with the following properties:
there exist a neighborhood $U$ of $x$ and a $\beta$-smooth function $g : U \to \mathbb{R}$ such that $g$ is Lipschitz continuous on $U$ with modulus $\lambda$, $\nabla_\beta g(x) = x^*$, and $\varphi - g$ attains a local minimum at $x$.

Both notions in Definition 2.1 are slight modifications of the corresponding definitions proposed in [2] where the concept of $\beta$-smoothness does not require that $\nabla_\beta g : X \to X_\beta^*$ is continuous around $x$. The given continuity requirement seems to be important for some applications including those in this paper.

One can see that the union

$$\partial_\beta \varphi(x) := \bigcup_{\lambda > 0} \partial^\lambda_\beta \varphi(x) \quad (2.1)$$

reduces to the viscosity $\beta$-subdifferential of $\varphi$ at $x$ defined in [6] as a modification of the corresponding definition in [11] where $g$ is not required to be Lipschitzian. The latter requirement is essential for bornologies weaker than the Fréchet bornology since the $\beta$-smoothness of $g$ may not imply its Lipschitz continuity. This never happens in the case of $\beta = F$ due to the required norm-to-norm continuity of $\nabla_F g : X \to X^*$ that automatically ensures the Lipschitz continuity of $g$. Actually the usage of controlled ranks in the Fréchet case does not provide any advantages in comparison with the full viscosity subdifferential (2.1); see below Proposition 3.6 and related discussions. On the contrary, the controlled rank narrowing of the viscosity $\beta$-subdifferential is a crucial tool of our analysis in the general non-Fréchet case.

In this paper we need to consider $\beta$-subdifferentials of controlled rank for lower semicontinuous (l.s.c.) functions. It follows from Definition 2.1(ii) that we may always assume that $g(x) = \varphi(x)$. Also one can easily see that

$$\partial^\lambda_\beta \varphi(x, y) = (\partial^\lambda_\beta \varphi_1(x), \partial^\lambda_\beta \varphi_2(y)) \text{ for } \varphi(x, y) = \varphi_1(x) + \varphi_2(y). \quad (2.2)$$

As it was observed in [2], for the function $g$ in Definition 2.1(i) one has

$$g(u) \leq \lambda \text{ dist}(u; \Omega) \quad \text{for all } u \in U$$

where $\text{dist}(\cdot; \Omega)$ signifies the distance function associated with $\Omega$. The latter inequality allows us to derive the relationship

$$N^\lambda_{\beta}(x; \Omega) = \partial^\lambda_{\beta} \{ \lambda \text{ dist}(x; \Omega) \} = \lambda \partial^1_{\beta} \text{ dist}(x; \Omega) \quad \text{for all } x \in \Omega \quad (2.3)$$

used in the sequel. On the other hand, one always has the relationship

$$N^\lambda_{\beta}(x; \Omega) = \partial^\lambda_{\beta} \delta(x; \Omega) \quad \text{for all } x \in \Omega \quad (2.4)$$

involving the indicator function $\delta(\cdot; \Omega)$ of $\Omega$ which equals 0 if $x \in \Omega$ and $\infty$ otherwise. Note that the union

$$N_{\beta}(x; \Omega) := \bigcup_{\lambda > 0} N^\lambda_{\beta}(x; \Omega) \quad (2.5)$$
is a cone, called the $\beta$-normal cone to $\Omega$ at $x$, which relates to the $\beta$-subdifferential (2.1) as in (2.4).

It easy follows from (2.2) and (2.4) that

$$N^\lambda_\beta((x_1, x_2); \Omega_1 \times \Omega_2) = N^\lambda_\beta(x_1; \Omega_1) \times N^\lambda_\beta(x_2; \Omega_2)$$

for any $x_1 \in \Omega_1 \subset X_1$ and $x_2 \in \Omega_2 \subset X_2$.

For the proof of the main result in Section 3 we need more subtle relationships between $\beta$-normals and $\beta$-subdifferentials of controlled rank that involve epigraphs of l.s.c. functions. We establish them below in Proposition 2.3 based on the following $\beta$-smooth version of the implicit function theorem that is certainly of some independent interest.

**2.2. Theorem.** Assume that $X$ is a Banach space and $X \times \mathbb{R}$ is equipped with a norm which restricts to the original norm on the subspace $X$. Suppose further that $W$ is a neighborhood of $(x_0, \alpha_0) \in X \times \mathbb{R}$ and $f : W \to \mathbb{R}$ is $\beta$-smooth with the $\beta$-derivative $\nabla_\beta f = (\nabla_{\beta x} f, \nabla_{\beta \alpha} f)$ and Lipschitz continuous with modulus $\gamma$. Let $f(x_0, \alpha_0) = 0$ and $\nabla_{\beta \alpha} f(x_0, \alpha_0) \neq 0$. Then for any $\lambda > \gamma$ there exist a neighborhood $U$ of $x_0$, a positive number $\delta$, and a unique $\beta$-smooth and Lipschitz continuous function $g : U \to [\alpha_0 - \delta, \alpha_0 + \delta]$ with modulus $\lambda / |\nabla_{\beta \alpha} f(x_0, \alpha_0)|$, such that $f(x, g(x)) = 0$ for all $x \in U$ and

$$\nabla_\beta g(x) = - (\nabla_{\beta \alpha} f(x, g(x)))^{-1} \nabla_{\beta x} f(x, g(x))$$

for all $x \in U$. (2.6)

**Proof.** Let

$$\varphi(x, \alpha) := \alpha - \alpha_0 - \frac{f(x, \alpha)}{\nabla_{\beta \alpha} f(x_0, \alpha_0)}$$

for all $(x, \alpha) \in W$.

Due to the assumptions made, $\varphi$ is Lipschitz continuous on $W$ satisfying

$$\varphi(x_0, \alpha_0) = 0 \quad \text{and} \quad \nabla_{\beta \alpha} \varphi(x_0, \alpha_0) = 0.$$ (2.7)

Since $f : W \to \mathbb{R}$ is $\beta$-smooth, $\nabla_\beta f : W \to X'_\beta \times \mathbb{R}$ is continuous that implies the continuity of the partial derivative $\nabla_{\beta \alpha} f$ on $W$. Using the second equality in (2.7), one can find a neighborhood $U_1$ of $x_0$ and a number $\delta > 0$ such that $U_1 \times [\alpha_0 - \delta, \alpha_0 + \delta] \subset W$ and

$$|\nabla_{\beta \alpha} \varphi(x, \alpha)| < l$$

for all $x \in U_1$ and $|\alpha - \alpha_0| \leq \delta$

where $0 < l < 1 - (\gamma / \lambda)$ is a constant. Now let us construct a sequence $\{g_n\}$ of functions on $U$

$$g_1(x) := \alpha_0 + \varphi(x, \alpha_0) \quad \text{and} \quad g_n(x) := \alpha_0 + \varphi(x, g_{n-1}(x))$$

for all $x \in U$ and $n = 1, 2, \ldots$ (2.8)

First we justify by induction that there is a neighborhood $U \subset U_1$ of $x_0$ such that

$$|g_n(x) - \alpha_0| < \delta$$

for all $x \in U$ and $n = 1, 2, \ldots$ (2.9)

Using the first equality in (2.7) and continuity of $\varphi$ at $(x_0, \alpha_0)$, we find a neighborhood $U \subset U_1$ of $x_0$ with $|\varphi(x, \alpha_0)| < (1 - l)\delta$ on $U$. This gives $|g_1(x) - \alpha_0| < \delta$ for all $x \in U$.  

6
Now assuming that $|g_{n-1}(x) - \alpha_0| < \delta$ for all $x \in U$ and employing the classical mean value theorem, one has

$$
|g_n(x) - \alpha_0| = |\varphi(x, g_{n-1}(x)) - \varphi(x, \alpha_0)| + |\varphi(x, \alpha_0)|
= |\nabla_\alpha \varphi(x, \theta_1)(g_{n-1}(x) - \alpha_0)| + |\varphi(x, \alpha_0)| \quad (\theta_1 \text{ lies between } \alpha_0 \text{ and } g_{n-1}(x))
< l\delta + (1 - l)\delta = \delta.
$$

In this way we get (2.9) for all $n$ by induction.

Next let us show that $\{g_n\}$ converges uniformly in $U$. Again using the mean value theorem, we get

$$
|g_{n+1}(x) - g_n(x)| = |\varphi(x, g_n(x)) - \varphi(x, g_{n-1}(x))|
= |\nabla_\alpha \varphi(x, \theta_2)(g_n(x) - g_{n-1}(x))| \quad (\theta_2 \text{ lies between } g_{n-1}(x) \text{ and } g_n(x))
< l|g_n(x) - g_{n-1}(x)| \quad \text{for all } x \in U \text{ and } n = 1, 2, \ldots.
$$

Therefore, one has

$$
|g_{n+1}(x) - g_n(x)| < l|g_n(x) - g_{n-1}(x)| < \cdots < l^n|g_1(x) - \alpha| < l^n|\varphi(x, \alpha_0)| < l^n(1 - \gamma)\delta \quad \text{for all } x \in U \text{ and } n = 1, 2, \ldots
$$

that yields the uniform convergence of $\{g_n\}$ in $U$. Setting $g(x) := \lim_{n \to \infty} g_n(x)$ and passing to the limit in (2.8), we conclude that $g(x) = \alpha_0 + \varphi(x, g(x))$ for all $x \in U$. It follows from (2.9) that $\alpha_0 - \delta \leq g(x) \leq \alpha_0 + \delta$ and, moreover, $g(x_0) = \alpha_0$ since $g_n(x_0) = \alpha_0$ for all $n = 1, 2, \ldots$.

Now we verify that $g$ is Lipschitz continuous on $U$ with modulus $\lambda/|\nabla_\alpha \varphi(x_0, \alpha_0)|$. To this end we observe that the function $\varphi$ defined above is Lipschitz continuous in $x$ with modulus $\gamma/|\nabla_\alpha \varphi(x_0, \alpha_0)|$ on $U$. Picking any $x_1, x_2 \in U$ and employing again the mean value theorem, one has

$$
|g(x_1) - g(x_2)| \leq |\varphi(x_1, g(x_1)) - \varphi(x_2, g(x_1))| + |\varphi(x_2, g(x_1)) - \varphi(x_2, g(x_2))|
\leq (\gamma/|\nabla_\alpha \varphi(x_0, \alpha_0)|)|x_1 - x_2| + |\nabla_\alpha \varphi(x_2, \theta_3)| \cdot |g(x_1) - g(x_2)|
\quad (\theta_3 \text{ lies between } g(x_1) \text{ and } g(x_2))
< (\gamma/|\nabla_\alpha \varphi(x_0, \alpha_0)|)|x_1 - x_2| + l|g(x_1) - g(x_2)|.
$$

This gives $|g(x_1) - g(x_2)| \leq (\lambda/|\nabla_\alpha \varphi(x_0, \alpha_0)|)|x_1 - x_2|$ and justifies the desired Lipschitz property.

Next let us show that $\alpha = g(x)$ is a unique solution to the equation $\alpha = \alpha_0 + \varphi(x, \alpha)$ as $x \in U$. Arguing by contradiction, we assume that there exist two solutions $g$ and $h$ on $U$ satisfying

$$
\alpha_0 - \delta \leq g(x) \leq \alpha_0 + \delta \quad \text{and} \quad \alpha_0 - \delta \leq h(x) \leq \alpha_0 + \delta.
$$
Then one has
\[
|g(x) - h(x)| = \left| [\alpha_0 + \varphi(x, g(x))] - [\alpha_0 + \varphi(x, h(x))] \right| = |\varphi(x, g(x)) - \varphi(x, h(x))| \\
\leq |\nabla_{\beta_0} \varphi(x, \theta_4)| \cdot |g(x) - h(x)| \leq \epsilon |g(x) - h(x)|
\]
for all \( x \in U \) and \( \theta_4 \) lies between \( g(x) \) and \( h(x) \).

that is a contradiction unless \( g(x) = h(x) \) for all \( x \in U \).

Finally we prove that \( g \) is \( \beta \)-smooth on \( U \) and (2.6) holds. Taking arbitrary \( \epsilon > 0 \), \((x, \alpha) \in W \), and \( V \in \beta \) with \( 0 \in V \subset X \) and using the \( \beta \)-differentiability of \( f \) on \( W \), we find \( \nu > 0 \) such that
\[
|f(x + h, \alpha + \xi) - f(x, \alpha) - \nabla_{\beta_x} \varphi(x, \alpha) h - \nabla_{\beta_0} \varphi(x, \alpha) \xi| \leq \epsilon \|h, \xi\| \tag{2.10}
\]
for any \( h \in V \) and \( \|h, \xi\| \leq \nu \). Let us substitute \( \alpha = g(x) \) and \( \xi = g(x + h) - g(x) \) into (2.10) and remember that \( f(x + h, g(x + h)) = f(x, g(x)) = 0 \) whenever \( x \in U \) and \( h \) is sufficiently small. Then (2.10) yields
\[
|\nabla_{\beta_0} f(x, \alpha)| \cdot |g(x + h) - g(x)| + \frac{\nabla_{\beta_x} f(x, g(x)) h}{\nabla_{\beta_0} f(x, g(x))} \\
= |\nabla_{\beta_x} f(x, g(x)) h + \nabla_{\beta_0} f(x, g(x)) [g(x + h) - g(x)]| \\
= |f(x + h, g(x + h)) - f(x, g(x)) + \nabla_{\beta_x} f(x, g(x)) [g(x + h) - g(x)]| \\
\leq \epsilon \|h, |g(x + h) - g(x)|\| \leq \epsilon \|(h, \lambda \|\nabla_{\beta_0} f(x, \alpha)\| \|h\|\|
\]
for all \( x \in U \) and small \( h \in V \). The latter implies that \( g \) is \( \beta \)-differentiable at each point \( x \in U \) and (2.6) holds. The continuity of \( \nabla_{\beta} g : U \to X^*_\beta \) follows directly from (2.6) due to the \( \beta \)-smoothness of \( f \). This is all we need to prove in the theorem. \( \square \)

Now we are ready to establish important relationships between \( \beta \)-normals and \( \beta \)-subdifferentials of controlled rank. Part (ii) of the following proposition provides a substantial impact to the proof of the main theorem in the next section.

2.3. Proposition. Let \( X \) be a Banach space and let \( \varphi : X \to (-\infty, \infty] \) be l.s.c. around \( x \in \operatorname{dom} \varphi \). Then one has:
(i) \( \partial_{\beta}^\lambda \varphi(x) \subseteq \{ x^* \in X^* \mid (x^*,-\lambda) \in N_{\lambda}^\gamma((x,\varphi(x));\operatorname{epi} \varphi) \} \) where \( \gamma := \max\{\lambda, 1\} \) and \( X \times \mathbb{R} \) is equipped with the norm \( \|(x,r)\| := \|x\| + |r| \).
(ii) \( \{ x^* \in X^* \mid (x^*,-\lambda) \in N_{\lambda}^\gamma((x,\varphi(x));\operatorname{epi} \varphi) \} \subseteq \partial_{\beta}^\lambda \varphi(x) \) for any \( \lambda \geq \gamma \), whenever \( X \times \mathbb{R} \) is equipped with a norm which restricts to the original norm on the subspace \( X \).

Proof. (i) Let \( x^* \in \partial_{\beta}^\lambda \varphi(x) \) for any given \( \lambda > 0 \). According to Definition 2.1(ii) we find a neighborhood \( U \) of \( x \) and a \( \beta \)-smooth function \( g : U \to \mathbb{R} \) such that \( g \) is Lipschitz continuous on \( U \) with modulus \( \lambda \), \( g(x) = \varphi(x) \), \( x^* = \nabla_{\beta} g(x) \), and \( \varphi(u) - g(u) \) attains a local minimum at \( x \). Considering the function
\[
m(u,\alpha) := g(u) - \alpha \quad \text{for} \quad u \in U \quad \text{and} \quad \alpha \in \mathbb{R},
\]

\[\]
one can easily observe that \( m \) is Lipschitz continuous around \((x, \varphi(x))\) with modulus \( \gamma = \{\lambda, 1\} \). Moreover, \( m \) is \( \beta \)-smooth around \((x, \varphi(x))\) and one has \((x^*, -1) = \nabla_- m(x, \varphi(x))\). Furthermore, it is evident that \( m(u, \alpha) \leq 0 \) for all \((u, \alpha) \in \text{epi} \varphi \cap (U \times \mathbb{R})\). Due to Definition 2.1(i) we arrive at the inclusion \((x^*, -1) \in N^\beta_m((x, \varphi(x)); \text{epi} \varphi)\) which completes the proof of part (i) of the proposition.

To prove (ii), we assume that \((x^*, -1) \in N^\beta_m((x, \varphi(x)); \text{epi} \varphi)\) for any given \( \gamma > 0 \). By definition there are a neighborhood \( W \) of \((x, \varphi(x))\) and a \( \beta \)-smooth function \( f : W \to \mathbb{R} \) such that \( f \) is Lipschitz continuous on \( W \) with modulus \( \gamma \) and one has the properties:

\[
\begin{align*}
(a) & \quad f(u, \alpha) \leq 0 \text{ if } (u, \alpha) \in W \cap \text{epi } f; \\
(b) & \quad f(x, \varphi(x)) = 0; \\
(c) & \quad x^* = \nabla_x f(x, \varphi(x)) \text{ and } -1 = \nabla_{\alpha} f(x, \varphi(x)).
\end{align*}
\]

Now taking any \( \lambda > \gamma \) and employing Theorem 2.2, we find a neighborhood \( U \) of \( x \), a positive number \( \delta \), and a \( \beta \)-smooth function \( g : U \to [\varphi(x) - \delta, \varphi(x) + \delta] \) such that \( g \) is Lipschitz continuous on \( U \) with modulus \( \lambda \), \( g(x) = \varphi(x), x^* = \nabla_x g(x), \) and \( f(u, g(u)) = 0 \) for all \( u \in U \).

Since \( \nabla_{\alpha} f(x, \varphi(x)) = -1 \) and \( f \) is \( \beta \)-smooth around \((x, \varphi(x))\), we may assume that \( f(u, \alpha) \) is strictly decreasing in \( \alpha \) near \( \varphi(x) \) for each \( u \in U \). This yields

\[
g(u) = \sup\{\alpha \mid f(u, \alpha) > 0\} \quad \forall u \in U. \tag{2.11}
\]

It follows from condition (a) that \( \varphi(u) \geq \alpha \) if \( f(u, \alpha) > 0 \). The latter implies \( g(u) \leq \varphi(u) \) for all \( u \in U \) which is the last step to justify \( x^* \in \partial^\beta \varphi(x). \) This completes the proof of the proposition. \( \square \)

3. MAIN RESULTS

In this section we establish the main results of the paper providing an equivalence between appropriate versions of the smooth variational principle, the extremal principle, and the fuzzy sum rule in Banach spaces with bornologically smooth renorms. Given a bornology \( \beta \) on the Banach space \( X \), we say that \( X \) is \( \beta \)-smooth if it admits an equivalent norm which is \( \beta \)-differentiable away from the origin. It is well known that many spaces important for applications to optimization and variational analysis happen to be \( \beta \)-smooth with respect to some bornology. In particular, every reflexive space is Fréchet-smooth and every separable space is Hadamard-smooth (equivalently Gâteaux-smooth). Considering further \( \beta \)-smooth Banach spaces, we always deal with \( \beta \)-differentiable norms on them.

To formulate the principal equivalence result below, we need to recall the concept of set extremality introduced in [22]. Given closed sets \( \Omega_1 \) and \( \Omega_2 \) with a common point \( \bar{x} \in \Omega_1 \cap \Omega_2 \) in a Banach space \( X \), we say that \( \bar{x} \) is a locally extremal point of the set system \( \{\Omega_1, \Omega_2\} \) if for any \( \varepsilon > 0 \) there are \( b \in X \) with \( \|b\| < \varepsilon \) and a neighborhood \( U \) of \( \bar{x} \) such that \((\Omega_1 + b) \cap \Omega_2 \cap U = \emptyset \).
It is clear that any boundary point $\bar{x}$ of a closed set $\Omega$ is a locally extremal point of the pair $\{\Omega, \bar{x}\}$. There are also close connections between extremality and separability of set systems. Furthermore, the given geometric concept of extremality covers conventional notions of optimal solutions to general constrained problems in both scalar and vector optimization; see [3, 21, 22, 26–29] for various examples, discussions, and more references. By “extremal principle” we mean necessary conditions for local extremal points of set systems which can be treated as generalized Euler equations in an abstract geometric setting and provide a proper extremal analog of the separation theorem for nonconvex sets. In the main result as follows we present a variant of the extremal principle in $\beta$-smooth spaces which is proved to be equivalent to appropriate versions of both smooth variational principle and enhanced fuzzy sum rule.

3.1. Theorem. The following principles hold in any $\beta$-smooth spaces $X$ and are equivalent to each other:

(i) (smooth variational principle) Let $f : X \to (-\infty, \infty]$ be a l.s.c. function bounded from below. Suppose that $\varepsilon > 0$ and $\bar{x} \in X$ are given satisfying

$$f(\bar{x}) < \inf_X f + \varepsilon. \quad (3.1)$$

Then for any $\gamma > 0$ one can find $x_0 \in X$ and a function $g : X \to (-\infty, \infty]$, which is $\beta$-smooth and Lipschitz continuous around $x_0$ with modulus $\lambda = \varepsilon / \gamma$, such that

(a) $\|x_0 - \bar{x}\| < \gamma$;
(b) $f(x_0) < \inf_X f + \varepsilon$;
(c) $f + g$ attains a local minimum at $x_0$.

(ii) (extremal principle) Let $\bar{x} \in \Omega_1 \cap \Omega_2 \subset X$ be a locally extremal point of the closed set system $\{\Omega_1, \Omega_2\}$. Then for any positive numbers $\varepsilon$, $\nu$, and $\xi$ there exist $x_i \in \Omega_i \cap B_\varepsilon(\bar{x})$ and $x_i^* \in N^\lambda_\beta(x_i; \Omega_i) + \varepsilon B^*$ with $\lambda = (\nu + \xi)/2$, $i = 1, 2$, such that

$$\|x_i^*\| + \|x_i^*\| = \nu \quad \text{and} \quad x_i^* + x_i^* = 0. \quad (3.2)$$

(iii) (enhanced fuzzy sum rule) Let $\varphi_i : X \to \mathbb{R}$, $i = 1, 2$, be l.s.c. functions finite at $\bar{x}$. Suppose that $\varphi_i$ is Lipschitz continuous around $\bar{x}$ with modulus $l$ and that $\varphi_1 + \varphi_2$ attains a local minimum at $\bar{x}$. Then for any $\varepsilon > 0$, $\delta > 0$, and $\lambda > l$ there exist $x_i \in B_\varepsilon(\bar{x})$ with $|\varphi_i(x_i) - \varphi_i(\bar{x})| \leq \delta$, $i = 1, 2$, such that

$$0 \in \partial^\lambda_\beta \varphi_1(x_1) + \partial^\lambda_\beta \varphi_2(x_2) + \varepsilon B^*. \quad (3.3)$$

Proof. The smooth variational principle in (i) follows from the variational principle of Borwein and Preiss [4] in any $\beta$-smooth space. Our goal is to show that (i) is equivalent to each of the other principles (ii) and (iii) with the same type of bornology. Let us prove the theorem following the scheme (i)$\implies$(ii)$\implies$(iii)$\implies$(i).
First we show that the smooth variational principle in (i) implies the extremal principle in (ii). Let $\bar{x}$ be a locally extremal point of the set system $\{\Omega_1, \Omega_2\}$. According to the definition, for any positive numbers $\nu$, $\xi$, and $\varepsilon$ given in (ii) we take $0 < \varepsilon' < \min\{\varepsilon, \xi\}/2$ and select $b \in X$ such that

$$\nu\|b\| < 2(\varepsilon')^2 \quad \text{and} \quad (\Omega_1 + b) \cap \Omega_2 \cap U = \emptyset$$

for some neighborhood $U$ of $\bar{x}$. On localizing, one may always assume that $U = X$. Thus considering a $\beta$-smooth norm $\| \cdot \|$ on $X$ and setting the function

$$\varphi(u, v) := (\nu/2)\|u - v + b\|, \quad (3.4)$$

we conclude that $\varphi(u, v) > 0$ for any $u \in \Omega_1$ and $v \in \Omega_2$ with $\varphi(\bar{x}, \bar{x}) < (\varepsilon')^2$.

Let us endow the Cartesian product $X \times X$ with the norm $\|(u, v)\| := (\|u\|^2 + \|v\|^2)^{1/2}$. This norm is obviously differentiable away from the origin with respect to the product bornology on $X \times X$ generated by $\beta$. Using the same symbol $\beta$ for this product bornology, we see that the Banach space $X \times X$ is $\beta$-smooth. Since the smooth variational principle of (i) holds in any bornologically smooth space, we can apply it to the extended-real-valued function

$$f(u, v) := \varphi(u, v) + \delta((u, v); \Omega_1 \times \Omega_2)$$

on $X \times X$ involving the indicator function $\delta(\cdot; \Omega_1 \times \Omega_2)$ of the set $\Omega_1 \times \Omega_2$. Taking into account that

$$f(\bar{x}, \bar{x}) < \inf_{X \times X} f + (\varepsilon')^2$$

and employing (i) with the localization constant $\gamma = \varepsilon'$, we find a point $(x_1, x_2) \in \Omega_1 \times \Omega_2$, a convex neighborhood $W$ of $(x_1, x_2)$, and a $\beta$-smooth function $\Delta : W \to \mathbb{R}$ such that $\|x_i - \bar{x}\| < \varepsilon'$, $\Delta$ is Lipschitz continuous on $W$ with modulus $\varepsilon' < \xi/2$, and $f + \Delta$ attains a local minimum at $(x_1, x_2)$. Due to $\varepsilon' < \varepsilon/2$ one gets

$$\|\nabla_\beta \Delta(x_1, x_2)\| \leq \varepsilon' < \varepsilon/2 \quad \text{and} \quad \|x_i - \bar{x}\| < \varepsilon/2 \quad \text{for} \quad i = 1, 2. \quad (3.5)$$

Let us further observe that since the norm function (3.4) never vanishes on $\Omega_1 \times \Omega_2$, it is $\beta$-differentiable at the point $(x_1, x_2)$ and its neighborhood $W$ and, moreover,

$$\nabla_\beta \varphi(x_1, x_2) = (\nu/2)(\hat{x}_*, -\hat{x}_*) \quad \text{for} \quad \hat{x}_* \in X^* \quad \text{with} \quad \|\hat{x}_*\| = 1. \quad (3.6)$$

Taking into account that $\varphi$ is convex on $W$, we conclude that $\varphi$ is $\beta$-smooth around $(x_1, x_2)$. Therefore, the sum function $\varphi + \Delta$ is $\beta$-smooth around $(x_1, x_2)$ and Lipschitz continuous on $W$ with modulus $\lambda = (\nu + \xi)/2$ generated by modulus $\nu/2$ for $\varphi$ in (3.4) and modulus $\varepsilon' < \xi/2$ for $\Delta$ from the above. Moreover, the function

$$(\varphi(u, v) + \Delta(u, v)) + \delta((u, v); \Omega_1 \times \Omega_2)$$

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attains a local minimum at \((x_1, x_2)\). Without loss of generality one can always assume that 
\(\varphi(x_1, x_2) + \Delta(x_1, x_2) = 0\). By virtue of Definition 2.1(ii) and relationships (2.4) and (2.2) we have

\[
\nabla_{\beta}(-\varphi - \Delta)(x_1, x_2) \in N^\lambda_{\beta}(x_1, x_2; \Omega_1 \times \Omega_2)
\]

\[
= N^\lambda_{\beta}(x_1; \Omega_1) \times N^\lambda_{\beta}(x_2; \Omega_2).
\]

Letting \(x_1^* := -\nu/2\tilde{x}^*\) and \(x_2^* := \nu/2\tilde{x}^*\), we obtain from (3.5)–(3.7) that

\[
x_i^* \in N^\lambda_{\beta}(x_i; \Omega_i) + \varepsilon B^*, \quad i = 1, 2.
\]

This gives (3.2) and ends the proof of (i) \(\Rightarrow\) (ii).

Next we are going to prove that (ii) \(\Rightarrow\) (iii). Given numbers \(l\) and \(\lambda\) in (iii), we pick

\(0 < a < 1\) with \(a\lambda > l\) and denote \(p := a\lambda - l\). Now let us define a norm on \(X \times \mathbb{R}\) by

\[
\|(x, r)\| := (\|x\|^2 + p^{-2}r^2)^{\frac{1}{2}}.
\]

Clearly this norm satisfies the norm assumption in Proposition 2.3(ii). Moreover, the dual space \((X \times \mathbb{R})^*\) can be identified with \(X^* \times \mathbb{R}\) by the pairing \(\langle (x^*, r^*), (x, r) \rangle := \langle x^*, x \rangle + r^* \cdot r\) that generates the norm

\[
\|(x^*, r^*)\| = (\|x^*\|^2 + p^2(r^*)^2)^{\frac{1}{2}}
\]

on \((X \times \mathbb{R})^*\) essential in the sequel.

Fix numbers \(\varepsilon > 0\) and \(\delta > 0\) in (iii) and suppose, without loss of generality, that \(\varphi_1\) is Lipschitz continuous on \(B_\delta(\bar{x})\) with modulus \(l\). For the remainder we choose a number \(\epsilon > 0\) satisfying

\[
\epsilon < \min \left\{ \frac{3}{8(a\lambda + 1)}, \frac{1 - a}{2a(1 + \lambda) + 2}, \frac{\varepsilon p}{8a\lambda(p + 8a\lambda)}, \delta, \frac{\delta}{p} \right\}.
\]

Assume for simplicity that \(\bar{x} = 0\) is a local minimizer for \(\varphi_1 + \varphi_2\) with \(\varphi_1(0) = \varphi_2(0) = 0\). We define closed sets \(\Omega_1\) and \(\Omega_2\) as

\[
\Omega_1 := \text{epi } \varphi_1 \quad \text{and} \quad \Omega_2 := \{ (x, \alpha) \in X \times \mathbb{R} \mid \varphi_2(x) \leq -\alpha \}.
\]

Then one can easily check that \((0, 0)\) is a locally extremal point of the set system \(\{\Omega_1, \Omega_2\}\) in the space \(X \times \mathbb{R}\). Now we are going to apply the extremal principle in (ii) to this system. First we observe that if \(\| \cdot \|\) is a \(\beta\)-smooth norm on \(X\), then (3.8) provides a smooth norm on \(X \times \mathbb{R}\) with respect to the bornology consisting of all sets \(V \times [-s, s]\) where \(V \in \beta\) and \(s > 0\). Since the extremal principle of (ii) holds in an arbitrary bornologically smooth space, we apply it to the given system with \(\epsilon\) chosen in (3.10), \(\nu = 1\), and \(\xi = 2\epsilon\). This gives us \((x_i, \mu_i) \in \Omega_i, x_i^* \in X^*\), and \(\eta_i \in \mathbb{R}\) such that

\[
\|(x_i, \mu_i)\| \leq \epsilon \quad \text{for} \quad i = 1, 2,
\]

(3.11)
\[
1/2 - \epsilon \leq \| (x_1^*, \eta_1) \| \leq 1/2 + \epsilon, \quad 1/2 - \epsilon \leq \| (x_2^*, \eta_2) \| \leq 1/2 + \epsilon,
\]
\[
(x_1^*, -\eta_1) \in N^{\frac{1}{2} + \epsilon}_\beta((x_1, \mu_1)); \Omega_1), \quad (-x_2^*, \eta_2) \in N^{\frac{1}{2} + \epsilon}_\beta((x_2, \mu_2)); \Omega_2),
\]
\[
\text{and} \quad \| (x_1^*, -\eta_1) + (-x_2^*, \eta_2) \| \leq \epsilon.
\]

It easily follows from (3.13) and the structure of \( \Omega_i \) that \( \eta_i \geq 0 \) for \( i = 1, 2 \). Our goal is to prove that actually \( \eta_i > 0 \) and, moreover, \( \mu_1 = \varphi_1(x_1) \) and \( \mu_2 = -\varphi_2(x_2) \). To furnish this, we employ Definition 2.1(i) in (3.13) and find a neighborhood \( U \) of \( (x_1, \mu_1) \) and a \( \beta \)-smooth function \( g : U \to \mathbb{R} \) such that \( g(x_1, \mu_1) = 0, \) \( g \) is Lipschitz continuous on \( U \) with modulus \( \frac{1}{2} + \epsilon, \) \( g(x, \mu) \leq 0 \) for all \( (x, \mu) \in \Omega_1 \cap U, \) and \( \nabla g(x_1, \mu_1) = (x_1^*, -\eta_1) \). The latter implies that for any \( \alpha > 0 \) and \( (h, l) \in X \times \mathbb{R} \) with \( \| h \| = 1 \) there exists \( t_0 > 0 \) such that
\[
\frac{g(x_1 + th, \mu_1 + tl)}{t} - \langle (x_1^*, -\eta_1), (h, l) \rangle < -\alpha
\]
whenever \( 0 < t < t_0, \) where \( l \) is the given Lipschitz modulus of \( \varphi_1 \) in (iii). Since
\[
\varphi_1(x_1 + th) \leq lt + \mu_1,
\]
one has \( g(x_1 + th, \mu_1 + tl) \leq 0 \) whenever \( t \) is sufficiently small. This fact and (3.13) yield
\[
\langle x_1^*, h \rangle < \alpha \quad \text{for all} \quad h \in X \quad \text{with} \quad \| h \| = 1
\]
that gives \( \langle x_1^*, h \rangle \leq \alpha \eta_1 \) for all such \( h \) since \( \alpha > 0 \) is arbitrary. In this way we arrive at
\[
\| x_1^* \| \leq l \eta_1.
\]
Now using (3.9), (3.16), and the definition of \( p \) above, one has
\[
\| (x_1^*, \eta_1) \| \leq l \eta_1 + p \eta_1 = l \eta_1 + (a \lambda - l) \eta_1 = a \lambda \eta_1.
\]
The latter implies the estimates
\[
\eta_1 \geq \frac{1}{2} \epsilon \frac{1}{a \lambda} > \frac{1}{8a \lambda} > 0
\]
due to the first inequality in (3.12) with \( \epsilon < 3/8 \) in (3.10). Moreover, from (3.14), (3.17), and the choice of \( \epsilon < 3/8(a \lambda + 1) \) in (3.10) we get the following estimates for \( \eta_2 \):
\[
\eta_2 \geq \eta_1 - \epsilon \geq \frac{1}{2} \epsilon \frac{1}{a \lambda} - \epsilon > \frac{1}{8a \lambda} > 0.
\]

Now we are able to show that \( \mu_1 = \varphi_1(x_1) \) and \( \mu_2 = -\varphi_2(x_2) \). Let us do it only for the first case since the other one is symmetric. Assuming that \( \mu_1 > \varphi_1(x_1) \) and picking any \( \alpha > 0 \), one can find, similarly to (3.15), a positive number \( t_1 < (\mu_1 - \varphi_1(x_1))/2 \) such that
\[
\frac{g(x_1, \mu_1 - t)}{t} - \langle (x_1^*, -\eta_1), (0, -1) \rangle > -\alpha
\]
whenever $0 < t < t_1$. Taking into account that $\varphi_1(x_1) < \mu_1 - t$, we get $g(x_1, \mu_1 - t) \leq 0$ if $t$ is sufficiently small. Then (3.19) implies $\eta_1 < \alpha$ that contradicts (3.17). Therefore, the inclusions in (3.13) can be written as follows

\[ (x_1^*/\eta_1, -1) \in \mathcal{N}_{\beta}(\frac{1}{2} + \epsilon)/\eta_1((x_1, \varphi_1(x_1)); \text{epi } \varphi_1), \]  

\[ (-x_2^*/\eta_2, -1) \in \mathcal{N}_{\beta}(\frac{1}{2} + \epsilon)/\eta_2((x_2, \varphi_2(x_2)); \text{epi } \varphi_2) \]  

due to the structure of the sets $\Omega_i$, $i = 1, 2$, and the definition of $\beta$-normals with controlled rank. Denote by

\[ \gamma := \max \left\{ \left( \frac{1}{2} + \epsilon \right)/\eta_1, \left( \frac{1}{2} + \epsilon \right)/\eta_2 \right\} \]  

and observe that

\[ \frac{1}{\eta_i} \leq \frac{1}{(\frac{1}{2} - \epsilon)\frac{1}{\alpha\lambda} - \epsilon} \quad \text{for } i = 1, 2 \]  

by virtue of (3.17) and (3.18). Therefore, one has the estimates

\[ \gamma \leq \frac{(1 + 2\epsilon)a\lambda}{1 - 2\epsilon(1 + a\lambda)} < \lambda \]  

since $\epsilon < (1 - a)/(2a(1 + \lambda) + 2)$ in (3.10). Now employing Proposition 2.3(ii), we deduce from (3.20)–(3.23) that

\[ \tilde{x}_1^* := x_1^*/\eta_1 \in \partial_{\beta}^\lambda \varphi_1(x_1) \quad \text{and} \quad \tilde{x}_2^* := -x_2^*/\eta_2 \in \partial_{\beta}^\lambda \varphi_2(x_2). \]  

To establish (3.3), we need to estimate $\|\tilde{x}_1^* + \tilde{x}_2^*\|$. One immediately has

\[ \|\tilde{x}_1^* + \tilde{x}_2^*\| = \left\| \frac{x_1^*}{\eta_1} - \frac{x_2^*}{\eta_2} \right\| \leq \left\| \frac{x_1^* - x_2^*}{\eta_1} \right\| + \frac{\eta_2}{\eta_1} \cdot \frac{\eta_2}{\eta_1} \cdot \|x_2^*\|. \]  

To proceed, let us first observe that $\|x_2^*\| < 1$ due to the last inequality in (3.12) and $\epsilon < 3/8$ in (3.10). Furthermore, it follows from (3.14) and the form of dual norm (3.9) that

\[ \|x_1^* - x_2^*\| \leq \epsilon \quad \text{and} \quad |\eta_2 - \eta_1| \leq \epsilon/p. \]  

Now using (3.17), (3.18), (3.25), (3.26), and the estimate

\[ \epsilon < \frac{\epsilon p}{8a\lambda(p + 8a\lambda)} \]  

in (3.10), we conclude that $\|\tilde{x}_1^* + \tilde{x}_2^*\| < \epsilon$. This justifies the fuzzy sum rule inclusion (3.3).

To finish the proof of (ii) $\Rightarrow$ (iii), it remains to show that

\[ \|x_i - \tilde{x}\| \leq \delta \quad \text{and} \quad |\varphi_i(x_i) - \varphi_i(\tilde{x})| \leq \delta \quad \text{for } i = 1, 2 \]  

(3.27)
where \( \delta > 0 \) is given in (iii), \( \bar{x} = 0 \), and \( \varphi_1(0) = \varphi_2(0) = 0 \). Using the norm definition in (3.8) and the choice of \( \epsilon < \min \{\delta, \delta/p\} \) in (3.10), we deduce (3.27) directly from (3.11) with \( |\mu_i| = |\varphi_i(x_i)| \) for \( i = 1, 2 \).

Finally let us prove the implication (iii) \( \Rightarrow \) (i) in Theorem 3.1. Given \( \varepsilon \) and \( \bar{x} \) satisfying (3.1), one always has \( \nu \in (0, \varepsilon) \) such that

\[
f(\bar{x}) < \inf_X f + (\varepsilon - \nu).
\]

(3.28)

Now we employ Ekeland’s variational principle [13] in (3.28) with the localization constant \( \gamma \) given in (i). According to [13] there exists \( x_1 \in X \) satisfying the properties

\[
f(x_1) < \inf_X f + (\varepsilon - \nu), \quad \|x_1 - \bar{x}\| < \gamma, \quad \text{and}
\]

\[
f(x_1) < f(x) + \frac{(\varepsilon - \nu)}{\gamma}\|x - x_1\| \quad \text{for all } x \in X \text{ with } x \neq x_1.
\]

(3.29)

(3.30)

Observe that the function

\[
\varphi(x) := \frac{(\varepsilon - \nu)}{\gamma}\|x - x_1\|
\]

is Lipschitz continuous on \( X \) with modulus

\[
l := \frac{\varepsilon - \nu}{\gamma} < \frac{\varepsilon}{\gamma}
\]

(3.31)

and the sum \( f + \varphi \) attains its minimum at \( x_1 \). Employing the enhanced sum rule (iii) with

\[
0 < \delta < \min \{\gamma - \|x_1 - \bar{x}\|, \nu\},
\]

(3.32)

we find \( x_0 \in X \) such that

\[
\|x_0 - x_1\| \leq \delta, \quad |f(x_0) - f(x_1)| \leq \delta, \quad \text{and}
\]

\[
\partial_\beta f(x_0) \neq \emptyset \quad \text{with} \quad \lambda := \varepsilon/\gamma
\]

(3.33)

(3.34)

since \( \lambda > l \) in (3.31). It follows from (3.29), (3.33), and the choice of \( \delta \) in (3.32) that

\[
\|x_0 - \bar{x}\| \leq \|x_1 - \bar{x}\| + \delta < \gamma \quad \text{and}
\]

\[
f(x_0) \leq f(x_1) + \delta < \inf_X f + \varepsilon,
\]

i.e., one gets both properties (a) and (b) in (i). Furthermore, using (3.34) and applying Definition 2.1(ii) to \( -x^* \in \partial_\beta f(x_0) \), we have a function \( g : X \to (-\infty, \infty] \), which is \( \beta \)-smooth and Lipschitz continuous around \( x_0 \) with modulus \( \lambda = \varepsilon/\gamma \), such that (c) holds.
This justifies the implication \((iii) \implies (i)\) and completes the proof of the theorem. \(\square\)

3.2. Remark. Theorem 3.1 establishes that the three basic principles under consideration are equivalent in the class of all bornologically smooth Banach spaces. Given a Banach space \(X\) smooth with respect to the underlying bornology \(\beta\), we actually prove the following:

(a) the smooth variational principle \((i)\) in \(X \times X\) equipped with the product bornology implies the extremal principle \((ii)\) in \(X\);

(b) the extremal principle in \(X \times \mathbb{R}\) equipped with the product bornology implies the enhanced fuzzy sum rule \((iii)\) in \(X\);

(c) the enhanced fuzzy sum rule \((iii)\) in \(X\) implies the smooth variational principle \((i)\) in the same space \(X\).

Since \(X \times \mathbb{R}\) is a subspace of \(X^2\), we can conclude that the three principles in Theorem 3.1 are equivalent being considered in product spaces \(X^{2n}\) for all \(n = 1, 2, \ldots\).

3.3. Remark. One can observe that the proof of \((i) \implies (ii)\) in Theorem 3.1 does not use property \((b)\) in \((i)\). On the other hand, we verify the implications \((ii) \implies (iii) \implies (i)\) including this property. Therefore, the smooth variational principle \((i)\) with property \((b)\) is equivalent to \((i)\) without this property.

3.4. Remark. The proof of Theorem 3.1 given above allows us to obtain an analog of the equivalence result with a convex \(\beta\)-smooth and Lipschitz continuous function \(g(\cdot)\) in the variational principle \((i)\). To have the corresponding extremal principle and enhanced fuzzy sum rule equivalent to such a convex smooth variational principle, we need to modify Definition 2.1 of \(\beta\)-normals and \(\beta\)-subdifferentials of controlled rank. One can observe from the proof of Theorem 3.1 that all we need for this purpose is to require supporting functions in both parts of Definition 2.1 to be concave.

Indeed, if we assume the convex smooth variational principle in \((i)\), then the function \(\varphi + \Delta\) in the proof of \((i) \implies (ii)\) is convex, and one has the required version of the extremal principle \((ii)\). To prove \((ii) \implies (iii)\) for the modifications made, it suffices to check that in Proposition 2.3(ii) concave supporting functions for normals generate concave supporting functions for subdifferentials according to Definition 2.1. The latter follows directly from the proof of Proposition 2.3(ii) due to the explicit representation (2.11). Finally the proof of the convex/concave modification in \((iii) \implies (i)\) is obviously contained in the proof of Theorem 3.1.

Note that the Borwein-Preiss smooth variational principle ensures the fulfillment of \((i)\) in Theorem 3.1 with a convex function \(g(\cdot)\) in any \(\beta\)-smooth Banach space. Therefore, both extremal principle \((ii)\) and enhanced fuzzy sum rule \((iii)\) hold in this setting with concave supporting functions in Definition 2.1.

Now let us consider the case of the Fréchet bornology \(\beta = F\) and obtain further refinements of the equivalence results in this case. First we observe the following relationship between viscosity subdifferentials of controlled rank in Definition 2.1(ii) and the (full) \(F\)-
subdifferential (2.1) for locally Lipschitz functions.

3.5. **Proposition.** Let $X$ be a $F$-smooth space and let $\varphi : X \to \mathbb{R}$ be Lipschitz continuous around $\bar{x}$ with modulus $l$. Then one has

$$\partial^\lambda_F \varphi(\bar{x}) = \partial_F \varphi(\bar{x}) \text{ for all } \lambda > l. \quad (3.35)$$

**Proof.** The inclusion “$\subset$” in (3.35) follows directly from (2.1) for any viscosity $\beta$-subdifferentials of l.s.c. functions with any $\lambda > 0$. Let us justify the opposite inclusion based on our assumptions that $\beta = F$ and $\varphi$ is Lipschitz continuous with modulus $l < \lambda$.

It is well known that $\|x^*\| \leq l$ for any $x^* \in \partial_F \varphi(\bar{x})$. Since $x^*$ is a viscosity subgradient of $\varphi$ at $\bar{x}$ in the Fréchet bornology, there exist a neighborhood $U$ of $\bar{x}$ and a Fréchet-differentiable function $g : U \to \mathbb{R}$ such that $\nabla_F g(\bar{x}) = x^*$, $\nabla_F g : X \to X^*$ is norm-to-norm continuous around $\bar{x}$, and $\varphi - g$ attains a local minimum at $\bar{x}$. Taking into account that $\|\nabla_F g(\bar{x})\| \leq l < \lambda$ and $\nabla_F g(\cdot)$ is norm-to-norm continuous, one gets $\|\nabla_F g(x)\| < \lambda$ in some neighborhood of $\bar{x}$. Now employing the classical mean value theorem (again due to the norm-to-norm continuity of $\nabla_F g(\cdot)$), we conclude that $g$ is Lipschitz continuous around $\bar{x}$ with modulus $\lambda$. This yields $x^* \in \partial^\lambda_F \varphi(\bar{x})$ and ends the proof of the proposition. \(\square\)

Next we establish that in the case of $F$-smooth spaces the enhanced fuzzy sum rule (iii) in Theorem 3.1 is equivalent to the conventional fuzzy sum rule in terms of full $F$-subdifferentials (2.1) without using controlled ranks.

3.6. **Proposition.** Let $X$ be an $F$-smooth space and let $\varphi_i : X \to \mathbb{R}$, $i = 1, 2$, be l.s.c. functions finite at $\bar{x}$. Suppose that $\varphi_1$ is Lipschitz continuous around $\bar{x}$ with modulus $l$ and that $\varphi_1 + \varphi_2$ attains a local minimum at $\bar{x}$. Then for any $\varepsilon > 0$ and $\delta > 0$ there exist $x_i \in B_\delta(\bar{x})$ with $|\varphi_i(x_i) - \varphi_i(\bar{x})| \leq \delta$, $i = 1, 2$, such that

$$0 \in \partial_{F,1} \varphi_1(x_1) + \partial_{F,2} \varphi_2(x_2) + \varepsilon B^*. \quad (3.36)$$

**Proof.** Obviously the enhanced fuzzy sum rule (3.3) implies the one in (3.36), and this is true for any bornology $\beta$. Let us prove the opposite implication which is specific for the Fréchet case. Due to Proposition 3.5 the first terms of the sums in (3.3) and (3.36) are identical for sufficiently small $\delta$ since $\lambda > l$. We are going to show that (3.36) implies (3.3) despite the difference between $\partial_{F,2} \varphi$ and $\partial^\lambda_F \varphi_2$ for general l.s.c. functions.

Indeed, let (3.36) hold, i.e., given positive numbers $\varepsilon$ and $\delta$ we find $x_i \in B_\delta(\bar{x})$ with $|\varphi_i(x_i) - \varphi_i(\bar{x})| \leq \delta$ and $x_i^* \in \partial_{F} \varphi_i(x_i)$, $i = 1, 2$, such that $\|x_i^* + x_2^*\| \leq \varepsilon$. The latter yields

$$\|x_2^*\| \leq \|x_1^*\| + \|x_2^* + x_1^*\| \leq \|x_1^*\| + \varepsilon. \quad (3.37)$$

Taking an arbitrary $\lambda > l$, we suppose, without loss of generality, that $\delta$ and $\varepsilon$ are sufficiently small to satisfy our requirements in what follows. Thus $\varphi_1$ is locally Lipschitzian around $x_1$ with modulus $l$ and, therefore, $\|x_1^*\| \leq l$. This implies $x_1^* \in \partial_{F} \varphi_1^\lambda(x_1)$ due to Proposition 3.5 and also $\|x_2^*\| < \lambda$ due to (3.37). Based on the latter fact and arguing similarly to the proof
of Proposition 3.5, we conclude that \( x_2^* \in \partial F \varphi_2(x_2) \). Thus we arrive at (3.3) and end the proof of the proposition. \( \square \)

Combining results in Theorem 3.1 and Proposition 3.6 with an equivalence result in [29] for the Fréchet case, we are able to establish the equivalence between the smooth variational principle formulated above and versions of the extremal principle and the fuzzy sum rule involving the full viscosity constructions (2.1) and (2.5) with \( \beta = F \).

3.7. Theorem. In the case of \( F \)-smooth spaces, the smooth variational principle in Theorem 3.1 is equivalent to the fuzzy sum rule in Proposition 3.6 as well as the following \( F \)-extremal principle:

For any locally extremal point \( \bar{x} \in \Omega_1 \cap \Omega_2 \) of the closed set system \( \{ \Omega_1, \Omega_2 \} \) in a \( F \)-smooth space and for any \( \varepsilon > 0 \) there exist \( x_i \in \Omega_i \cap B_\varepsilon(\bar{x}) \) and \( x_i^* \in N_F(x_i; \Omega_i) + \varepsilon B^* \) such that

\[
\|x_1^*\| + \|x_2^*\| = 1 \quad \text{and} \quad x_1^* + x_2^* = 0.
\]

Proof. The equivalence between the smooth variational principle and the fuzzy sum rule (3.36) follows from (i)\( \Rightarrow \) (iii) in Theorem 3.1 and Proposition 3.6. On the other hand, it has been proved in [29] that the \( F \)-extremal principle is always equivalent to the fuzzy sum rule (3.36). Note that in the case of \( F \)-smooth spaces under consideration the definitions of the Fréchet subdifferential and normal cone used in [29] are equivalent to those in (2.1) and (2.5) for \( \beta = F \); see [12]. This completes the proof of the theorem. \( \square \)

3.8. Remark. The results in Propositions 3.5, 3.6 and Theorem 3.7 are based on properties of the Fréchet bornology and do not seem to be true in the case of weaker bornologies. The most essential property of the Fréchet bornology we use is that for \( \beta = F \) the topology in \( X_\beta^* \) agrees with the norm topology in \( X^* \) and, therefore, the continuity of \( \nabla_\beta g : X \to X_\beta^* \) required in Definition 2.1 reduces to the norm-to-norm continuity. The latter fact, which does not hold for weaker bornologies \( \beta \), is crucial in the proofs of the mentioned results.

3.9. Remark. It follows from the proofs given above that analogs of the results in 3.5–3.7 hold if one imposes stronger (than Fréchet) differentiability properties on supporting functions in the definitions of viscosity normals/subdifferentials and the smooth variational principle. Indeed, we can assume that these supporting functions are simultaneously \( s \)-Hölder smooth and \( X \) has a power modulus of smoothness \( t^{s+1} \) for some \( s \in (0, 1] \), in particular, superreflexive (see [4, 12]). The case of \( s = 1 \) corresponds to proximal normals and subgradients in Hilbert spaces [7, 33].

3.10. Remark. As we mentioned before, the equivalence between the extremal principle and the fuzzy sum rule has been established by Mordukhovich and Shao [29] in the case of the Fréchet bornology. This result is extended by Zhu [35] to general bornological spaces who also expands the list of equivalent results including a (non-Lipschitz) \( \beta \)-local fuzzy sum rule under an appropriate qualification condition, a \( \beta \)-nonlocal fuzzy sum rule, and a \( \beta \)-multidirectional mean value inequality formulated in terms of the full viscosity constructions.
(2.1) and (2.5); see [35] for more details and references. The proofs of all these equivalent results are based on the smooth variational principle not being equivalent to the latter. As we showed above, the usage of a proper control rank narrowing allows us to establish such an equivalence in general bornologically smooth spaces. At the same time, one does not need a control rank narrowing in the case of the Fréchet bornology as well as stronger s-Hölder smoothness concepts.

4. SOME APPLICATIONS

In the concluding section of the paper we consider some applications of the main results obtained in Section 3. First we present a corollary of the extremal principle in Theorem 3.1 that provides a refined nonconvex generalization of the Bishop-Phelps density theorem [32] in term of viscosity $\beta$-normals of controlled rank. Then we show that the enhanced fuzzy sum rule allows us to give a simplified proof and clarification of the main representation results in Borwein and Ioffe [2]. Sharper representations, recovered results in Mordukhovich and Shao [30], are obtained in the case of Fréchet-smooth spaces.

4.1. Theorem. Let $X$ be a $\beta$-smooth Banach space, $\Omega$ be a nonempty closed subset of $X$, and $\lambda > 0$. Then the set of $(\lambda, \beta)$-proper points, defined by

$$x \in \Omega \text{ with } N_\beta^\lambda(x; \Omega) \neq \{0\},$$

is dense in the boundary of $\Omega$.

Proof. Let $\bar{x}$ be a boundary point of the set $\Omega$. Then it is a locally extremal point of the system $\{\Omega_1, \Omega_2\}$ where $\Omega_1 := \Omega$ and $\Omega_2 := \{\bar{x}\}$. Let us apply to this system the extremal principle (ii) in Theorem 3.1. Given $\lambda > 0$, we take positive numbers $\varepsilon, \nu, \xi$ such that

$$\varepsilon < \lambda/2 \text{ and } \nu = \xi = \lambda.$$ 

According to Theorem 3.1(ii) there exist $x_{\lambda \varepsilon} \in \Omega$ and $x_{1\lambda \varepsilon}^*, x_{2\lambda \varepsilon}^* \in X^*$ satisfying

$$x_{\lambda \varepsilon} \in B_\varepsilon(\bar{x}), \quad x_{1\lambda \varepsilon}^* \in N_{\beta}^\lambda(x_{\lambda \varepsilon}; \Omega) \oplus \varepsilon B^*,$$

$$\|x_{1\lambda \varepsilon}^*\| + \|x_{2\lambda \varepsilon}^*\| = \lambda, \quad \text{and} \quad x_{1\lambda \varepsilon}^* + x_{2\lambda \varepsilon}^* = 0.$$ (4.2)

From the second inclusion in (4.2) we find $x_{\lambda \varepsilon}^* \in N_{\beta}^\lambda(x_{\lambda \varepsilon}; \Omega)$ such that $x_{\lambda \varepsilon}^* = x_{1\lambda \varepsilon}^* + \varepsilon e$ with $\|e\| \leq 1$. Since $\|x_{1\lambda \varepsilon}^*\| = \lambda/2$ due to (4.3) and $\varepsilon < \lambda/2$ by our assumption, one has

$$\|x_{\lambda \varepsilon}^*\| \geq \|x_{1\lambda \varepsilon}^*\| - \varepsilon = \lambda/2 - \varepsilon > 0.$$ (4.3)

This yields that $x_{\lambda \varepsilon} \in B_\varepsilon(\bar{x})$ belongs to the set (4.1) whenever $\varepsilon$ is sufficiently small.
To finish the proof of the theorem, we need to conclude that $x_\lambda$ is a boundary point of the closed set $\Omega$. The latter follows from the fact that $N^\lambda_\beta(x; \Omega) = \{0\}$ for any interior point $x$ of $\Omega$. Indeed, if

$$x^* \in N^\lambda_\beta(x; \Omega) \text{ with } x \in \text{int } \Omega,$$

then the corresponding $\beta$-smooth function $g(\cdot)$ in Definition 2.1(i) attains a local minimum on $\Omega$ at an interior point $x$. Therefore, $x^* = \nabla_\beta g(x) = 0$ due to the Fermat stationary principle which works in this setting. \qed

Next let us consider an application of the enhanced fuzzy sum rule to prove refined representations of nonconvex limiting normal and subdifferential constructions in smooth Banach spaces. First we recall the notions of $A$- and $G$-subdifferentials developed by Ioffe in general Banach spaces; see [18] and its references. If $X$ is a smooth space (admitting a Gâteaux-smooth renorm), then the $A$-subdifferential of a l.s.c. function $\varphi : X \to \mathbb{R}$ at $\bar{x} \in \text{dom } \varphi$ can be defined as follows

$$\partial_A \varphi(\bar{x}) := \operatorname{Limsup}_{x \to \bar{x}, \varepsilon \downarrow 0} \partial^-_{\varepsilon} \varphi(x)$$  \hspace{1cm} (4.4)

where “Limsup” means the topological Kuratowski-Painlevé upper limit (in the norm topology of $X$ and the weak-star topology of $X^*$) of the Dini subdifferential constructions

$$\partial^-_{\varepsilon} \varphi(x) := \{x^* \in X^* | \langle x^*, v \rangle \leq d(\varphi(x); v) + \varepsilon \|v\| \text{ for all } v \in X\}$$

with

$$d(\varphi(x); v) := \liminf_{u \to v, t \downarrow 0} \frac{1}{t} (\varphi(x + tv) - \varphi(x)).$$

Then the $G$-normal cone of $\Omega \subset X$ at $\bar{x} \in \text{cl } \Omega$ and the $G$-subdifferential of $\varphi$ at $\bar{x} \in \text{dom } \varphi$ are defined, respectively, by

$$N_G(\bar{x}; \Omega) := \bigcup_{\lambda > 0} \lambda \partial_A \text{dist}(\bar{x}; \Omega) \text{ and }$$

$$\partial_G \varphi(\bar{x}) := \{x^* \in X^* | (x^*, -1) \in N_G((\bar{x}, \varphi(\bar{x})); \text{epi } \varphi)\}$$  \hspace{1cm} (4.5)

through the $A$-subdifferential (4.4) of the (Lipschitz continuous) distance function. Note that constructions (4.5) and (4.6) were introduced in [18] as nuclei of the $G$-normal cone and the $G$-subdifferential therein which were obtained from (4.5) and (4.6) by using the weak-star topological closure. The terminology used herein was first adopted in [2]. In finite dimensions, constructions (4.4)–(4.6) reduce to the limiting normal cone and subdifferential defined in Mordukhovich [25].

We also consider the sequential $A$-subdifferential

$$\partial_A \varphi(\bar{x}) := \limsup_{x \to \bar{x}, \varepsilon \downarrow 0} \partial^-_{\varepsilon} \varphi(x)$$  \hspace{1cm} (4.7)
that may be strictly smaller than its topological counterpart (4.4) even for locally Lipschitz functions. Relationships between weak-star topological and sequential limits are studied in Borwein and Fitzpatrick [1] where one can find the following important result.

4.2. Proposition. Let \( X \) be a Banach space and let \( \{ S_k \} \) be a sequence of bounded subsets of \( X^* \) such that \( S_{k+1} \subset S_k \) for all \( k = 1, 2, \ldots \). If the unit ball of \( X^* \) is weak-star sequentially compact, then one has

\[
\bigcap_{k=1}^{\infty} \text{cl}^* S_k = \text{cl}^* \{ \lim_{k \to \infty} x^*_k \mid x^*_k \in S_k \text{ for all } k \}
\]

where the topological closure \( \text{cl}^* \) and the sequential limit are taken in the weak-star topology of \( X^* \).

Based on the enhanced fuzzy sum rule (3.3), now we can prove a refined sequential representation of the (topological) \( A \)-subdifferential (4.4) for Lipschitz continuous functions through their viscosity subdifferentials of controlled rank.

4.3. Theorem. Let \( X \) be a \( \beta \)-smooth Banach space and let \( \varphi : X \to \mathbb{X} \) be Lipschitz continuous around \( \bar{x} \) with modulus \( l \). Then for any \( \lambda > l \) one has

\[
\partial_A \varphi(\bar{x}) = \text{cl}^* \{ \limsup_{x \to \bar{x}} \partial_{\beta}^* \varphi(x) \}.
\]  

(4.8)

Proof. It follows directly from the definitions that \( \partial_{\beta}^* \varphi(x) \subset \partial_{\varepsilon}^* \varphi(x) \) for any \( x \in \text{dom} \varphi \), \( \lambda > 0 \), and \( \varepsilon \geq 0 \). This gives the inclusion “\( \supset \)” in (4.8). To prove the opposite inclusion, we first observe that the \( A \)-subdifferentials (4.4) and (4.7) can be represented as

\[
\partial_A \varphi(\bar{x}) = \bigcap_{k=1}^{\infty} \text{cl}^* S_k \quad \text{and} \quad \partial_{A}^* \varphi(\bar{x}) = \{ \lim_{k \to \infty} x^*_k \mid x^*_k \in S_k \text{ for all } k \}
\]

where \( S_k := \bigcup \{ \partial_{1/k}^* \varphi(x) \mid \|x - \bar{x}\| \leq 1/k \} \). Obviously \( S_{k+1} \subset S_k \) for all \( k = 1, 2, \ldots \) and, since \( \varphi \) is Lipschitz continuous around \( \bar{x} \), the sets \( S_k \) are bounded in \( X^* \) when \( k \) is sufficiently large. Now taking into account that for any smooth space \( X \) the unit ball in \( X^* \) is weak-star sequentially compact [32], we conclude from Proposition 4.2 that

\[
\partial_A \varphi(\bar{x}) = \text{cl}^*(\partial_A^* \varphi(\bar{x})).
\]

Therefore, representation (4.8) follows from the inclusion

\[
\partial_A^* \varphi(\bar{x}) \subset \text{cl}^* \{ \limsup_{x \to \bar{x}} \partial_{\beta}^* \varphi(x) \}.
\]

(4.9)

To establish (4.9), we are going to verify that

\[
\partial_A^* \varphi(\bar{x}) \subset \limsup_{x \to \bar{x}} \partial_{\beta}^* \varphi(\bar{x}) + V^*
\]

(4.10)
for any weak-star neighborhood $V^*$ of the origin in $X^*$. To this end we observe that for any such $V^*$ there exist a finite dimensional subspace $L \subset X$ and a number $\gamma > 0$ satisfying

$$L^\perp + 2\gamma B^* \subset V^* \text{ where } L^\perp := \{x^* \in X^* | \langle x^*, x \rangle = 0 \ \forall x \in L\}. \quad (4.11)$$

Now picking any $x^* \in \partial^*_\lambda \varphi(\bar{x})$ and using definition (4.7), one gets sequences $x_k \to \bar{x}$, $\varepsilon_k \downarrow 0$, and $x_k^* \rightharpoonup x^*$ with $x_k^* \in \partial^*_{\varepsilon_k} \varphi(x_k)$ for all $k$. We take $k$ to be sufficiently large to ensure that $0 < \varepsilon_k < \gamma$. It follows from the definition of the Dini $\varepsilon$-subdifferential that the function

$$f_k(x) := \varphi(x) - \langle x_k^*, x - x_k \rangle + \gamma\|x - x_k\| + \delta(x - x_k; L) \quad (4.12)$$

attains a local minimum at $x_k$ for each $k$. Applying the enhanced fuzzy sum rule (iii) of Theorem 3.1 to the sum of two functions in (4.12) with

$$\varphi_1(x) := \varphi(x) \quad \text{and} \quad \varphi_2(x) := -\langle x_k^*, x - x_k \rangle + \gamma\|x - x_k\| + \delta(x - x_k; L),$$

we find $x_{1k} \in B_{1/k}(x_k)$ and $x_{2k} \in B_{1/k}(x_k) \cap (x_k + L)$ such that

$$0 \in \partial^*_\lambda \varphi_1(x_{1k}) + \partial^*_{\varepsilon_k} \varphi_2(x_{2k}) + \gamma B^* \quad (4.13)$$

for any $\lambda > 1$ and large $k$. Note that $\varphi_2$ is convex and its first two summands are continuous. Thus one can employ the classical Moreau-Rockafellar theorem to compute the subdifferential of convex analysis for $\varphi_2$ and conclude that

$$\partial^*_\lambda \varphi_2(x_{2k}) \subset -x_k^* + \gamma B^* + L^\perp. \quad (4.14)$$

Now combining (4.13) with (4.11) and (4.14), we get

$$x_k^* \in \partial^*_\lambda \varphi(x_{1k}) + V^*. \quad (4.15)$$

It follows directly from Definition 2.1(ii) that the sets $\partial^*_\lambda \varphi(x_{1k})$ are uniformly bounded by $\lambda$ for all $k$ sufficient large. Using again the weak-star sequential compactness of bounded sets in $X^*$ for smooth spaces $X$ and passing to the limit in (4.15) as $k \to \infty$, we finally arrive at (4.10) and complete the proof of the theorem. \qed

Theorem 4.3 implies the main representation results in Borwein and Ioffe [2] for the $G$-normal cone (4.5) and the $G$-subdifferential (4.6) in smooth Banach spaces.

4.4. Corollary. Let $X$ be a $\beta$-smooth Banach space. Then the following hold:

(i) For any closed set $\Omega \subset X$ and any $\bar{x} \in \Omega$ one has

$$N_G(\bar{x}; \Omega) = \bigcup_{\lambda > 0} \text{cl}^* \{\limsup_{x \to \bar{x}} N^*_\lambda(x; \Omega)\}. \quad (4.16)$$

(ii) For any function $\varphi : X \to \mathbb{R}$ l.s.c. around $\bar{x} \in \text{dom } \varphi$ one has

$$\partial^*_G \varphi(\bar{x}) = \bigcup_{\lambda > 0} \text{cl}^* \{\limsup_{x \to \bar{x}} \partial^*_\lambda \varphi(x)\}. \quad (4.17)$$
Proof. To prove (i), we just observe that
\[
\limsup_{x \to \bar{x}, \lambda > 0} N^\lambda_{\beta}(x;\Omega) = \limsup_{x \to \bar{x}} \partial^\lambda_{\beta} \{ \lambda \dist(x;\Omega) \} = \limsup_{x \to \bar{x}} \lambda \partial^1_{\beta} \dist(x;\Omega)
\] (4.18)
due to (2.3). Therefore, representation (4.16) follows directly from (4.5), (4.8), and (4.18). Representation (4.17) follows from (4.16) due to (4.6) and Proposition 2.3. □

4.5. Remark. One can easily observe from Remark 3.4 and the proofs given above that the supporting functions \( g(\cdot) \) for \( \beta \)-normals and subdifferentials of controlled rank in representations (4.8), (4.16), and (4.17) can always be chosen to be \textit{concave}.

Now let us confine our treatment to the case of Fréchet-smooth spaces and recall the notions of limiting normal cone and subdifferential introduced by Kruger and Mordukhovich [22] as extensions of the corresponding finite dimensional constructions in [25]. Based on (Fréchet) \( \varepsilon \)-normals and subdifferentials
\[
\hat{N}_\varepsilon(x;\Omega) := \{ x^* \in X^* \mid \limsup_{u \to x, \Omega} \frac{\langle x^*, u - x \rangle}{\| u - x \|} \leq \varepsilon \},
\] (4.19)
\[
\hat{\partial}_\varepsilon \varphi(x) := \{ x^* \in X^* \mid \liminf_{u \to x} \frac{\varphi(u) - \varphi(x) - \langle x^*, u - x \rangle}{\| u - x \|} \geq -\varepsilon \}, \varepsilon \geq 0,
\] (4.20)
we define these \textit{sequential} limiting constructions as follows
\[
N(\bar{x};\Omega) := \limsup_{x \to \bar{x}, \varepsilon \downarrow 0} \hat{N}_\varepsilon(x;\Omega),
\] (4.21)
\[
\partial \varphi(\bar{x}) := \limsup_{x^* \to \bar{x}, \varepsilon \downarrow 0} \hat{\partial}_\varepsilon \varphi(x).
\] (4.22)

A recent comprehensive study of the limiting constructions (4.21) and (4.22) in the framework of Asplund spaces (including spaces with Fréchet renorms) has been conducted in [30] where the reader can find detailed discussions and further references. Our purpose here is to show how one can easily prove results of [30] related to the description of (4.21) and (4.22) through viscosity normals and subdifferentials of controlled rank.

4.6. Theorem. Let \( X \) be an \( F \)-smooth Banach space. Then the following hold:
(i) For any closed set \( \Omega \subset X \) and any \( \bar{x} \in \Omega \) one has
\[
N(\bar{x};\Omega) = \bigcup_{\lambda > 0} \limsup_{x \to \bar{x}} N^\lambda_{F}(x;\Omega).
\] (4.23)
(ii) For any function \( \varphi : X \to \mathbb{R} \) l.s.c. around \( \bar{x} \in \text{dom} \varphi \) one has
\[
\partial \varphi(\bar{x}) = \bigcup_{\lambda > 0} \limsup_{x^* \to x} \partial^\lambda_{F, \varphi}(x^*).
\] (4.24)
Proof. First we consider the case when \( \varphi \) is a function Lipschitz continuous around \( \bar{x} \) with modulus \( l \). Let us show that in this case the limiting subdifferential (4.22) admits the representation

\[
\partial \varphi(\bar{x}) = \limsup_{x \to \bar{x}} \partial^\lambda_F \varphi(x) \quad \forall \lambda > l.
\]  

(4.25)

Indeed, due to Proposition 3.5 the \( F \)-subdifferential \( \partial^\lambda_F \varphi(x) \) of controlled rank coincides with the full viscosity \( F \)-subdifferential (2.1) for any \( \lambda > l \). Therefore, (4.25) reduces to

\[
\partial \varphi(\bar{x}) = \limsup_{x \to \bar{x}} \partial F \varphi(x).
\]

It is shown in [12] that \( \partial F \varphi \) agrees with definition (4.20) as \( \varepsilon = 0 \) for any \( F \)-smooth space. To finish the proof of (4.25), it suffices to recall that the original limiting construction (4.22) is equivalent to the one with \( \varepsilon = 0 \) when \( X \) is an \( F \)-smooth (even Asplund) space; cf. [19, 30].

Based on (4.25), we establish representation (4.23) by using the relationship

\[
N(\bar{x}; \Omega) := \bigcup_{\lambda > 0} \lambda \partial \text{dist}(\bar{x}; \Omega)
\]

between the limiting constructions (4.21) and (4.22).

It remains to justify representation (4.24) for general l.s.c. functions. This follows from (4.23) due to the formula

\[
\partial \varphi(\bar{x}) = \{ x^* \in X^* \mid (x^*, -1) \in N((\bar{x}, \varphi(\bar{x})); \text{epi } \varphi) \}
\]

and its counterpart for the Fréchet constructions (4.19) and (4.20) with \( \varepsilon = 0 \). \( \square \)

One can observe that the only difference between the limiting constructions (4.21), (4.22) and the corresponding \( G \)-constructions (4.5), (4.6) in the case of \( F \)-smooth spaces is the absence of the weak-star topological closure in representations (4.23) and (4.24). We refer the reader to Section 9 of [30] for more discussions in this direction.

REFERENCES


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